Atmospheric Flight Dynamics Summary

1. Stochastic variables and processes

In this chapter, we're going to examine stochastic variables and stochastic processes. First, we look at one stochastic variable. Then, we examine multiple stochastic variables. Finally, we're going to examine stochastic processes. What kind of properties can such processes have?

1.1 Stochastic variables and their properties

1.1.1 Probability distribution and probability density functions

Let's examine a **stochastic variable** $\bar{x}$, also called a **random variable**. A stochastic variable can be seen as a normal variable x, but with uncertainty concerning its value. For example, in 1/3 of the cases its value may be 2, but in the other 2/3 of the cases, its value may be 3.

Every random variable has an associated **probability distribution function** $F_{\bar{x}}(x)$, also known as the **cumulative distribution function**. This function is defined as the probability that $\bar{x} \leq x$. So, in formal notation,

$$F_{\bar{x}}(x) = \Pr\{\bar{x} \leq x\}. \tag{1.1.1}$$

Such a function has several evident properties. We have $F_{\bar{x}}(-\infty) = 0$ and $F_{\bar{x}}(+\infty) = 1$. Also, the function is monotonically increasing. So, if $a \leq b$ then also $F_{\bar{x}}(a) \leq F_{\bar{x}}(b)$.

There is also the **probability density function** $f_{\bar{x}}(x)$, abbreviated as PDF. (Note that PDF does not mean probability distribution function!) The PDF is defined as the derivative of the probability distribution function. So,

$$f_{\bar{x}}(x) = \frac{dF_{\bar{x}}(x)}{dx}. \tag{1.1.2}$$

It immediately follows that $f_{\bar{x}}(x) \geq 0$. We also have

$$\int_{-\infty}^{\infty} f_{\bar{x}}(x) = 1, \qquad \int_{-\infty}^{b} f_{\bar{x}}(x) = F_{\bar{x}}(b) \qquad \text{and} \qquad \int_{-a}^{b} f_{\bar{x}}(x) = F_{\bar{x}}(b) - F_{\bar{x}}(a). \tag{1.1.3}$$

1.1.2 Moments of distributions

Often, it is very hard, if not impossible, to exactly determine $F_{\bar{x}}(x)$ and $f_{\bar{x}}(x)$. But we may try to determine other quantities. For example, we have defined the i**th moment** of the PDF as

$$m_i = \mathrm{E}\left\{\bar{x}^i\right\} = \int_{-\infty}^{\infty} x^i f_{\bar{x}}(x)\, dx. \tag{1.1.4}$$

So, we have $m_0 = 1$. Also, $m_1 = \mu_{\bar{x}}$ is the **mean** or **average** of $\bar{x}$. A similar and even more important quantity is the i**th central moment** m_i'. It is defined as

$$m_i' = \mathrm{E}\left\{(\bar{x} - \mu_{\bar{x}})^i\right\} = \int_{-\infty}^{\infty} (x - \mu_{\bar{x}})^i f_{\bar{x}}(x)\, dx. \tag{1.1.5}$$

Now we have $m_1' = 0$. Also, $m_2' = \sigma_{\bar{x}}^2$ is the **variance** of the stochastic process. The square root of the variance, being $\sigma_{\bar{x}}$, is called the **standard deviation**.

1.1.3 The normal distribution

There is one very important and common type of distribution. This is the **normal distribution**, also known as the **Gaussian distribution**. Its PDF is given by

$$f_{\bar{x}}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{1.1.6}$$

Quite trivially, this distribution has $\mu = \mu_{\bar{x}}$ as mean and $\sigma^2 = \sigma_{\bar{x}}^2$ as variance.

The **central limit theorem** states that the PDF of a process, caused by a large number of other random processes, approximates the PDF of the Gaussian distribution. Such situations often occur in real life. So, in the remainder of this summary, we will mostly assume that unknown stochastic processes are normally distributed. All that is then left for us to do is to determine the mean $\mu_{\bar{x}}$ and the variance $\sigma_{\bar{x}}^2$.

1.2 Multiple stochastic variables

1.2.1 Definitions for multiple random variables

Let's examine the case where we have two random variables $\bar{x}$ and $\bar{y}$. The **joint probability distribution function** $F_{\bar{x}\bar{y}}(x, y)$ is now defined as

$$F_{\bar{x}\bar{y}}(x, y) = \Pr\{\bar{x} \leq x \wedge \bar{y} \leq y\}, \tag{1.2.1}$$

where the $\wedge$ operator means 'and'. We thus have $F_{\bar{x}\bar{y}}(-\infty, b) = F_{\bar{x}\bar{y}}(a, -\infty) = 0$, $F_{\bar{x}\bar{y}}(+\infty, +\infty) = 1$, $F_{\bar{x}\bar{y}}(a, +\infty) = F_{\bar{x}}(a)$ and $F_{\bar{x}\bar{y}}(+\infty, b) = F_{\bar{y}}(b)$.

Similarly, the **joint probability density function** (joint PDF) $f_{\bar{x}\bar{y}}(x, y)$ is defined as

$$f_{\bar{x}\bar{y}}(x, y) = \frac{\partial^2 F_{\bar{x}\bar{y}}(x, y)}{\partial x \partial y}. \tag{1.2.2}$$

The joint PDF has as properties

$$\int_{-\infty}^{a}\int_{-\infty}^{b} f_{\bar{x}\bar{y}}(x, y)\, dx\, dy = F_{\bar{x}\bar{y}}(a, b), \qquad \int_{-\infty}^{\infty} f_{\bar{x}\bar{y}}(x, y)\, dy = f_{\bar{x}}(x) \quad \text{and} \quad \int_{-\infty}^{\infty} f_{\bar{x}\bar{y}}(x, y)\, dx = f_{\bar{y}}(y). \tag{1.2.3}$$

It may occur that the value of one of the two random variables is known. Let's suppose that it is given that $\bar{y} = y_1$. We can then find the **conditional distribution** of $\bar{x}$ given $\bar{y}$ using

$$f_{\bar{x}}(x|\bar{y} = y_1) = \frac{f_{\bar{x}\bar{y}}(x, y_1)}{f_{\bar{y}}(y_1)}. \tag{1.2.4}$$

1.2.2 Moments of joint distributions

The **joint moment** m_{ij} of two random variables $\bar{x}$ and $\bar{y}$ is defined as

$$m_{ij} = \mathrm{E}\left\{\bar{x}^i \bar{y}^j\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^i y^j f_{\bar{x}\bar{y}}(x, y)\, dx\, dy. \tag{1.2.5}$$

The sum $n = i + j$ is called the **order** of the joint moment. It can be noted that $m_{10} = \mu_{\bar{x}}$ and $m_{01} = \mu_{\bar{y}}$. Also, the second order moment m_{11} is called the **average product** $R_{\bar{x}\bar{y}}$.

Of course, there is also a **joint central moment** m'_{ij}. It is defined as

$$m'_{ij} = \mathrm{E}\left\{(\bar{x} - \mu_{\bar{x}})^i (\bar{y} - \mu_{\bar{y}})^j\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\bar{x} - \mu_{\bar{x}})^i (\bar{y} - \mu_{\bar{y}})^j f_{\bar{x}\bar{y}}(x, y)\, dx\, dy. \tag{1.2.6}$$

The second order joint moments m'_{20} and m'_{02} are equal to the variances of $\bar{x}$ and $\bar{y}$: $m'_{20} = \sigma_{\bar{x}}^2$ and $m'_{02} = \sigma_{\bar{y}}^2$. The other second order joint moment m'_{11} is called the **covariance** $C_{\bar{x}\bar{y}}$. It satisfies

$$m'_{11} = C_{\bar{x}\bar{y}} = R_{\bar{x}\bar{y}} - \mu_{\bar{x}}\mu_{\bar{y}} = m_{11} - m_{10}m_{01}. \tag{1.2.7}$$

Finally, we can define the **correlation** $K_{\bar{x}\bar{y}}$ as

$$K_{\bar{x}\bar{y}} = \frac{C_{\bar{x}\bar{y}}}{\sigma_{\bar{x}}\sigma_{\bar{y}}} = \frac{m'_{11}}{\sqrt{m_{10}m_{01}}}. \tag{1.2.8}$$

1.2.3 Properties of multiple random variables

Let's examine two random variables $\bar{x}$ and $\bar{y}$. We say that $\bar{x}$ and $\bar{y}$ are...

- **orthogonal** if $\mathrm{E}\{\bar{x}\bar{y}\} = 0$.
- **fully correlated** if $\mathrm{E}\{\bar{x}\bar{y}\} = \mu_{\bar{x}}\mu_{\bar{y}} \pm \sigma_{\bar{x}}\sigma_{\bar{y}}$ or, equivalently, if $K_{\bar{x}\bar{y}} = \pm 1$.
- **uncorrelated** if $\mathrm{E}\{\bar{x}\bar{y}\} = \mathrm{E}\{\bar{x}\}\,\mathrm{E}\{\bar{y}\} = \mu_{\bar{x}}\mu_{\bar{y}}$ or, equivalently, if $K_{\bar{x}\bar{y}} = 0$. We now have $C_{\bar{x}\bar{y}} = 0$ and $\sigma_{\bar{x}+\bar{y}}^2 = \sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2$. Also, $\bar{x} - \mu_{\bar{x}}$ and $\bar{y} - \mu_{\bar{y}}$ are orthogonal.
- **independent** if $f_{\bar{x}\bar{y}}(x, y) = f_{\bar{x}}(x)f_{\bar{y}}(y)$. This also implies that $\bar{x}$ and $\bar{y}$ are uncorrelated. (Though the converse is not always true.)

1.3 Stochastic processes

1.3.1 Basics of stochastic processes

Let's suppose that we are doing an experiment several times. It could occur that the output signal $x(t)$ of the experiment is always the same: it is a **deterministic function**. However, often uncertainty is involved. In this case, the output $x(t)$ is a bit different every time. The output signal $\bar{x}(t)$ is then called a **stochastic function** or a **stochastic process**. At every time τ, the value of $\bar{x}(\tau)$ is a stochastic variable.

Every time we run the experiment, we get a certain output $x(t)$. This output is called a **realization** of the stochastic process $\bar{x}(t)$. The set of all realizations is called the **ensemble** of the process.

There always is a certain chance that a stochastic process $\bar{x}(t)$ results in a certain realization $x(t)$. If this chance is constant in time (that is, the **distribution** of $\bar{x}(t)$ is constant), then we call the process **stationary**. It is very hard, if not impossible, to show that a process is stationary. So it is often simply assumed that stochastic processes are stationary.

1.3.2 The distribution of stochastic processes

Previously, we talked about a stochastic process $\bar{x}(t)$. Every stochastic process also has a probability distribution and probability density function, which are defined as

$$F_{\bar{x}}(x;t) = \Pr\{\bar{x}(t) \leq x\} \qquad \text{and} \qquad f_{\bar{x}}(x;t) = \frac{\partial F_{\bar{x}}(x;t)}{\partial x}. \tag{1.3.1}$$

Now let's examine two stochastic processes $\bar{x}(t)$ and $\bar{y}(t)$. The joint distribution of these two processes is defined as

$$F_{\bar{x}\bar{y}}(x, y; t_1, t_2) = \Pr\{\bar{x}(t_1) \leq x \wedge \bar{y}(t_2) \leq y\} \qquad \text{and} \qquad f_{\bar{x}\bar{y}}(x, y; t_1, t_2) = \frac{\partial^2 F_{\bar{x}\bar{y}}(x, y; t_1, t_2)}{\partial x \partial y}. \tag{1.3.2}$$

Often, it is assumed that the processes $\bar{x}(t)$ and $\bar{y}(t)$ are stationary. This means that not the times t_2 and t_1 theirselves are important, but only the time difference $\tau = t_2 - t_1$. We can thus write

$$F_{\bar{x}\bar{y}}(x, y; \tau) = \Pr\{\bar{x}(t) \leq x \wedge \bar{y}(t+\tau) \leq y\} \qquad \text{and} \qquad f_{\bar{x}\bar{y}}(x, y; \tau) = \frac{\partial^2 F_{\bar{x}\bar{y}}(x, y; \tau)}{\partial x \partial y}. \tag{1.3.3}$$

1.3.3 Properties of stochastic processes

Let's suppose that we know the joint distribution function $f_{\bar{x}\bar{y}}(x, y; \tau)$ of two stationary processes $\bar{x}(t)$ and $\bar{y}(t)$. We can now define the **moment function** $m_{ij}(\tau)$ of these processes as

$$m_{ij}(\tau) = \mathrm{E}\left\{\bar{x}(t)^i \bar{y}(t+\tau)^j\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^i y^j f_{\bar{x}\bar{y}}(x, y; \tau)\, dx\, dy. \tag{1.3.4}$$

Similarly, we can define the **central moment function** $m'_{ij}(\tau)$ as

$$m'_{ij}(\tau) = \mathrm{E}\left\{(\bar{x}(t) - \mu_{\bar{x}})^i (\bar{y}(t+\tau) - \mu_{\bar{y}})^j\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \mu_{\bar{x}})^i (y - \mu_{\bar{y}})^j f_{\bar{x}\bar{y}}(x, y; \tau)\, dx\, dy. \tag{1.3.5}$$

Several of these moments have special names. The **cross product function** $R_{\bar{x}\bar{y}}(\tau)$ is equal to $m_{11}(\tau)$ and the **cross covariance function** $C_{\bar{x}\bar{y}}(\tau)$ is equal to $m'_{11}(\tau)$. Also, the **cross correlation function** $K_{\bar{x}\bar{y}}(\tau)$ is defined as

$$K_{\bar{x}\bar{y}}(\tau) = \frac{C_{\bar{x}\bar{y}}(\tau)}{\sigma_{\bar{x}}\sigma_{\bar{y}}}. \tag{1.3.6}$$

Next to these three cross-functions, we also have three auto-functions. They are the **auto product function** $R_{\bar{x}\bar{x}}(\tau)$, the **auto covariance function** $C_{\bar{x}\bar{x}}(\tau)$ and the **auto correlation function** $K_{\bar{x}\bar{x}}(\tau)$. They are defined identically as the cross-functions, with the only difference that we substitute $\bar{y}(t+\tau)$ by $\bar{x}(t+\tau)$.

The cross correlation function $K_{\bar{x}\bar{y}}(\tau)$ is an indication of the correlation between two stochastic processes $\bar{x}(t)$ and $\bar{y}(t+\tau)$. But you might be wondering, what is the auto correlation function $K_{\bar{x}\bar{x}}(\tau)$ good for? Well, it gives an indication of how much the value of $\bar{x}(t+\tau)$ at time $t+\tau$ depends on the value of $\bar{x}(t)$ at time t. We'll examine how this works.

First, we can note that $K_{\bar{x}\bar{x}}(\tau)$ gives the correlation between the random variables $\bar{x}(t)$ and $\bar{x}(t+\tau)$. Generally, if τ becomes big, then the signals $\bar{x}(t)$ and $\bar{x}(t+\tau)$ will be uncorrelated: $K_{\bar{x}\bar{x}}(\tau)$ will go to zero. But for small (absolute) values of τ, the signals $\bar{x}(t)$ and $\bar{x}(t+\tau)$ are correlated a lot. (Especially if $\tau = 0$, because $K_{\bar{x}\bar{x}}(0) = 1$.) How fast $K_{\bar{x}\bar{x}}(\tau)$ goes to zero now determines how fast the signal $\bar{x}(t)$ loses its influence on $\bar{x}(t+\tau)$.

1.3.4 Ergodic processes

Let's examine a stochastic process $\bar{x}(t)$. We can examine all possible realizations $x(t)$ of this process. If we then take the (weighted) average of these realization values, we will find the **ensemble average** $\mu_{\bar{x}}(t)$ at time t.

However, in real life, we don't know all realizations of a stochastic process $\bar{x}(t)$. All we have is one realization $x(t)$. The average value μ_x of this realization is called the **time average**. An **ergodic process** is now defined as a process in which these averages are equal. Or, more formally, it is defined as a process for which, for every function $g(x)$, we have

$$\mathrm{E}\{g(\bar{x}(t))\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(\bar{x}(t))\, dt. \tag{1.3.7}$$

In real life, because we only have one realization, we often assume that a process is ergodic. This implies that our single realization is representative for the entire process. In other words, we can use it to derive the process properties. For this, we can use

$$\mu_{\bar{x}} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt, \qquad \sigma_{\bar{x}}^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (x(t) - \mu_{\bar{x}})^2\,dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)^2\,dt - \mu_{\bar{x}}^2, \quad (1.3.8)$$

$$R_{\bar{x}\bar{y}}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau)\,dt, \quad \text{and} \quad C_{\bar{x}\bar{y}}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau)\,dt - \mu_{\bar{x}}\mu_{\bar{y}}. \quad (1.3.9)$$

1.3.5 White noise

One special type of a stochastic process is **white noise** $\bar{w}(t)$. White noise has a zero mean: $\mu_{\bar{w}} = 0$. Next to this, the value of $\bar{w}(t)$ has absolutely no influence on the value of $\bar{w}(t+\tau)$ with $\tau \neq 0$. Thus, $C_{\bar{w}\bar{w}}(\tau) = 0$ for $\tau \neq 0$. To be more precise, the auto covariance function of white noise is defined as

$$C_{\bar{w}\bar{w}}(\tau) = W\delta(\tau), \quad (1.3.10)$$

where W is called the **intensity** of the white noise and $\delta(\tau)$ is the **Dirac delta function**. However, white noise is only a convenient theoretical trick. In real life, white noise as defined above does not occur. To show this, we can look at the variance $\sigma_{\bar{w}}^2$ of $\bar{w}(t)$. It is given by $\sigma_{\bar{w}}^2 = C_{\bar{w}\bar{w}}(0) = \infty$. This is physically of course impossible. So instead, in real life, we usually call a stochastic process $\bar{x}(t)$ white noise if $C_{\bar{x}\bar{x}}(\tau) \approx 0$ for $|\tau| > \epsilon$ for some sufficiently small ϵ.

2. Spectral analysis of continuous processes

In the previous chapter, we have examined systems in the time-domain. In both this chapter and the next one, we're going to look at the frequency domain. To do that, we first examine Fourier series and the Fourier transform. With this theory, we can then examine the properties of systems in the frequency domain. This chapter concerns the continuous-time case, while the next chapter deals with discrete time.

2.1 Fourier series

2.1.1 Continuous-time Fourier series

Let's examine a periodic function $x(t)$. (**Periodic** means that there is some T such that $x(t) = x(t+T)$ for all t.) We can approximate $x(t)$ by summing up several **basis functions**. In the **continuous-time Fourier series** (CTFS) approximation, we use sines and cosines as basis functions. So, we approximate $x(t)$ like

$$\tilde{x}(t) = \sum_{k=0}^{N-1} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right) = a_0 + \sum_{k=1}^{N-1} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right). \tag{2.1.1}$$

The **fundamental frequency** is generally chosen to be $\omega_0 = 2\pi/T$. (The frequency $k\omega_0$ is now called the k**th harmonic**.) In this case, all basis functions are **orthogonal** on the interval $[t_0, t_0+T]$. This means that, if k and l are positive integers, we have

$$\int_{t_0}^{t_0+T} \sin(k\omega_0 t)\cos(l\omega_0 t)\,dt = 0, \tag{2.1.2}$$

$$\int_{t_0}^{t_0+T} \sin(k\omega_0 t)\sin(l\omega_0 t)\,dt = \begin{cases} 0 & \text{if } k \neq l \\ \frac{T}{2} & \text{if } k = l, \end{cases} \tag{2.1.3}$$

$$\int_{t_0}^{t_0+T} \cos(k\omega_0 t)\cos(l\omega_0 t)\,dt = \begin{cases} 0 & \text{if } k \neq l \\ \frac{T}{2} & \text{if } k = l. \end{cases} \tag{2.1.4}$$

We can use the above equations to find the coefficients a_0, a_k and b_k. We will then find that

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt, \tag{2.1.5}$$

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(k\omega_0 t)\,dt, \tag{2.1.6}$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(k\omega_0 t)\,dt. \tag{2.1.7}$$

It is interesting to note that a_0 is, in fact, the average of the signal $x(t)$.

2.1.2 Continuous-time Fourier series in complex form

Using complex numbers, we can write the equations of the previous paragraph in a much easier form. Let's denote $j = \sqrt{-1}$ as the complex number. As you know, we can write $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$, so

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \quad \text{and} \quad \sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right). \tag{2.1.8}$$

We can now rewrite equation (2.1.1) to

$$\tilde{x}(t) = \sum_{k=-(N-1)}^{N-1} c_k e^{jk\omega_0 t}, \qquad \text{with} \qquad c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t}\,dt. \tag{2.1.9}$$

Note that the coefficients c_k are complex numbers as well. However, they are set such that the approximation $\tilde{x}(t)$ is a real-valued function. In the above equation, the left relation is called the **synthesis equation**. This is because it constructs/synthesizes the approximation $\tilde{x}(t)$ from basis functions. The right relation is called the **analysis equation**. This is because it analyses how to approximate $x(t)$ using basis functions.

We can also find the relationships between the coefficients a_k, b_k and c_k. These are

$$c_0 = a_0, \qquad c_k = \frac{1}{2}(a_k - jb_k), \qquad c_{-k} = \frac{1}{2}(a_k + jb_k), \tag{2.1.10}$$

$$a_0 = c_0, \qquad a_k = 2\text{Re}\{c_k\} = 2\text{Re}\{c_{-k}\}, \qquad b_k = -2\text{Im}\{c_k\} = 2\text{Im}\{c_{-k}\}. \tag{2.1.11}$$

The latter relations follow from the fact that c_k and c_{-k} are complex conjugates. We denote this by $c_k = c_{-k}^*$.

2.1.3 Properties of the Fourier series

The Fourier transform has several properties. We'll mention a couple of them. First of all, let's examine N. When N increases, then the approximation $\tilde{x}(t)$ of $x(t)$ becomes better. And, if $N \to \infty$, then $\tilde{x}(t) \to x(t)$.

When we find the Fourier transform of an even function, then we will only get cosine terms. So, $b_k = 0$ for all k. (An **even function** $x(t)$ satisfies $x(t) = x(-t)$.) Similarly, when we find the Fourier transform of an odd function, we only get sine terms. So, $a_k = 0$ for all k. (An **odd function** $x(t)$ satisfies $x(t) = -x(-t)$.)

Let's look at the average of the squared signal $x(t)^2$. It can be shown that this equals the sum of the squared Fourier series coefficients. So,

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(t)^2\,dt = a_0^2 + \sum_{k=1}^{\infty} \frac{1}{2}\left(a_k^2 + b_k^2\right) = \sum_{k=-\infty}^{\infty} |c_k|^2. \tag{2.1.12}$$

This relation is called **Parseval's theorem for the Fourier series expansion**.

Finally, we can consider the Fourier series expansion of the nth derivative of $x(t)$. We then find that it equals

$$\frac{d^n x(t)}{dt^n} = \sum_{k=-\infty}^{\infty} (jk\omega_0)^n c_k e^{jk\omega_0 t}. \tag{2.1.13}$$

2.2 The continuous-time Fourier transform

2.2.1 The Fourier transform equations

Previously, we have derived the Fourier series of periodic functions. However, now we examine an aperiodic function. This function can, in fact, be seen as a periodic function with period $T = \infty$. So we can approximate it using a Fourier series. If we take $N = \infty$ and $t_0 = -\frac{1}{2}T$, then we get

$$\tilde{x}(t) = \lim_{T\to\infty}\left(\sum_{k=-\infty}^{+\infty} \frac{\omega_0}{2\pi}\left(\int_{-\frac{1}{2}T}^{+\frac{1}{2}T} x(t) e^{-jk\omega_0 t}\,dt\right) e^{jk\omega_0 t}\right). \tag{2.2.1}$$

However, if $T \to \infty$, then $\omega_0 = 2\pi/T$ becomes infinitesimally small. So, we rewrite it as $d\omega$. This then turns the sum into an integral. We should then also denote $k\omega_0$ simply as ω. This turns the above equation into

$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \, dt \right) e^{j\omega t} \, d\omega. \tag{2.2.2}$$

The inner integral from the above equation is called the **Fourier integral**. In fact, it is the **Fourier transform** $X(\omega)$ of the signal $x(t)$. This Fourier transform is denoted as

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \, dt. \tag{2.2.3}$$

The outer integral is the **inverse Fourier transform**. It is written as

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} \, d\omega. \tag{2.2.4}$$

It must be noted that, in literature, there is no general consensus on where to place the $1/2\pi$ term in the above two equations. Some people put it in the first, other people put it in the second, and other people put the term $\sqrt{1/2\pi}$ in both terms. Also, when you use the frequency in Hertz instead of rad/s, the whole term vanishes altogether. In this summary, however, we'll simply use the notation of the above two equations.

2.2.2 Fourier transforms of basic functions

It can be worthwhile to remember the Fourier transform of several basic functions. When remembering them, it is convenient to keep in mind that $X(\omega)$ is an indication of how 'strong' the frequency ω is present in the signal. This may make it easier to remember. Now we'll list a couple of basic transforms.

- $\mathcal{F}\{1\} = 2\pi\delta(\omega)$, where $\delta(\omega)$ is again the Dirac delta function. So basically, only the frequency $\omega = 0$ is present in the signal $x(t) = 1$.
- $\mathcal{F}\{\cos(\omega_0 t)\} = \pi\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right)$. So, the frequencies $\omega = \omega_0$ and $\omega = -\omega_0$ are present in the signal $x(t) = \cos(\omega_0 t)$.
- Let's define the **block function** $b(t)$ with width T and the sinc function according to

$$b(t) = \begin{cases} 1 & \text{if } |t| < T/2, \\ 1/2 & \text{if } |t| = T/2, \\ 0 & \text{if } |t| > T/2, \end{cases} \qquad \text{and} \qquad \text{sinc}(x) = \frac{\sin(x)}{x}. \tag{2.2.5}$$

  Now, we have $B(\omega) = \mathcal{F}\{b(t)\} = T\text{sinc}\left(\omega\frac{T}{2}\right)$. It is interesting to note that, if $T \to \infty$, then $b(t) = 1$ for all t. Thus, $B(\omega) \to 2\pi\delta(\omega)$.
- Let's consider the block function $b(\omega)$ with width W in the frequency domain. Now let's take the inverse Fourier transform. We then get $\mathcal{F}^{-1}\{b(\omega)\} = \frac{W}{2\pi}\text{sinc}\left(\frac{W}{2}t\right)$.

By the way, the sinc function is quite an important function. This function has a big peak of $\text{sinc}(x) = 1$ at $x = 0$. For the rest, it is zero if $\omega = 2\pi k/T$, with k a nonzero integer. Also, the sinc function is an even function. So, $\text{sinc}(x) = \text{sinc}(-x)$.

It is interesting to note that transforming a block function gives a sinc-function, while transforming the sinc-function gives a block-function. This is due to the **duality property** of the Fourier transform. This property states that

$$\text{if} \quad \mathcal{F}\{x(t)\} = X(\omega) \quad \text{or, equivalently,} \quad x(t) = \mathcal{F}^{-1}\{X(\omega)\} \quad \text{then} \quad \mathcal{F}\{X(t)\} = 2\pi x(-\omega). \tag{2.2.6}$$

2.2.3 Making a function periodic

Let's suppose that we have some function $x(t)$. We can make this function periodic by 'copying' it and moving it by integer multiples of T_0. This gives us the periodic function $x_p(t)$, being

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(t + nT_0). \tag{2.2.7}$$

Because the above function is periodic with period T_0, we can find the Fourier series. But now it can be shown that the coefficients c_k of this series actually equal $X(k\omega_0)/T_0$, where $X(\omega) = \mathcal{F}\{x(t)\}$ and $\omega_0 = 2\pi/T_0$. So, the coefficients c_k can simply be derived from $X(\omega)$.

Alternatively, we can also find the continuous-time Fourier transform of $x_p(t)$. This then becomes

$$X_p(\omega) = \sum_{n=-\infty}^{+\infty} \frac{X(\omega)}{T_0} \delta(\omega - n\omega_0). \tag{2.2.8}$$

Note that this is a discrete function: it only has values at certain points.

Let's see how this trick works for the block function $b(t)$. First, we define the **periodic block function** $b_p(t)$ as

$$b_p(t) = \begin{cases} 1 & \text{if } |t| < T/2 + nT_0, \\ 1/2 & \text{if } |t| = T/2 + nT_0, \\ 0 & \text{if } |t| > T/2 + nT_0. \end{cases} \tag{2.2.9}$$

We can now find the Fourier series coefficients c_k. They will turn out to be equal to $\frac{T}{T_0}\text{sinc}\left(k\omega\frac{T}{2}\right)$, which is exactly what the above trick predicts them to be.

2.3 Spectral analysis applied to systems

2.3.1 Spectral analysis

Let's examine a stochastic process $\bar{x}(t)$. If we try to analyze it in the frequency domain, we run into a problem. The resulting Fourier transform will be different for every realization $x(t)$. However, usually we aren't interested in $x(t)$. Instead, we are interested in the **energy** of the process $\bar{x}(t)$. This energy is generally proportional to $\bar{x}(t)^2$ or, when two processes are involved, to $\bar{x}(t)\bar{y}(t+\tau)$. (Here, we do assume that $\bar{x}(t)$ and $\bar{y}(t)$ have zero mean. If not, they can be **normalized** by subtracting the mean from the process.)

We know that the product $\bar{x}(t)\bar{y}(t+\tau)$ is related to $C_{\bar{x}\bar{y}}(\tau)$. So, let's examine this parameter. We assume that both $\bar{x}(t)$ and $\bar{y}(t)$ are ergodic processes. $x(t)$ and $y(t)$ are realizations of these processes, with corresponding Fourier transforms $X(\omega)$ and $Y(\omega)$. It can now be shown that $C_{\bar{x}\bar{y}}(\tau)$ equals

$$C_{\bar{x}\bar{y}}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(\omega)X(-\omega)e^{j\omega\tau}\, d\omega. \tag{2.3.1}$$

We can define the **power spectral density function** (PSD function) $S_{\bar{x}\bar{y}}(\omega)$ as

$$S_{\bar{x}\bar{y}}(\omega) = \lim_{T\to\infty} \frac{1}{2T} Y(\omega)X(-\omega). \tag{2.3.2}$$

The relation between the covariance function $C_{\bar{x}\bar{y}}(\tau)$ and the power spectral density function $S_{\bar{x}\bar{y}}(\omega)$ is then very easy. It is given by

$$S_{\bar{x}\bar{y}}(\omega) = \mathcal{F}\{C_{\bar{x}\bar{y}}(\tau)\} = \int_{-\infty}^{+\infty} C_{\bar{x}\bar{y}}(\tau)e^{-j\omega\tau}\,d\tau, \tag{2.3.3}$$

$$C_{\bar{x}\bar{y}}(\tau) = \mathcal{F}^{-1}\{S_{\bar{x}\bar{y}}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{\bar{x}\bar{y}}(\omega)e^{j\omega\tau}\,d\omega. \tag{2.3.4}$$

By the way, if $S_{\bar{x}\bar{y}}(\omega)$ concerns two different processes, then we call it the **cross-power spectral density function**. If it concerns only one process, then we have the **auto-power spectral density function** $S_{\bar{x}\bar{x}}(\omega)$.

2.3.2 The Laplace transform versus the Fourier transform

Let's examine a system. This system has input $\bar{u}(t)$, output $\bar{y}(t)$ and an impulse response function $h(t)$. When dealing with systems, people often confuse the Laplace transform with the Fourier transform. These two transforms are, respectively, defined as

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \qquad \text{and} \qquad X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,d\omega. \tag{2.3.5}$$

The Laplace transform is more general than the Fourier transform. (If you insert the special case $s = j\omega$ in the Laplace transform, you get the Fourier transform.) The Laplace transform $H(s) = Y(s)/U(s)$ of the impulse response function $h(t)$ is called the **transfer function**. It can be used very well to investigate the transient responses of the system. (Think of the final value theorem and such.)

However, the Fourier transform has its qualities as well. The Fourier transform $H(\omega) = Y(\omega)/U(\omega)$ of $h(t)$ is called the **frequency response function** (FRF). It can be used very well to examine the frequency response of the system. Since, in this chapter, we're examining the frequency response of time-invariant processes, we will use the FRF.

2.3.3 System analysis in the frequency domain

Let's suppose that we know the properties of the stochastic input process $\bar{u}(t)$ which we put into a system. We also know the system dynamics, in the form of the impulse response function $h(t)$ or, alternatively, its Fourier transform $H(\omega)$. Can we then find the properties of the stochastic output process $\bar{y}(t)$?

The answer is simple: yes we can. First of all, we can find the mean $\mu_{\bar{y}}$ of $\bar{y}(t)$. It is given by $\mu_{\bar{y}} = H(0)\mu_{\bar{u}}$. However, usually we assume that the mean is zero. (If not, then we can normalize the signals by subtracting the mean.) If this is the case, then we can find the covariance function for $\bar{u}$ and $\bar{y}$. We have

$$C_{\bar{u}\bar{y}}(\tau) = C_{\bar{u}\bar{u}}(\tau) * h(\tau) = \int_{-\infty}^{+\infty} C_{\bar{u}\bar{u}}(\tau-\theta)h(\theta)\,d\theta. \tag{2.3.6}$$

The $*$ operator indicates the **convolution integral**, which is defined as shown above. Also,

$$C_{\bar{y}\bar{u}}(\tau) = C_{\bar{u}\bar{y}}(-\tau) = C_{\bar{u}\bar{u}}(\tau) * h(-\tau) \qquad \text{and} \qquad C_{\bar{y}\bar{y}}(\tau) = C_{\bar{u}\bar{u}}(\tau) * h(\tau) * h(-\tau). \tag{2.3.7}$$

To find the power spectral density function, we can simply take the Fourier transform. And luckily, the convolution integral in the time domain is simply multiplication in the frequency domain. So,

$$S_{\bar{u}\bar{y}}(\omega) = \mathcal{F}\{C_{\bar{u}\bar{y}}(\tau)\} = H(\omega)S_{\bar{u}\bar{u}}(\omega), \qquad S_{\bar{y}\bar{u}}(\omega) = H(-\omega)S_{\bar{u}\bar{u}}(\omega) \quad \text{and} \quad S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega). \tag{2.3.8}$$

The variance of the output process can now be found using

$$\sigma_{\bar{y}}^2 = C_{\bar{y}\bar{y}}(\tau=0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{\bar{x}\bar{x}}(\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{+\infty} S_{\bar{x}\bar{x}}(\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{+\infty} |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega)\,d\omega. \tag{2.3.9}$$

2.3.4 White and colored noise in the frequency domain

Previously, we have defined white noise $\bar{w}(t)$. The covariance function was $C_{\bar{w}\bar{w}}(\tau) = W\delta(\tau)$. The power spectral density function now becomes

$$S_{\bar{w}\bar{w}}(\omega) = \int_{-\infty}^{+\infty} C_{\bar{w}\bar{w}}(\tau)e^{-j\omega\tau}\,d\tau = W. \tag{2.3.10}$$

In real life, this of course isn't possible. (A signal can't have energy at all frequencies.) So instead, we call a signal white noise if $S_{\bar{w}\bar{w}}(\omega) = W$ for $-\omega_1 < \omega < \omega_1$, with ω_1 sufficiently big.

Now let's suppose that we use white noise as the input $\bar{u}(t)$ of a system. The resulting output has a power spectral density function of $S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 W$. The variance of the output can then be found using

$$\sigma_{\bar{y}}^2 = W\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} |H(\omega)|^2\,d\omega\right). \tag{2.3.11}$$

The output can be seen as filtered white noise, also known as **colored noise**, with the FRF $H(\omega)$ as the **shaping filter**.

2.3.5 A system with noise

Let's consider a system with input $\bar{u}(t)$ and output $\bar{x}(t)$. Assume that we don't know the frequency response function $H(\omega)$ of the system. But luckily, we can measure the output. However, the measured output $\bar{y}(t)$ is distorted by a noise $\bar{n}(t)$. Thus, $\bar{y}(t) = \bar{x}(t) + \bar{n}(t)$. The question is, can we find $H(\omega)$? Yes, we can. After some derivation, we can find that

$$H(\omega) = \frac{S_{\bar{u}\bar{y}}(\omega)}{S_{\bar{u}\bar{u}}(\omega)}. \tag{2.3.12}$$

We can also find information about the noise $\bar{n}(t)$. Its PSD function is given by

$$S_{\bar{n}\bar{n}}(\omega) = S_{\bar{y}\bar{y}}(\omega) - |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega) = S_{\bar{y}\bar{y}}(\omega) - \frac{|S_{\bar{u}\bar{y}}(\omega)|^2}{S_{\bar{u}\bar{u}}(\omega)}. \tag{2.3.13}$$

Finally, we can also compare the real output signal $\bar{x}(t)$ to the measured output signal $\bar{y}(t)$. We then find that

$$\frac{S_{\bar{x}\bar{x}}(\omega)}{S_{\bar{y}\bar{y}}(\omega)} = \frac{|H(\omega)^2|S_{\bar{u}\bar{u}}(\omega)}{S_{\bar{y}\bar{y}}(\omega)} = \frac{|S_{\bar{u}\bar{y}}(\omega)^2|}{S_{\bar{u}\bar{u}}(\omega)S_{\bar{y}\bar{y}}(\omega)} = \Gamma_{\bar{u}\bar{y}}(\omega)^2, \qquad \text{where} \qquad \Gamma_{\bar{u}\bar{y}}(\omega) = \sqrt{\frac{|S_{\bar{u}\bar{y}}(\omega)^2|}{S_{\bar{u}\bar{u}}(\omega)S_{\bar{y}\bar{y}}(\omega)}}. \tag{2.3.14}$$

The function $\Gamma_{\bar{u}\bar{y}}(\omega)$ is called the **coherence** between the system input $\bar{u}(t)$ and the measured output $\bar{y}(t)$. A value of 0 indicates no coherence, while a value of 1 indicates full coherence.

3. Spectral analysis of discrete processes

In the previous chapter, we have examined spectral analysis for continuous-time processes. However, when working with computers, we can only deal with discrete-time processes. So, that's we'll focus on in this chapter. First, we'll look at how we can make a signal discrete. Second, we examine multiple ways to transform such discrete signals. Finally, we find out how can actually estimate the PSD function from a transformed discrete signal.

3.1 Making a signal discrete: sampling

3.1.1 The working principle of sampling

Let's suppose that we have a continuous signal $x(t)$. The Fourier transform of this function is $X(\omega)$. But we don't want a continuous signal. Instead, we want a discrete signal. To acquire one, we first define the **pulse train** $y(t)$ as

$$y(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n\Delta t), \tag{3.1.1}$$

where n is an integer, Δt is the **sampling period** and $\omega_s = \omega_0 = 2\pi/\Delta t$ is the **sampling frequency**. According to a trick from the previous chapter, this pulse train has a Fourier transform of

$$Y(\omega) = \frac{2\pi}{\Delta t} \sum_{n=-\infty}^{+\infty} \delta\left(\omega - n\frac{2\pi}{\Delta t}\right). \tag{3.1.2}$$

We can use $y(t)$ to create the discrete signal $z(t)$ of $x(t)$. To do this, we multiply $x(t)$ by $y(t)$. Thus,

$$z(t) = x(t)y(t) = \sum_{n=-\infty}^{+\infty} x(t)\,\delta(t - n\Delta t). \tag{3.1.3}$$

Now we would like to find the Fourier transform $Z(\omega)$ of $z(t)$. Since multiplication in the time domain means convolution in the frequency domain, we get

$$Z(\omega) = \frac{1}{2\pi} X(\omega) * Y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\xi) Y(\omega - \xi)\, d\xi. \tag{3.1.4}$$

By inserting the relation for $Y(\omega)$ and by working things out further, we can then find that

$$Z(\omega) = \frac{1}{\Delta t} \sum_{n=-\infty}^{+\infty} X(\omega - n\omega_0). \tag{3.1.5}$$

So basically, to find $Z(\omega)$, we take $X(\omega)$, scale it by $1/\Delta t$, make copies of it and then shift it by integer multiples of ω_0. This thus makes $Z(\omega)$ periodic. By the way, the above function $Z(\omega)$ is the **continuous-time Fourier transform** (CTFT) of the discrete signal. We will examine the discrete-time Fourier transform later in this chapter.

It is interesting to note the duality with the 'making-a-function-periodic' trick from the last chapter. There, we made a normal function $x(t)$ periodic by copying it and shifting these copies by integer multiple of T_0. The result was a discrete version of the original Fourier transform, scaled by a factor $1/T_0$. Here, we make a function discrete with time step Δt. The result is a copied-and-shifted version of the original Fourier transform, scaled by a factor $1/\Delta t$.

3.1.2 Losing data due to sampling

Of course, the downside of sampling is that data is lost. In the time domain, we only have data at the points $z(k\Delta t)$ (or alternatively, $z[k]$). The data in between these points is lost. In the frequency domain, something different occurs. Let's examine the function $Z(\omega)$ in the interval $[-\omega_s/2, \omega_s/2]$. In this interval, we don't only have the original function $X(\omega)$. Instead, copies of shifted versions of $X(\omega)$ are also added. This often makes it impossible to extract the original function $X(\omega)$ in this interval. This phenomenon is called **aliasing**.

Luckily, there is a trick to reduce this problem. We simply take the original signal $X(\omega)$ and, before we alias it, we remove all frequencies higher than the so-called **Nyquist frequency** $\omega_n = \omega_s/2$. (Alternatively, we can select the sampling frequency ω_s to be twice as high as the highest frequency that already exists.) This does mean that we lose some data about the higher frequencies. However, the resulting adjusted signal $X(\omega)$ won't be effected by the sampling. In other words, we keep perfect information about the more important lower frequencies.

Based on this information, we can also answer the question when we can reconstruct the exact original signal $x(t)$ from the sampled signal $z(t)$. Let's suppose that $X(\omega)$ is a **bandlimited signal**. This means that there is a frequency ω_M such that if $|\omega| > \omega_M$, then $X(\omega) = 0$. If we now choose the sampling frequency ω_s such that $\omega_s > 2\omega_M$, then we can perfectly reconstruct the original signal $x(t)$ from the sampled signal $z(t)$. If this is not the case, then there will be errors. (This theorem is called the **(Shannon) sampling theorem**.)

3.1.3 Reconstruction of the original signal

The transformation of the discrete-time signal $z(t)$ back to the continuous-time signal $x(t)$ is called **signal reconstruction**. To do this, we should simply take $Z(\omega)$ and only use the values in the interval $[-\omega_s/2, \omega_s/2]$. In other words, if we take as **reconstruction filter** $R(\omega)$ a block function with width ω_s, then we will simply have as reconstructed signal

$$X_r(\omega) = Z(\omega)R(\omega). \tag{3.1.6}$$

If we put this equation in the time-domain, then the reconstruction filter will be a sinc function. So, $r(t) = \text{sinc}\left(\frac{\omega_s t}{2}\right) = \text{sinc}\left(\frac{\pi t}{\Delta t}\right)$. To find the reconstructed signal $x_r(t)$ in the time domain, we can then use the convolution integral

$$x_r(t) = z(t) * r(t) = \int_{-\infty}^{+\infty} x(\theta)\delta(\theta - n\Delta t)\,\text{sinc}\left(\frac{\omega_s}{2}(t-\theta)\right). \tag{3.1.7}$$

The above integrand only gives a nonzero value if $\theta = n\Delta t$. Thus, we find that

$$x_r(t) = \sum_{-\infty}^{+\infty} x(n\Delta t)\,\text{sinc}\left(\frac{\omega_s}{2}(t - n\Delta t)\right). \tag{3.1.8}$$

So, the reconstructed signal is a sum of sinc functions. The nth sinc function has its center at $t = n\Delta t$ (so, at the nth sample) and has magnitude $x(n\Delta t)$ (which is the magnitude of the nth sample). Also, the sinc function is zero at the position of the other samples. So, at least we can always be sure that $x(n\Delta t) = x_r(n\Delta t)$ for all n. That is, the reconstructed signal equals the original signal at the position of the samples. In between these samples. $x_r(t)$ usually only approximates $x(t)$.

3.1.4 Reconstruction with insufficient data

The previous reconstruction method requires us to have $z(t)$ available at all times t. But in practice, this often isn't the case, as we can't look into the future. To solve this problem, we use a different reconstruction method, called the **zero-order hold** (ZOH). We now simply say that the reconstructed signal $x_r(t)$ equals the last sample that we have found. So,

$$x_r(t) = z(k\Delta t) \quad \text{for} \quad k\Delta t \leq t < (k+1)\Delta t. \tag{3.1.9}$$

This method is actually equivalent with using as reconstruction signal $r(t)$ a block with width Δt, shifted in time by $\Delta t/2$. So, $r(t) = b(t - \Delta t/2)$. In the frequency domain, we then have

$$R(\omega) = e^{-j\omega\Delta t/2} B(\omega) = e^{-j\omega\Delta t/2} \Delta t \,\mathrm{sinc}\left(\frac{\omega\Delta t}{2}\right). \tag{3.1.10}$$

Alternatively, there is also the **first-order hold** (FOH), but we won't discuss that method here.

3.2 Discrete Fourier transforms

3.2.1 The discrete-time Fourier transform

It is time to really start to work in discrete time. First, we will denote $x(n\Delta t)$ simply as $x[n]$, and similarly, $z(n\Delta t) = z[n]$. (Note that $x[n]$ and $z[n]$ in fact are the same.) Now we define the **discrete-time Fourier transform** (DTFT) of a discrete signal $x[n]$ as

$$X(\Omega) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} \qquad \text{and} \qquad x[n] = \mathcal{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega) e^{j\Omega n}\, d\Omega. \tag{3.2.1}$$

The DTFT $X(\Omega)$ is very similar to the CTFT $Z(\omega)$ of the discrete signal $z(t)$. The fundamental difference is that we now don't use ω but Ω. The relation between the two is given by $\Omega = \omega\Delta t$. Also, as can be seen from the above relation, the DTFT $X(\Omega)$ has a period of 2π, whereas the CTFT $Z(\omega)$ has a period of $2\pi/\Delta t$. And finally, we don't use times $n\Delta t$ anymore, but we simply use indices n. So, the DTFT can be seen as a 'normalized' version of the CTFT, such that the factor Δt has been taken out.

3.2.2 The discrete Fourier transform

A different type of transform is the **discrete Fourier transform** (DFT). (Don't confuse the DTFT with the DFT!) Let's suppose that we have N samples $x[n]$ $(0 \leq n < N)$, taken with a sampling time Δt over a measurement time T. (So, $T = N\Delta t$.) The DFT $X[k]$ of $x[n]$ is now given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \qquad \text{and, as inverse,} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{jn\frac{2\pi}{N}k}. \tag{3.2.2}$$

Just like the DTFT resembles the CTFT, so does the DFT resemble the CTFS. To see how, compare the above equation with the relations for the CTFS, given by

$$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jk\frac{2\pi}{T}t}\, dt \qquad \text{and} \qquad x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\frac{2\pi}{T}t}. \tag{3.2.3}$$

So, the period T from the CTFS resembles the number of measurements N in the DFT. In fact, the function $X[k]$ is periodic with period N. So, $X[k] = X[k+N]$ for all k. Also, it can be noted that $X[-k] = X[n-k] = X^*[k]$, where $X^*[k]$ is the complex conjugate of $X[k]$. (Remember that we also had $c_{-k} = c_k^*$ with the CTFS.)

Let's denote $f_s = 1/\Delta t$ as the sampling frequency (in Hertz). Also, we say that $f_s/N = 1/N\Delta t = 1/T$ is the **frequency resolution** (FR) of the DFT in Hertz. (The FR in radians per second is $2\pi f_s/N$.) The

DFT can now be used to measure how 'strong' the frequency $f = mf_s/N$ (with $-N/2 < m < N/2$) is present in the signal $x[n]$. To see how this works, let's examine a signal with frequency f_0, like

$$x(t) = \cos(2\pi f_0 t), \qquad \text{or equivalently,} \qquad x[k] = \cos(2\pi f_0 k\Delta t). \tag{3.2.4}$$

Let's first assume that we can write $f_0 = mf_s/N$ for some integer m. (That is, f_0 is a multiple of the frequency resolution. We do ought to have $m < N/2$ though, since higher frequencies can't be measured with only N measurements.) In this case, $|X[\pm m]|$ will have a relatively high value, while $X[k] = 0$ if $k \neq \pm m$. But now assume that we can't write $f_0 = mf_s/N$ for some integer m. In this case, **spectral leakage** occurs. This means that $X[k]$ will have a value for about all k. This makes it hard to determine the actual frequency of the signal.

3.2.3 Applying a window

The DFT has leakage, while the DTFT does not. So, to prevent leakage, we can in a way draw inspiration from the DTFT. We start with a signal $x[n]$. (For example, the one of equation (3.2.4).) When we use the DFT to acquire $X[k]$, we only consider the signal $x[n]$ for $0 \leq n < N$. We want to do a similar thing with the DTFT. So, we define the **rectangular time window** $w[n]$ as

$$w[n] = \begin{cases} 1 & \text{if } 0 \leq n < N, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.5}$$

We then define the signal $y[n] = w[n]x[n]$. We put this signal $y[n]$ into the DTFT and find $Y(\Omega)$. It can now be shown that $X[k]$ actually equals $Y(\Omega)$ on the corresponding points. (That is, if $\Omega = 2\pi k/N$.)

Although the above is interesting, it doesn't solve the leakage problem. However, things are different if we use a different time window. Several good time windows are available. We could, for example, try the **Hanning window**

$$h[k] = \begin{cases} \frac{1}{2}\left(1 - \cos\left(\frac{2\pi k}{N}\right)\right) = \sin^2\left(\frac{\pi k}{N}\right) & \text{if } 0 \leq k < n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.6}$$

If we apply this window to the signal $x[k]$ (and thus get $y[k] = h[k]x[k]$) and put this new signal into the DFT, then there generally is less leakage. So, this significantly reduces our problem.

3.2.4 The fast Fourier transform

Let's suppose that we have a signal $x[n]$ and we want to find the DFT $X[k]$. We could simply use equation (3.2.2). However, this would require N^2 computations. The **fast Fourier transform** (FFT) is a very efficient algorithm that finds the DFT with only $N \log_2 N$ computations. Especially for big N, this saves a lot of computation time.

To apply the FFT, we first split the sequence $x[n]$ up into even terms $y[r]$ and odd terms $z[r]$. So,

$$y[r] = x[2r] \qquad \text{and} \qquad z[r] = x[2r+1] \qquad \text{for } 0 \leq r < N/2. \tag{3.2.7}$$

For each of the sequences $y[r]$ and $z[r]$, we now find the DFTs $Y[k]$ and $Z[k]$ for $0 \leq k < N/2$. We then put these two together to find $X[k]$, according to

$$X[k] = Y[k] + e^{-j\frac{2\pi k}{N}} Z[k]. \tag{3.2.8}$$

There is, however, a problem. During computations, we only have the values for $Y[k]$ and $Z[k]$ for $0 \leq k < N/2$. So the above equation only works for $0 \leq k < N/2$. But we need to find $X[k]$ for $0 \leq K < N$. To solve this problem, we make use of the fact that $Y[k]$ and $Z[k]$ are, in reality, periodic functions with period $N/2$. So, $Y[k] = Y[k+N/2]$ and $Z[k] = Z[k+N/2]$. By using this fact, we can find that, for $N/2 \leq K < N$, we have

$$X[k+N/2] = Y[k+N/2] + e^{-j\frac{2\pi(k+N/2)}{N}} Z[k+N/2] = Y[k] - e^{-j\frac{2\pi k}{N}} Z[k]. \tag{3.2.9}$$

(Note that we have used the fact that $e^{-j\pi} = -1$.) In this way, we can derive $X[k]$ from $Y[k]$ and $Z[k]$ in a very efficient way. The question remains, how do we find $Y[k]$ and $Z[k]$? Well, this is simply done recursively: we again use the FFT.

3.3 Calculating spectral estimates

3.3.1 An initial estimate

The DFT is quite popular, because it can be used to approximate the PSD function. Let's suppose that we have a stochastic process $\bar{x}(t)$. We have several samples $x[n]$ of a realization of this process. We have also found the DFT $X[k]$ of $x[n]$. The **Periodogram** $I_{N_{\bar{x}\bar{x}}}[k]$ of this signal $x[n]$ is now defined as

$$I_{N_{\bar{x}\bar{x}}}[k] = \frac{1}{N} X[-k]X[k] = \frac{1}{N} X^*[k]X[k] = \frac{1}{N}|X[k]|^2. \tag{3.3.1}$$

This periodogram is an unbiased estimate of the discrete-time PSD function $S_{\bar{x}\bar{x}}[k]$. That is,

$$\lim_{N\to\infty} \mathrm{E}\left\{I_{N_{\bar{x}\bar{x}}}[k]\right\} = S_{\bar{x}\bar{x}}(\omega). \tag{3.3.2}$$

However, it is not a consistent estimate: the variance doesn't go to zero as $N \to \infty$. Instead, we have

$$\lim_{N\to\infty} \mathrm{var}\left\{I_{N_{\bar{x}\bar{x}}}[k]\right\} = \sigma_{I_N}^2 = \sigma_{\bar{x}}^4 \neq 0. \tag{3.3.3}$$

3.3.2 Improving the estimate

We would like to reduce the variance of our estimate $I_{N_{\bar{x}\bar{x}}}[k]$. One way to do this is to take multiple signals $x[n]$ and derive an estimate $I_{N_{\bar{x}\bar{x}}}[k]$ for each one of them. However, the problem is that we usually only have one signal $x[n]$. According to **Bartlett's procedure**, we then simply divide this signal into K signals, each having M samples, with $N = KM$. The ith signal would thus be $x^{(i)}[n] = x[n+(i-1)M]$, with $0 \leq i < K$ and $0 \leq n < M$. The estimate $\hat{S}_{\bar{x}\bar{x}}[k]$ of the discrete-time PSD function now becomes

$$\hat{S}_{\bar{x}\bar{x}}[k] = \frac{1}{K}\sum_{i=1}^{K} I_M^{(i)}[k], \qquad \text{with} \qquad I_M^{(i)}[k] = \frac{1}{M}\left|\sum_{n=0}^{M-1} x^{(i)}[n]e^{-j\frac{2\pi k}{M}n}\right|^2. \tag{3.3.4}$$

The variance of the estimate $\hat{S}_{\bar{x}\bar{x}}[k]$ is now $1/K$ times the variance of each periodogram. So, that's a significant improvement. However, there of course is a downside. The frequency resolution equals $2\pi f_s/M$. So, decreasing the variance of the estimate means that you decrease the frequency resolution as well. When applying Bartlett's procedure, you thus have to make a tradeoff between these two parameters.

Let's suppose that we've finally found a satisfactory estimate $\hat{S}_{\bar{x}\bar{x}}[k]$ of the discrete-time PSD function $S_{\bar{x}\bar{x}}[k]$. Now we want to find an estimate of the actual continuous-time PSD function $\hat{S}_{\bar{x}\bar{x}}(\omega)$. What is the relation between the two? Well, at the end of paragraph 3.1.1, we've seen that transforming a discrete version of a signal with time step Δt is the same as transforming the continuous signal, copying-and-shifting it and scaling it by $1/\Delta t$. To reverse this, we thus need to scale the function $\hat{S}_{\bar{x}\bar{x}}[k]$ back by a factor Δt. So,

$$\hat{S}_{\bar{x}\bar{x}}(\omega) = \Delta t \hat{S}_{\bar{x}\bar{x}}[k]. \tag{3.3.5}$$

Here, we have $\omega\Delta t = \frac{2\pi k}{N}$. Do note though, that the estimate $\hat{S}_{\bar{x}\bar{x}}(\omega)$ is only valid for $-2\pi f_s/2 < \omega < 2\pi f_s/2$. Also, the resulting frequency resultion is given by $2\pi f_s/M$.

4. Multivariable stochastic processes

Previously, we have only dealt with single-input single-output systems. But what happens if you insert a stochastic vector into a multi-input system? That's what we'll look at in this chapter. First, we'll look at some multivariable probability theory. After that, we're going to examine the properties of signals as they are passed through a system. Finally, we discuss how we can use the impulse response function.

4.1 Multivariable probability theory

4.1.1 Distribution functions of stochastic vectors

Let's examine a **stochastic vector** $\bar{\mathbf{x}}$. This is simply a vector of stochastic variables. So, we have

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_n \end{bmatrix}^T. \tag{4.1.1}$$

The probability distribution function and the probability density function simply equal the joint probability distribution/density functions of the variables $\bar{x}_i$. So,

$$F_{\bar{\mathbf{x}}}(\mathbf{x}) = \Pr\{\bar{x}_1 \leq x_1 \wedge \bar{x}_2 \leq x_2 \wedge \dots \wedge \bar{x}_n < x_n\} \qquad \text{and} \qquad f_{\bar{\mathbf{x}}}(\mathbf{x}) = \frac{\partial^n F_{\bar{\mathbf{x}}}(\mathbf{x})}{\partial x_1 \, \partial x_2 \dots \partial x_n}. \tag{4.1.2}$$

4.1.2 Properties of stochastic vectors

In practice, we generally can't determine the exact distribution functions. Instead, we'll simply look at important parameters. For example, the **mean** (or **average**) $\mu_{\bar{\mathbf{x}}}$ is defined as

$$\mu_{\bar{\mathbf{x}}} = \mathrm{E}\{\bar{\mathbf{x}}\} = \begin{bmatrix} \mathrm{E}\{\bar{x}_1\} & \mathrm{E}\{\bar{x}_2\} & \dots & \mathrm{E}\{\bar{x}_n\} \end{bmatrix}^T. \tag{4.1.3}$$

So, applying the expectation operator E to a vector or matrix simply means applying it to every individual element of the vector/matrix. Similarly to the mean, we also have the $n \times m$ **covariance matrix** $C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}$ of two stochastic vectors $\bar{\mathbf{x}}$ (size n) and $\bar{\mathbf{y}}$ (size m). It is defined as

$$C_{\bar{\mathbf{x}}\bar{\mathbf{y}}} = \mathrm{E}\left\{(\bar{\mathbf{x}} - \mu_{\bar{\mathbf{x}}})(\bar{\mathbf{y}} - \mu_{\bar{\mathbf{y}}})^T\right\}. \tag{4.1.4}$$

In the above equation, we again simply have to take the expectation of every parameter $(\bar{x}_i - \mu_{\bar{x}_i})(\bar{y}_j - \mu_{\bar{y}_j})$ of the matrix $(\bar{\mathbf{x}} - \mu_{\bar{\mathbf{x}}})(\bar{\mathbf{y}} - \mu_{\bar{\mathbf{y}}})^T$ to find $C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}$.

Next to the covariance matrix, we of course also have the **autocovariance matrix** $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}$. It can be noted that this is a symmetric matrix ($C_{\bar{x}_i\bar{x}_j} = C_{\bar{x}_j\bar{x}_i}$). Also, its diagonal elements are the variances of the individual parameters ($C_{\bar{x}_i\bar{x}_i} = \sigma^2_{\bar{x}_i}$). We can use the autocovariance matrix to find the **correlation matrix** $K_{\bar{\mathbf{x}}\bar{\mathbf{x}}}$. This matrix is defined as

$$K_{\bar{\mathbf{x}}\bar{\mathbf{x}}} = \begin{bmatrix} \frac{C_{\bar{x}_1\bar{x}_1}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_1}} & \frac{C_{\bar{x}_1\bar{x}_2}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}} & \dots & \frac{C_{\bar{x}_1\bar{x}_n}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_n}} \\ \frac{C_{\bar{x}_1\bar{x}_2}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}} & \frac{C_{\bar{x}_2\bar{x}_2}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_2}} & \dots & \frac{C_{\bar{x}_2\bar{x}_n}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{C_{\bar{x}_1\bar{x}_n}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_n}} & \frac{C_{\bar{x}_2\bar{x}_n}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_n}} & \dots & \frac{C_{\bar{x}_n\bar{x}_n}}{\sigma_{\bar{x}_n}\sigma_{\bar{x}_n}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{C_{\bar{x}_1\bar{x}_2}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}} & \dots & \frac{C_{\bar{x}_1\bar{x}_n}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_n}} \\ \frac{C_{\bar{x}_1\bar{x}_2}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}} & 1 & \dots & \frac{C_{\bar{x}_2\bar{x}_n}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{C_{\bar{x}_1\bar{x}_n}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_n}} & \frac{C_{\bar{x}_2\bar{x}_n}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_n}} & \dots & 1 \end{bmatrix}. \tag{4.1.5}$$

Stochastic vectors can be transformed linearly. For example, we may have $\bar{\mathbf{y}} = A\bar{\mathbf{x}}$. Let's suppose that we know the properties of the vector $\bar{\mathbf{x}}$. The properties of $\bar{\mathbf{y}}$ can then be found using

$$\mu_{\bar{\mathbf{y}}} = A\mu_{\bar{\mathbf{x}}} \qquad \text{and} \qquad C_{\bar{\mathbf{y}}\bar{\mathbf{y}}} = AC_{\bar{\mathbf{x}}\bar{\mathbf{x}}}A^T. \tag{4.1.6}$$

4.1.3 Properties of multivariable stochastic processes

We can extend the above properties to stochastic processes. Let's examine the multivariable stochastic processes $\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{y}}(t)$. Once more, we assume that these properties are stationary. In a previous chapter, we defined the covariance function $C_{\bar{x}\bar{y}}(\tau)$ of the signals $\bar{x}(t)$ and $\bar{y}(t)$ as the covariance between $\bar{x}(t)$ and $\bar{y}(t+\tau)$. We do exactly the same to define the covariance function $C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\tau)$ of the two processes $\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{y}}(t)$. We thus get

$$C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\tau) = \mathrm{E}\left\{(\bar{\mathbf{x}}(t) - \mu_{\bar{\mathbf{x}}}(t))(\bar{\mathbf{y}}(t+\tau) - \mu_{\bar{\mathbf{y}}}(t+\tau))^T\right\}. \tag{4.1.7}$$

Once we have the covariance function, we can find the power spectral density $S_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\omega)$ function for multivariable stochastic processes. This is once more simply the Fourier transform of $C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\tau)$. So, $S_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\omega) = \mathcal{F}\left\{C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\tau)\right\}$. By the way, when you want to take the Fourier transform of a matrix, you simply transform all the individual elements of the matrix separately.

4.2 Stochastic processes in systems

4.2.1 Continuous-time and discrete-time systems

Let's examine a multivariable linear system. We denote the state vector by $\mathbf{x}$, the input vector by $\mathbf{u}$ and the output vector by $\mathbf{y}$. We can write the system in its state space form. This is done for continuous (left) and discrete systems (right) like

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \qquad \mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k], \tag{4.2.1}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t), \qquad \mathbf{y}[k] = C\mathbf{x}[k] + D\mathbf{u}[k]. \tag{4.2.2}$$

Now let's ask ourselves an interesting question. What will happen if we don't put a deterministic input vector $\mathbf{u}$ into the system, but a stochastic input vector $\bar{\mathbf{u}}$? Well, we usually assume that $\bar{\mathbf{u}}$ is a Gaussian vector. And in this case, it can be shown that $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ will be Gaussian vectors as well. How to find their properties will be discussed in the upcoming two sections.

4.2.2 Properties for continuous-time systems

Let's examine the state equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$ of a continuous system. This equation can be solved. We will then find that

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B\mathbf{u}(\tau)\,d\tau, \tag{4.2.3}$$

where $\Phi(t, t_0)$ is the **transition matrix**, defined as

$$\Phi(t, t_0) = e^{(t-t_0)A} = I + (t-t_0)A + \frac{(t-t_0)^2A^2}{2!} + \frac{(t-t_0)^3A^3}{3!} + \ldots = \sum_{n=0}^{+\infty} \frac{(t-t_0)^nA^n}{n!}. \tag{4.2.4}$$

Now let's suppose that we use white noise $\bar{\mathbf{w}}(t)$ as input. We thus have $\mu_{\bar{\mathbf{w}}} = \mathbf{0}$ and $C_{\bar{\mathbf{w}}\bar{\mathbf{w}}}(\tau) = W\delta(\tau)$, where W is the **intensity matrix**. We can use the above equations to find the mean $\mu_{\bar{\mathbf{x}}}(t)$ and the covariance matrix $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t)$ of the resulting stochastic state vector $\bar{\mathbf{x}}(t)$ at time t. We will have $\mu_{\bar{\mathbf{x}}}(t) = \Phi(t, t_0)\mu_{\bar{\mathbf{x}}}(t_0)$ and

$$C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t_1, t_2) = \Phi(t_1, t_0)C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t_0, t_0)\Phi(t_2, t_0)^T + \int_{t_0}^{\min(t_1,t_2)} \Phi(t_1,\tau)BWB^T\Phi(t_2,\tau)^T\,d\tau. \tag{4.2.5}$$

Note that we have used the notation $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t_1, t_2)$, instead of the normal notation $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(\tau)$. The reason for this is that the stochastic process $\bar{\mathbf{x}}(t)$ is not necessarily stationary. If we simply want to know the covariance matrix of $\bar{\mathbf{x}}(t)$ at time t, then we can insert $t_1 = t_2 = t$. We denote this matrix then as $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t)$. (Note that this is a different matrix function than $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(\tau)$.)

When dealing with systems, we usually aren't interested in transient behavior. Instead, it would be nice to know the steady state solution $C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss}$ of the above equation. By setting $dC_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t,t)/dt$ to zero, it can be derived that

$$0 = AC_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss} + C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss}A^T + BWB^T. \tag{4.2.6}$$

This is the **continuous-time Lyapunov equation**. A unique solution only exists if the matrix A is exponentially stable. (In other words, all eigenvalues are strictly negative.) If this is the case, then the solution is given by

$$C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss} = \int_{t_0}^{+\infty} e^{\tau A} BWB^T e^{\tau A^T} \, d\tau. \tag{4.2.7}$$

It is interesting to note that the above equation is equal to equation (4.2.5) when $t \to \infty$. The covariance matrix of exponentially stable systems thus always converges to the steady state covariance matrix.

4.2.3 Properties for discrete-time systems

In the previous paragraph, we considered a continuous system. Now, let's examine a discrete system. The state of this system satisfies the **linear difference equation** $\mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k]$. Let's suppose that we derived this discrete system from a continuous system. If Δt is the sampling time, then we have

$$\Phi = \Phi(t_{k+1}, t_k) = e^{\Delta t\, A} = I + \Delta t\, A + \frac{\Delta t^2 A^2}{2!} + \frac{\Delta t^3 A^3}{3!} + \ldots = \sum_{n=0}^{+\infty} \frac{\Delta t^n A^n}{n!}, \tag{4.2.8}$$

$$\Gamma = \Delta t\, B + \frac{\Delta t^2 AB}{2!} + \frac{\Delta t^3 A^2 B}{3!} + \ldots = \sum_{n=1}^{+\infty} \frac{\Delta t^n A^{n-1} B}{n!}. \tag{4.2.9}$$

Note the similarity between the discrete-time system matrix Φ and the continuous-time transition matrix $\Phi(t, t_0)$. (That's the reason why the same symbol is used for both parameters.) The direct equation for finding $\mathbf{x}[k]$ is now given by

$$\mathbf{x}[k] = \Phi^k \mathbf{x}[0] + \sum_{n=0}^{k-1} \Phi^n \Gamma \mathbf{u}[k-n-1]. \tag{4.2.10}$$

Let's suppose that we use white noise $\bar{\mathbf{w}}[k]$ as input. So, we have $\mu_{\bar{\mathbf{w}}} = \mathbf{0}$ and $C_{\bar{\mathbf{w}}\bar{\mathbf{w}}}[k] = W_d\delta[k]$. (By the way, $\delta[k]$ is the **Kronecker delta function**. We have $\delta[k] = 1$ if $k = 0$ and $\delta[k] = 0$ otherwise. Also, W_d is the intensity of the discrete noise.) With this data, the properties of $\mathbf{x}[k]$ can be derived. We find that $\mu_{\bar{\mathbf{x}}}[t] = \Phi^n \mu_{\bar{\mathbf{x}}}[0]$ and

$$C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}[k_1, k_2] = \Phi^{k_1} \bar{\mathbf{x}}[0,0] \left(\Phi^T\right)^{k_2} + \sum_{n=0}^{\min(k_1,k_2)-1} \Phi^n \Gamma W_d \Gamma^T \left(\Phi^T\right)^n. \tag{4.2.11}$$

Often, we only want to find the covariance matrix of the stochastic variable $\bar{\mathbf{x}}[k]$ at time k. We then simply take $k_1 = k_2 = k$. The resulting matrix is denoted as $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}[k]$.

Let's suppose that we have some continuous process, and we are turning this into a discrete process. We already know how to find the system matrices Φ and Γ. However, given that we know the continuous noise intensity matrix W, how do we find the discrete noise intensity matrix W_d? It can be shown that, for small time steps Δt, we approximately have $W_d = W/\Delta t$. If we use this intensity matrix, then our discrete system is a good approximation of our non-discrete system.

We remain with the question of how to find the steady state covariance matrix $C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss}$. This time, it can be shown that it must satisfy

$$C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss} = \Phi C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss} \Phi^T + \Gamma W_d \Gamma^T. \tag{4.2.12}$$

This is the **discrete-time Lyapunov equation**. A unique solution exists if Φ is exponentially stable. (That is, if all eigenvalues λ of Φ satisfy $|\lambda| < 1$.) However, no analytic solution is available. Instead, the solution is usually found using computational/numerical methods.

4.3 The impulse response function

4.3.1 Finding the impulse response function

When examining a system, it is always interesting to look at the relation between the input and the output. Let's suppose that this relation is given by the **impulse response matrix** $h_{\bar{\mathbf{y}}\bar{\mathbf{u}}}(t)$ or, in an abbreviated notation, simply $h(t)$. If we denote the Fourier transform of this matrix by $H(\omega)$, then we have

$$\bar{\mathbf{y}}(t) = h(t) * \bar{\mathbf{u}}(t) \qquad \text{and} \qquad \bar{\mathbf{Y}}(\omega) = H(\omega)\bar{\mathbf{U}}(\omega). \tag{4.3.1}$$

The question remains: how can we find the impulse response function? For that, we can use the equation

$$h(t) = C\Phi(t, t_0)B + D. \tag{4.3.2}$$

Let's suppose that we have a system of which we do not know the system matrices. However, we are able to experiment with the system. How do we now find the impulse response function? Well, first we set the initial state $\mathbf{x}(t_0)$ to zero. Then, we simply set all inputs to zero, except for one input $u_i(t)$. We put an impulse function on this input. (So, $u_i(t) = \delta(t)$.) The resulting output $\mathbf{y}(t)$ now equals the ith column $\mathbf{h_i}(t)$ of the impulse response matrix $h(t)$. Perform this trick for all inputs/columns i and we have found the impulse response matrix $h(t)$.

There is also a slightly alternative method. It can be shown that putting an impulse function on $u_i(t)$ is equivalent to giving the system an initial condition $\mathbf{x}(t_0) = \mathbf{B_i}$, with $\mathbf{B_i}$ the ith column of B. So, if we apply this trick for all inputs/columns i, then we have again found the impulse response matrix $h(t)$.

4.3.2 The covariance matrix and the PSD function

Having the impulse response matrix can be very convenient. We can use it to find the covariance matrices between $\bar{\mathbf{u}}(t)$ and $\bar{\mathbf{y}}(t)$. This is done using

$$C_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(\tau) = C_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(\tau) * h(\tau)^T, \qquad C_{\bar{\mathbf{y}}\bar{\mathbf{u}}}(\tau) = C_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(-\tau)^T = h(-\tau) * C_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(\tau), \tag{4.3.3}$$

$$C_{\bar{\mathbf{y}}\bar{\mathbf{y}}}(\tau) = h(-\tau) * C_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(\tau) * h(\tau)^T. \tag{4.3.4}$$

If we Fourier transform the above equations to the frequency domain, then we will find the power spectral density function. So,

$$S_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(\omega) = S_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(\omega)H(\omega)^T, \qquad S_{\bar{\mathbf{y}}\bar{\mathbf{u}}}(\omega) = S_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(-\omega)^T = H(-\omega)S_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(\omega), \tag{4.3.5}$$

$$S_{\bar{\mathbf{y}}\bar{\mathbf{y}}}(\omega) = H(-\omega)S_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(\omega)H(\omega)^T. \tag{4.3.6}$$

By the way, all the above tricks also work if you use $\bar{\mathbf{x}}$ instead of $\bar{\mathbf{y}}$. But you then of course need to use the impulse response matrix $h_{\bar{\mathbf{x}}\bar{\mathbf{u}}}(t)$ instead of $h_{\bar{\mathbf{y}}\bar{\mathbf{u}}}(t)$.

5. Describing atmospheric turbulence

The previous chapters only discussed theory. It is time to look at how we can apply this theory to aircraft. First, we need to get a mathematical framework of turbulence. That's what we'll derive in this chapter. We start by examining some basic information about turbulence. We then look at how we can model turbulence as a stochastic process. Finally, we examine possible covariance and PSD functions for turbulence.

5.1 Basics of turbulence

5.1.1 Causes of turbulence

Let's examine the atmosphere. This atmosphere is often subject to turbulence. We'll take a look at where this turbulence comes from. First of all, it depends on the **temperature lapse rate** $\lambda = dT/dh$ in the atmosphere. This is the temperature change for every meter which we go up. In the ICAO standard atmosphere, up to 11 km, we have $\lambda = -0.0065$ °C/m.

Now examine a small parcel of air in the atmosphere. When this parcel goes up, its temperature will decrease. The rate of temperature decrease during adiabatic ascent is denoted by β. This parameter mostly depends on how much water is in the air. For dry air, we roughly have $\beta_{dry} = 0.0098$ °/m. For saturated air, this is less, though the exact value strongly depends on the temperature and the pressure.

Let's suppose that $|\lambda| > |\beta|$. (The absolute sign is present to prevent confusion with minus signs and such.) When our parcel of air now goes up, it cools less than the surrounding air. So, its density ρ is lower than the surrounding air. This causes buoyancy, causing our parcel of air to go up faster. We thus have **vertical instability**. (If, however, we have $|\lambda| < |\beta|$, then we have **vertical stability**.) Vertical instability is a common cause of vertical gusts.

Another cause of turbulence is windshear. Let's assume that the wind vector is directed horizontally. We can now distinguish **horizontal windshear** ($\partial V_w/\partial x$ and $\partial V_w/\partial y$), where the wind velocity varies per horizontal position, and **vertical windshear** ($\partial V_w/\partial z$), where the wind velocity varies per vertical position. Windshear causes friction between layers of air, which in turn causes turbulence.

5.1.2 Types of turbulence

To be able to quantify turbulence, we introduce the **eddy energy equation**

$$\frac{dE}{dt} = S + H + B - D. \tag{5.1.1}$$

Here, E is the **turbulent kinetic energy**, S is a term relating to vertical windshear and H is a term relating to horizontal windshear. Both S and H are positive. The term B is related to vertical stability. If we have vertical stability, then it is negative. In case of vertical instability, it is positive. Finally, D represents heat dissipation. Although the term is always positive, its exact value depends on E.

Let's examine the above equation for some conditions. We start on the ground. Here, we have a relatively big value for S. However, if we go up, S quickly decreases. The parameter B is positive during the day and negative during clear nights. It doesn't change much with height. So, close to the ground, S is dominant, while a bit higher up, B is dominant.

In clouds, we have saturated air. Such air is vertically unstable. So, B is positive. This is especially the case for rain-producing **cumulonimbus** clouds. Also, once the vertical instability has caused air to go up/down, windshears will be present. So, S and H will be positive too.

The term **clear-air turbulence** concerns turbulence high up, in clear air. It often occurs around jet streams, at altitudes of 10.000 to 12.000 meters. For this kind of turbulence, the horizontal windshear term H is often the most important term.

Finally, there is **mountain wave turbulence**, which occurs in the vicinity of mountains. These mountains perturb the air flow, causing turbulence. This type of turbulence can become very strong.

We can distinguish four degrees of turbulence intensity. In **light** turbulence, objects in the aircraft still remain at rest. In **moderate** turbulence, unsecured objects start to move about. In **severe** turbulence, the aircraft may momentarily be out of control. Finally, in **extreme** turbulence, it is impossible to control the aircraft. Structural damage may very well be present.

5.2 Modeling turbulence as a stochastic process

5.2.1 Splitting up the wind velocity

In principle, turbulence is a deterministic process, just like everything else in nature. But, because it is so hard to predict, it is much easier to simply consider it as a stochastic process. In fact, let's consider the 'deterministic' **gust vector** $\mathbf{V_g}(\mathbf{r}, t)$, being the velocity of air with respect to the ground. We generally split it up into two parts. These are the average wind velocity $\mathbf{V}_{g,av}$ and the deviations $\bar{\mathbf{u}}(\mathbf{r}, t)$, which we consider to be stochastic. So,

$$\mathbf{V_g}(\mathbf{r}, t) = \mathbf{V}_{g,av} - \bar{\mathbf{u}}(\mathbf{r}, t). \tag{5.2.1}$$

(The minus sign is present due to convention.) We hereby declare the average wind velocity $\mathbf{V}_{g,av}$ a matter of navigation/guidance. We will only concern ourselves with the velocity deviation vector $\bar{\mathbf{u}}(\mathbf{r}, t)$. This velocity vector has three components $\bar{u}_1(\mathbf{r}, t)$, $\bar{u}_2(\mathbf{r}, t)$ and $\bar{u}_3(\mathbf{r}, t)$. Each of these components depends on four parameters: the position $\mathbf{r} = [\xi_1, \xi_2, \xi_3]^T$ and the time t.

An important parameter is the covariance matrix of the wind velocity. This matrix can be found using

$$C_{\bar{u}\bar{u}}(\mathbf{r}, \mathbf{t}; \mathbf{r} + \xi, t + \tau) = \mathrm{E}\left\{\bar{\mathbf{u}}(\mathbf{r}, t)\bar{\mathbf{u}}(\mathbf{r} + \xi, t + \tau)^T\right\}. \tag{5.2.2}$$

We can also find the power spectral density function $S_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{\Omega}, \omega)$. To do this, we simply take the Fourier transform of $C_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{r} + \xi, t + \tau)$. However, this is slightly more difficult now, since this matrix now depends on four parameters. Because of this, the Fourier transform has become

$$S_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{\Omega}, \omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{r} + \xi, t + \tau)e^{-j(\mathbf{\Omega}\xi^T + \omega t)}\, d\xi_1\, d\xi_2\, d\xi_3\, d\tau. \tag{5.2.3}$$

By the way, Ω is called the **spatial frequency**. It is related to the **wavelength** λ of the turbulence according to $\Omega = 2\pi/\lambda$.

5.2.2 Simplifying assumptions

Currently, atmospheric turbulence is still a bit too difficult to work with. So we have to make some simplifying assumptions.

- We assume that turbulence is normally distributed. So, $\bar{\mathbf{u}}(\mathbf{r}, t)$ has a normal distribution. Since $\bar{\mathbf{u}}(\mathbf{r}, t)$ has zero mean, we thus only need to know $C_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{r} + \xi, t + \tau)$ to fully describe $\bar{\mathbf{u}}(\mathbf{r}, t)$.
- We assume that turbulence is a stationary process. In fact, we assume that $\bar{\mathbf{u}}(\mathbf{r}, t)$ does not depend on time at all. We thus write $\bar{\mathbf{u}}(\mathbf{r})$, $C_{\bar{u}\bar{u}}(\mathbf{r}; \mathbf{r}+\xi)$ and $S_{\bar{u}\bar{u}}(\mathbf{r}; \mathbf{\Omega})$. (This assumption is called **Taylor's hypothesis**.)
- We assume that turbulence is homogeneous along the flight path. So, the turbulence does not depend on the position. We thus write $C_{\bar{u}\bar{u}}(\xi)$ and $S_{\bar{u}\bar{u}}(\mathbf{\Omega})$. This assumption also implies that turbulence is an ergodic process.

- We assume that turbulence is an isotropic process: the statistical properties are independent of direction. We thus have $\sigma^2_{\bar{u}_1} = \sigma^2_{\bar{u}_2} = \sigma^2_{\bar{u}_3} = \sigma^2$. This assumption seems to hold for high altitudes, but is not so accurate close to the ground.

5.2.3 The fundamental correlation functions

To see the effect of the assumptions that have been made, we will examine two points **a** and **b** in the atmosphere. We denote the relative position of these points by the vector $\xi = \mathbf{b} - \mathbf{a}$. Now examine the components of the velocities $\mathbf{u}(\mathbf{a})$ and $\mathbf{u}(\mathbf{b})$ in the direction of ξ in both points. (The so-called **longitudinal** components.) There is a correlation between these velocity components. Due to our assumptions, this correlation only depends on the distance $|\xi|$ and is denoted as $f(|\xi|)$.

In a similar way, we can look at each of the velocity components perpendicular to the vector ξ. (The so-called **lateral** components.) Once more, the correlation between these components only depends on the distance $|\xi|$. We denote this correlation by $g(|\xi|)$. Both **fundamental correlation functions** f and g can be found from the PSD function $S_{\bar{u}\bar{u}}(\mathbf{\Omega})$. Once we have them, we can find the covariance matrix of the turbulence. It is given by

$$C_{ij}(\xi) = \sigma^2 \left(\frac{(f(|\xi|) - g(|\xi|))\, \xi_i \xi_j}{|\xi|^2} + g(|\xi|)\delta_{ij} \right), \tag{5.2.4}$$

where δ_{ij} is the Kronecker delta function. (It equals 1 if $i = j$ and is zero otherwise.)

Turbulence occurs on many scales. An indication of the scale is the **integral scale of turbulence**. The longitudinal scale L_g and the lateral scale L'_g are, respectively, defined as

$$L_g = \int_0^\infty f(\xi)\, d\xi \qquad \text{and} \qquad L'_g = \int_0^\infty g(\xi)\, d\xi. \tag{5.2.5}$$

The continuity condition for incompressible fluids imposes a relation between these two scales. This relation is $L_g = 2L'_g$.

5.3 Finding the covariance and PSD functions for turbulence

5.3.1 The von Kármán spectra

The question remains what kind of PSD function we should use for turbulence. This is where the difficult mathematical equations come in. The **von Kármán functions** yield spectra that seem to match quite well with theoretical and experimental data on turbulence. So, let's examine it. The longitudinal and lateral spectra $S_{lo}(\Omega)$ and $S_{la}(\Omega)$ are, respectively, given by

$$S_{lo}(\Omega) = 2\sigma^2 L_g \frac{1}{\left(1 + (1.339 L_g \Omega)^2\right)^{5/6}} \qquad \text{and} \qquad S_{la}(\Omega) = \sigma^2 L_g \frac{1 + \frac{8}{3}(1.339 L_g \Omega)^2}{\left(1 + (1.339 L_g \Omega)^2\right)^{11/6}}. \tag{5.3.1}$$

If we take the inverse Fourier transform, then we find that

$$f(\xi) = \frac{2^{\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} \left(\frac{\xi}{1.339 L_g}\right)^{\frac{1}{3}} K_{\frac{1}{3}}\left(\frac{\xi}{1.339 L_g}\right), \tag{5.3.2}$$

$$g(\xi) = \frac{2^{\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} \left(\frac{\xi}{1.339 L_g}\right)^{\frac{1}{3}} \left(K_{\frac{1}{3}}\left(\frac{\xi}{1.339 L_g}\right) - \frac{1}{2}\left(\frac{\xi}{1.339 L_g}\right) K_{\frac{2}{3}}\left(\frac{\xi}{1.339 L_g}\right)\right). \tag{5.3.3}$$

In the above equation, $\Gamma(z)$ denots the Gamma function and $K_m(z)$ denotes that modified Bessel function of the second kind. For reasons of brevity, we won't examine their rather lengthy definitions. However, if the functions f and g are found, then the covariance matrix could be found using equation (5.2.4).

5.3.2 The von Kármán spectra applied to aircraft

Previously, we have considered an arbitrary reference frame and seperation vector ξ. Now, we can specify these for an aircraft. Let's use the aircraft stability reference frame. In this case, the turbulence velocity is $\bar{\mathbf{u}} = [\bar{u}_g, \bar{v}_g, \bar{w}_g]^T$, with $\bar{u}_g$ the longitudinal gust velocity (positive backwards), $\bar{v}_g$ the lateral gust velocity (positive to the left) and $\bar{w}_g$ the vertical gust velocity (positive upward). We also choose $\xi = [V\tau, 0, 0]^T$.

The result is that we can express the covariance matrix as a function of time τ again, instead of position ξ. (We simply use $\xi = V\tau$.) Also, we can express the PSD function in the angular frequency ω again, instead of the spatial frequency Ω. (We now use $\omega = V\Omega$.) The relations between the old and the new functions are given by

$$C_{\bar{u}\bar{u},new}(\tau) = C_{\bar{u}\bar{u},old}(\xi = V\tau) \qquad \text{and} \qquad S_{\bar{u}\bar{u},new}(\omega) = \frac{1}{V} S_{\bar{u}\bar{u},old}(\Omega = \omega/V). \tag{5.3.4}$$

If we apply this to the von Kármán spectra, then we find that

$$S_{\bar{u}_g\bar{u}_g}(\omega) = 2\sigma^2 \frac{L_g}{V} \frac{1}{\left(1 + \left(1.339 \frac{L_g\omega}{V}\right)^2\right)^{5/6}}, \tag{5.3.5}$$

$$S_{\bar{v}_g\bar{v}_g}(\omega) = S_{\bar{w}_g\bar{w}_g}(\omega) = \sigma^2 \frac{L_g}{V} \frac{1 + \frac{8}{3}\left(1.339 \frac{L_g\omega}{V}\right)^2}{\left(1 + \left(1.339 \frac{L_g\omega}{V}\right)^2\right)^{11/6}}. \tag{5.3.6}$$

In this special situation, the cross-PSD functions $S_{\bar{u}_g\bar{v}_g}(\omega)$, $S_{\bar{u}_g\bar{w}_g}(\omega)$ and $S_{\bar{v}_g\bar{w}_g}(\omega)$ are zero.

5.3.3 The Dryden spectral form

There is a problem with the von Kármán spectra. They are not rational functions. Having rational functions would simplify computations. To solve this, the **Dryden spectral form** is introduced. This function is a rational function. And furthermore, it more or less equals the von Kármán spectra on all frequencies except for really high ones.

In the Dryden spectral form, we again have $\xi = [V\tau, 0, 0]^T$. However, this time

$$S_{\bar{u}_g\bar{u}_g}(\omega) = 2\sigma^2 \frac{L_g}{V} \frac{1}{1 + \left(\frac{L_g\omega}{V}\right)^2} \qquad \text{and} \qquad S_{\bar{v}_g\bar{v}_g}(\omega) = S_{\bar{w}_g\bar{w}_g}(\omega) = \sigma^2 \frac{L_g}{V} \frac{1 + 3\left(\frac{L_g\omega}{V}\right)^2}{\left(1 + \left(\frac{L_g\omega}{V}\right)^2\right)^2}. \tag{5.3.7}$$

To find the covariance matrix, we can again use equation (5.2.4). But this time, we need to insert

$$f(\xi) = e^{-\frac{\xi}{L_g}} \qquad \text{and} \qquad g(\xi) = e^{-\frac{\xi}{L_g}} \left(1 - \frac{\xi}{2L_g}\right). \tag{5.3.8}$$

5.3.4 Generating a turbulence signal

Let's suppose that we have chosen which spectral form to use. We now want to generate a set of turbulence data. So how do we do that? The main idea is that we use the equation

$$S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega) \qquad \text{or} \qquad |H(\omega)|^2 = \frac{S_{\bar{y}\bar{y}}(\omega)}{S_{\bar{u}\bar{u}}(\omega)}. \tag{5.3.9}$$

The output PSD function $S_{\bar{y}\bar{y}}(\omega)$ is known: it is the spectral form which we selected. As input, we usually take white noise, so $S_{\bar{u}\bar{u}}(\omega) = 1$. Now we need to find the function $H(\omega)$ which satisfies the above equation. The solution $H(\omega)$ is called the **forming filter**. Once we have $H(\omega)$, we generate a white noise signal $U(\omega)$ and use $Y(\omega) = H(\omega)U(\omega)$ to form our turbulence signal $Y(\omega)$.

Once we have the forming filter $H(\omega)$, we can also put it in a state space form. This can, however, be quite complicated. So we won't treat this in depth here. Instead, we'll only mention that the forming filters for the Dryden spectral form are

$$H_{\bar{u}_g\bar{w}_1}(\omega) = \frac{\bar{u}_g(\omega)}{\bar{w}_1(\omega)} = \sigma\sqrt{\frac{2L_g}{V}}\frac{1}{1-\frac{L_g}{V}j\omega} \qquad \text{and} \qquad H_{\bar{w}_g\bar{w}_3}(\omega) = \frac{\bar{w}_g(\omega)}{\bar{w}_3(\omega)} = \sigma\sqrt{\frac{L_g}{V}}\frac{1+\sqrt{3}\frac{L_g}{V}j\omega}{\left(1-\frac{L_g}{V}j\omega\right)^2}. \tag{5.3.10}$$

If we subsitute/generalize $s = j\omega$, then we can put the above equations into state space form. If we define

$$w_g^*(t) = \dot{w}_g(t) - \sigma\sqrt{\frac{3V}{L_g}}w_3(t), \tag{5.3.11}$$

then we get

$$\dot{u}_g(t) = -\frac{V}{L_g}u_g(t) + \sigma\sqrt{\frac{2V}{L_g}}w_1(t), \tag{5.3.12}$$

$$\begin{bmatrix}\dot{w}_g(t)\\ \dot{w}_g^*(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{V^2}{L_g^2} & -2\frac{V}{L_g}\end{bmatrix}\begin{bmatrix}w_g(t)\\ w_g^*(t)\end{bmatrix} + \begin{bmatrix}\sigma\sqrt{\frac{3V}{L_g}}\\ (1-2\sqrt{3})\,\sigma\sqrt{\left(\frac{V}{L_g}\right)^3}\end{bmatrix}w_3(t). \tag{5.3.13}$$

5.3.5 Finding the quantative parameters close to the ground

When setting up a turbulence model, we do need the parameters $\sigma_{\bar{u}_g}$, $\sigma_{\bar{v}_g}$, $\sigma_{\bar{w}_g}$ and L_g. These parameters are based on experimental data. Tables are available to find the parameters. Using these tables for relatively low altitudes (below 450 m) can be difficult. This is mainly because, for these low altitudes, the homogeneous and isotropic flow assumptions don't hold anymore. To solve this problem, we need some data.

First of all, we need to know the temperature lapse rate λ. Second, we also need an indication of the wind speed. For this, usually the wind speed at a reference height is used. (30 ft/9.15 m is an often-used reference height.) With these two parameters, we can find $\sigma_{\bar{w}_g}$ from tables. The quantities $\sigma_{\bar{u}_g}$ and $\sigma_{\bar{v}_g}$ are a bit harder to find though. This is because they strongly depend on the height h at which we want to know the turbulence properties. A guideline that is often used to find them is

$$\underbrace{\frac{\sigma_{\bar{u}_g}}{\sigma_{\bar{w}_g}} = \frac{\sigma_{\bar{v}_g}}{\sigma_{\bar{w}_g}} = 2.5,}_{0\text{m}\leq h<15\text{m}} \qquad \underbrace{\frac{\sigma_{\bar{u}_g}}{\sigma_{\bar{w}_g}} = \frac{\sigma_{\bar{v}_g}}{\sigma_{\bar{w}_g}} = 1.25 - 0.001h}_{15\text{m}\leq h<250\text{m}} \qquad \text{and} \qquad \underbrace{\frac{\sigma_{\bar{u}_g}}{\sigma_{\bar{w}_g}} = \frac{\sigma_{\bar{v}_g}}{\sigma_{\bar{w}_g}} = 1}_{250\text{m}\leq h}. \tag{5.3.14}$$

We can then simply insert the values for $\sigma_{\bar{u}_g}$, $\sigma_{\bar{v}_g}$ and $\sigma_{\bar{w}_g}$ in the right positions of equations (5.3.5), (5.3.6) and (5.3.7).

To find the value of L_g, we have to use tables again. All we have to know for this parameter are the temperature lapse rate λ and the height h at which we want to know the turbulence properties.

6. Symmetric responses to turbulence

In the previous chapter, we've created a model for turbulence. Now, we're going to apply it to an aircraft. How will an aircraft in turbulent air behave? In this chapter we'll examine the symmetric response. (The asymmetric response is left for the next chapter.) We start by finding a way to include turbulence into our equations of motion. After that, we will turn these equations of motion into a state space form.

By the way, in this chapter, we will not use the formal notation for stochastic variables $\bar{u}$ anymore, since otherwise some of the formulas will be a bit unreadable. You'll just have to remember yourself which variables are stochastic.

6.1 The effects of turbulence on the aircraft

6.1.1 Definitions

Let's consider an aircraft in a steady symmetric flight. Its velocity with respect to the ground is denoted by $\mathbf{V}$. Its velocity with respect to the air is $\mathbf{V_a}$. Finally, the velocity of air with respect to the ground is $\mathbf{V_g}$ (the **gust velocity**). So, we have

$$\mathbf{V} = \mathbf{V_a} + \mathbf{V_g} = \mathbf{V_a} + \mathbf{V_{g,av}} - \mathbf{u}. \tag{6.1.1}$$

In this chapter, we're considering symmetric aircraft motions. So, we will be interested in the components u_g and w_g of $\mathbf{u}$, but not in v_g. (We leave that for the next chapter.) Also, for simplicity we assume that there is no wind. So, $\mathbf{V_{g,av}} = \mathbf{0}$.

As you probably know, the **pitch angle** θ is defined as the angle between the aircraft's X axis and the horizontal plane. The **flight path angle** γ is the angle between the velocity vector $\mathbf{V}$ and the horizontal plane. The **angle of attack** α is now defined as $\alpha = \theta - \gamma$. That is, it's the angle between the aircraft's X axis and the velocity vector $\mathbf{V}$.

Usually, α is the angle between the aircraft's X axis and the airflow. But when there are gusts, this is not the case anymore. So, assuming that u_g and w_g are small relative to the velocity V, we define the **gust angle of attack** $\alpha_g = w_g/V$. The **total angle of attack** is now defined as $\alpha_{tot} = \alpha + \alpha_g$. Finally, we define the **non-dimensional gust velocity** $\hat{u}_g = u_g/V$. This implies that $V_a = V(1 + \hat{u}_g)$.

6.1.2 Equations for forces and moments

If we want to consider the aircraft response, we need to look at the forces and moments that act on the aircraft. The symmetric forces and moments acting on the aircraft due to gusts are denoted by X_g, Z_g and M_g. These forces/moments are turned into non-dimensional coefficients using

$$C_{X_g} = \frac{X_g}{\frac{1}{2}\rho V^2 S}, \qquad C_{Z_g} = \frac{Z_g}{\frac{1}{2}\rho V^2 S} \qquad \text{and} \qquad C_{m_g} = \frac{M_g}{\frac{1}{2}\rho V^2 S}. \tag{6.1.2}$$

To find an expression for C_{X_g}, we can use linearization. This gives us

$$C_{X_g} = \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \hat{u}_g}\hat{u}_g + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \frac{\dot{\hat{u}}_g \bar{c}}{V}}\frac{\dot{\hat{u}}_g \bar{c}}{V} + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \alpha_g}\alpha_g + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \frac{\dot{\alpha}_g \bar{c}}{V}}\frac{\dot{\alpha}_g \bar{c}}{V}. \tag{6.1.3}$$

Using some coefficients, we can make the above equation a lot shorter. We then get

$$C_{X_g} = C_{X_{u_g}}\hat{u}_g + C_{X_{\dot{u}_g}}\frac{\dot{\hat{u}}_g \bar{c}}{V} + C_{X_{\alpha_g}}\alpha_g + C_{X_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g \bar{c}}{V}. \tag{6.1.4}$$

Here, $\bar{c}$ is the mean chord length of the aircraft. The coefficients C_{Z_g} and C_{m_g} can be written in a similar way. The partial derivatives $C_{X_{u_g}}$, $C_{X_{\dot{u}_g}}$ and such are called **gust derivatives**.

6.1.3 Implementing the turbulence model

Let's try to implement the turbulence model which was derived in the previous chapter. We often assume that the gust field only varies in the X direction. So, it does not vary in the Y and Z direction. We thus write

$$\hat{u}_g = \hat{u}_{g_{max}} e^{j\Omega x} = \hat{u}_{g_{max}} e^{j\frac{\omega x}{V}} \qquad \text{and} \qquad \alpha_g = \alpha_{g_{max}} e^{j\Omega x} = \alpha_{g_{max}} e^{j\frac{\omega x}{V}}. \tag{6.1.5}$$

The wavelength in this field is still given by $\lambda = 2\pi/\Omega = 2\pi V/\omega$. However, we often don't work with ω and x, but with the **non-dimensional distance** s_c and the **reduced frequency** k_c, defined as

$$s_c = \frac{x}{\bar{c}} = \frac{Vt}{\bar{c}} \qquad \text{and} \qquad k_c = \Omega\bar{c} = \frac{\omega\bar{c}}{V}. \tag{6.1.6}$$

Note that now $\Omega x = k_c s_c$. If we use this fact and combine it with equations (6.1.4) and (6.1.5), we find that

$$C_{X_g} = \left(C_{X_{u_g}} + C_{X_{\dot{u}_g}} jk_c\right)\hat{u}_g + \left(C_{X_{\alpha_g}} + C_{X_{\dot{\alpha}_g}} jk_c\right)\alpha_g. \tag{6.1.7}$$

Once more, a similar expression can be derived for C_{Z_g} and C_{m_g}.

6.2 Finding coefficients and state space representations

6.2.1 Finding the gust derivatives

It would be nice if we could find expressions for the gust derivatives. First, we will examine the **steady gust derivatives** like $C_{X_{u_g}}$, $C_{Z_{u_g}}$ and $C_{M_{u_g}}$. These coefficients simply represent the forces/moment acting on the aircraft when the velocity changes. However, these coefficients are already known from normal flight dynamics. In fact, we have

$$C_{X_{u_g}} = C_{X_u} \qquad C_{Z_{u_g}} = C_{Z_u} \qquad \text{and} \qquad C_{M_{u_g}} = C_{M_u}. \tag{6.2.1}$$

Now let's examine the **unsteady gust derivatives** $C_{X_{\dot{u}_g}}$, $C_{Z_{\dot{u}_g}}$ and $C_{M_{\dot{u}_g}}$. It can be shown that the term $C_{X_{\dot{u}_g}} jk_c$ (and also the term $C_{X_{\dot{\alpha}_g}} jk_c$) is very small. Next to this, it is also very hard to derive a relation for it. So, it is usually neglected.

Deriving expressions for $C_{Z_{\dot{u}_g}}$ and $C_{M_{\dot{u}_g}}$ is quite difficult as well. But it can be done for an aircraft with a normal wing-fuselage-horizontal tailplane configuration. In fact, there are two methods for it. Both methods use the fact that the turbulence first hits the main wing. A time $\tau = (x_h - x_w)/V$ later, it hits the horizontal tailplane. (This is called the **gust penetration effect**.) Here, x_w and x_h are the x-positions of the aerodynamic centers of the wing and the horizontal tailplane, respectively. Also, we denote the x-position of the aircraft center of gravity by x_{cg}.

In the first method, we look at dynamic pressures. When a gust hits the wing, the dynamic pressure at the wing changes. The same holds for the horizontal tailplane, but this happens a time τ later. By using this data, we can derive that

$$C_{Z_{\dot{u}_g}} = 2\left(C_{Z_w}\frac{x_{cg} - x_w}{\bar{c}} + C_{Z_h}\frac{x_{cg} - x_h}{\bar{c}}\right) = 2C_{m_{ac}}, \tag{6.2.2}$$

$$C_{m_{\dot{u}_g}} = 2\left(C_{m_w}\frac{x_{cg} - x_w}{\bar{c}} + C_{m_h}\frac{x_{cg} - x_h}{\bar{c}}\right) = -2C_{m_h}\frac{l_h}{\bar{c}}. \tag{6.2.3}$$

In the second method, we don't consider the change in dynamic pressure. Instead, we look at how much the wing changes the flow velocity. The gust hits the wing first. When this happens, the velocity of the gust is changed by an amount Δu. A time τ later, a gust with a velocity $\left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right)\hat{u}_g$ arrives at the horizontal tailplane. Based on this data, we can't only derive relations for the unsteady gust derivatives, but also for the steady gust derivatives. In fact, we will find that

$$
\begin{aligned}
C_{X_{u_g}} &= C_{X_u} = C_{X_{w_u}} + C_{X_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), && (6.2.4)\\
C_{X_{\dot{u}_g}} &= -C_{X_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), && (6.2.5)\\
C_{Z_{u_g}} &= C_{Z_u} = C_{Z_{w_u}} + C_{Z_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), && (6.2.6)\\
C_{Z_{\dot{u}_g}} &= -C_{Z_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), && (6.2.7)\\
C_{m_{u_g}} &= C_{m_u} = C_{m_{w_u}} + C_{Z_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), && (6.2.8)\\
C_{m_{\dot{u}_g}} &= -C_{Z_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h^2}{S\bar{c}^2} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right). && (6.2.9)
\end{aligned}
$$

Now let's try to find the coefficients for α_g. This time, we only use one method, which is similar to the second method which we just saw. The gust α_g that hits the wing causes a change in downwash ϵ. By using this knowledge, we can derive for the steady gust derivatives that

$$C_{X_{\alpha_g}} = C_{X_\alpha} \qquad C_{Z_{\alpha_g}} = C_{Z_\alpha} \qquad \text{and} \qquad C_{M_{\alpha_g}} = C_{M_\alpha}. \tag{6.2.10}$$

(Although usually, it is simply assumed that $C_{X_{\alpha_g}} = 0$.) For the unsteady gust derivates, we have

$$C_{X_{\dot{\alpha}_g}} = C_{X_{\dot{\alpha}}} - C_{X_q} \qquad C_{Z_{\dot{\alpha}_g}} = C_{Z_{\dot{\alpha}}} - C_{Z_q} \qquad \text{and} \qquad C_{M_{\dot{\alpha}_g}} = C_{M_{\dot{\alpha}}} - C_{m_q}. \tag{6.2.11}$$

6.2.2 The symmetric equations of motion for aircraft turbulence

Now that we have values for the gust derivatives, we can derive the equations of motion. If we take the equations of motion, known from flight dynamics, and add the gusts in the input vector, we find that

$$
\begin{bmatrix}
C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & 0\\
C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c) D_c & -C_{X_0} & 2\mu_c + C_{Z_q}\\
0 & 0 & -D_c & 1\\
C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c
\end{bmatrix}
\begin{bmatrix}\hat{u}\\ \alpha\\ \theta\\ \frac{q\bar{c}}{V}\end{bmatrix} =
$$

$$
- \begin{bmatrix}
C_{X_{\delta_e}} & C_{X_{u_g}} & 0 & C_{X_{\alpha_g}} & 0\\
C_{Z_{\delta_e}} & C_{Z_{u_g}} & C_{Z_{\dot{u}_g}} & C_{Z_{\alpha_g}} & C_{Z_{\dot{\alpha}_g}}\\
0 & 0 & 0 & 0 & 0\\
C_{m_{\delta_e}} & C_{m_{u_g}} & C_{m_{\dot{u}_g}} & C_{m_{\alpha_g}} & C_{m_{\dot{\alpha}_g}}
\end{bmatrix}
\begin{bmatrix}\delta_e\\ \hat{u}_g\\ D_c\hat{u}_g\\ \alpha_g\\ D_c\alpha_g\end{bmatrix}. \tag{6.2.12}
$$

The above equations can be transformed to a state space form. (To do this, you have to use the definition of the derivative operator $D_c = \frac{\bar{c}}{V}\frac{d}{dt}$.) And, if necessary, this state space form can also be combined with the (normalized) state space form of the forming filter for u_g and w_g. However, this is quite complicated, so we won't go into depth on that here.

6.2.3 Eigenmotions of the aircraft

Let's suppose that the pilot provides no input to the aircraft. So, $\delta_e = 0$. If the aircraft is in a gust, how does it behave? That's what we'll investigate now.

First, we'll examine the short period motion. During this motion, it is assumed that the velocity and the flight path angle don't change. Also, the forces in X direction are zero. Let's apply these assumptions to the state space matrix above. We then remain with

$$\begin{bmatrix} C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c)D_c & 2\mu_c + C_{Z_q} \\ C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = - \begin{bmatrix} C_{Z_{\alpha_g}} & C_{Z_{\dot{\alpha}_g}} \\ C_{m_{\alpha_g}} & C_{m_{\dot{\alpha}_g}} \end{bmatrix} \begin{bmatrix} \alpha_g \\ D_c \alpha_g \end{bmatrix}. \tag{6.2.13}$$

With this state space representation, the short period motion of an aircraft in turbulence can be modeled. A similar trick can be performed for the phugoid motion. This time, we assume that the angle of attack α remains constant. Furthermore, we neglect C_{Z_q} and C_{X_0} and we assume that the moment acting on the aircraft is approximately zero. We now remain with

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{Z_0} & 0 \\ C_{Z_u} & 0 & 2\mu_c \\ 0 & -D_c & 1 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = - \begin{bmatrix} C_{X_{u_g}} & 0 \\ C_{Z_{u_g}} & C_{Z_{\dot{u}_g}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ D_c \hat{u}_g \end{bmatrix}. \tag{6.2.14}$$

This state space representation can be used to model the phugoid motion of an aircraft in turbulence.

7. Asymmetric responses to turbulence

In the previous chapter we examined the symmetric response of an aircraft to turbulence. In this chapter, we'll focus on the asymmetric response. This is a bit more difficult than the symmetric response. Why this is the case will be examined first. After that, the asymmetric force and moment coefficients will be derived. At the end, the asymmetric PSD functions and the equations of motion will be examined.

7.1 The covariance and PSD function in two-dimensional space

7.1.1 Deriving the covariance matrix

Previously, we have assumed that the turbulence only varies in a longitudinal direction. This works when we're examining longitudinal motions. But when examining lateral motions, the lateral distance y also needs to be taken into account. So, in this chapter, the turbulence parameters u_g, v_g and w_g depend on x and y. The turbulence covariance matrix is thus given by

$$C_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(x,y) = \begin{bmatrix} C_{u_g u_g}(x,y) & 0 & 0 \\ 0 & C_{v_g v_g}(x,y) & 0 \\ 0 & 0 & C_{w_g w_g}(x,y) \end{bmatrix}. \tag{7.1.1}$$

Note that, due to the isotropic assumption, the parameters u_g, v_g and w_g are mutually independent. We also have $C_{u_g u_g}(x,y) = \mathrm{E}\{u_g(0,0), u_g(x,y)\}$ and the same for v_g and w_g.

Let's denote the distance to the point $P = (x,y)$ by $r = \sqrt{x^2+y^2}$. When the functions $f(r)$ and $g(r)$ are known, the terms $C_{u_g u_g}(x,y)$ and $C_{v_g v_g}(x,y)$ of the covariance matrix can be found using

$$C_{u_g u_g}(x,y) = \sigma_{u_g}^2 \left(f(r)\left(\frac{x}{r}\right)^2 + g(r)\left(\frac{y}{r}\right)^2 \right) \quad \text{and} \quad C_{v_g v_g}(x,y) = \sigma_{v_g}^2 \left(f(r)\left(\frac{y}{r}\right)^2 + g(r)\left(\frac{x}{r}\right)^2 \right). \tag{7.1.2}$$

Also, we have $C_{w_g w_g}(x,y) = \sigma_{w_g}^2 g(r)$. However, often the covariance matrices are expressed, not in x and y, but in the dimensionless parameters x/L_g and y/L_g. It could be worthwhile to keep this in mind when reading other texts on atmospheric flight dynamics.

7.1.2 Deriving the power spectral density function

To derive the PSD function, we simply take the Fourier transform of the covariance matrix. This time, the covariance matrix is a function of two variables. The Fourier transform thus becomes

$$S(\Omega_x L_g, \Omega_y L_g) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C\left(\frac{x}{L_g}, \frac{y}{L_g}\right) e^{-j(\Omega_x x \Omega_y y)}\, d\frac{x}{L_g}\, d\frac{y}{L_g}. \tag{7.1.3}$$

Note that, for the PSD matrix, we have also used dimensionless parameters. This time they are $\Omega_x L_g$ and $\Omega_y L_g$.

For the Dryden spectral form, the above integral can be solved. The obtained results are

$$S_{u_g u_g}(\Omega_x L_g, \Omega_y L_g) = \pi\sigma_{u_g}^2 \frac{1 + \Omega_x^2 L_g^2 + 4\Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}}, \tag{7.1.4}$$

$$S_{v_g v_g}(\Omega_x L_g, \Omega_y L_g) = \pi\sigma_{v_g}^2 \frac{1 + 4\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}}, \tag{7.1.5}$$

$$S_{w_g w_g}(\Omega_x L_g, \Omega_y L_g) = \pi\sigma_{w_g}^2 \frac{3\Omega_x^2 L_g^2 + 3\Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}}. \tag{7.1.6}$$

There is a relation with the one-dimensional variant $S'_{u_g u_g}(\Omega_x L_g)$ of the PSD function which we've used in earlier chapters. To find it from the above equation, we apply the inverse Fourier transform for the parameter $(\Omega_y L_g)$. We thus have

$$S'_{u_g u_g}(\Omega_x L_g) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{u_g u_g}(\Omega_x L_g, \Omega_y L_g)\, d(\Omega_y L_g). \tag{7.1.7}$$

The same relation holds for $S'_{v_g v_g}(\Omega_x L_g)$ and $S'_{w_g w_g}(\Omega_x L_g)$.

The turbulence field resulting from the PSD function can be seen as a superpositioning of multiple turbulence fields. Each turbulence field has spatial frequencies Ω_x and Ω_y and corresponding wavelengths $\lambda_x = 2\pi/\Omega_x$ and $\lambda_y = 2\pi/\Omega_y$. The turbulence velocity is now given by

$$u_g = u_{g_{max}} \mathrm{Re}\left(e^{j(\Omega_x x + \Omega_y y)}\right), \tag{7.1.8}$$

with the same for v_g and w_g. The above function can be seen as a sinusoid wave in two-dimensional space. The direction in which the waves 'run' is then given by $\arctan(\Omega_y/\Omega_x)$.

7.2 Finding the asymmetric forces and moments

7.2.1 Asymmetric forces and moments caused by longitudinal gusts u_g

To investigate the aircraft response to turbulence, we need to look at the forces and moments caused by turbulence. We'll do that now. First, we will examine the asymmetric forces and moments caused by longitudinal gusts u_g. For this, we use equation (7.1.8). In fact, we rewrite it to

$$u_g(x,y) = u_{g_{max}}\cos(\Omega_x x)\cos(\Omega_y y) + u_{g_{max}}\sin(\Omega_x x)\sin(\Omega_y y) = u_{g_1}(x,y) + u_{g_2}(x,y). \tag{7.2.1}$$

The first part $u_{g_1}(x,y)$ of the above equation is symmetric. It will thus not cause any asymmetric forces and moments. So, we will only examine the asymmetric function $u_{g_2}(x,y)$. To do this, we look at a small strip of the wing. For this small strip, we calculate the change in lift dL. We can then integrate $y\,dL$ over the entire wing to calculate the rolling moment caused by the gust. This gives us the coefficient of rolling motion due to gust

$$C_{l_g} = C_{l_{u_g}}\left(\Omega_y \frac{b}{2}\right)\hat{u}_g, \qquad \text{where} \qquad C_{l_{u_g}}\left(\Omega_y\frac{b}{2}\right) = -\frac{4}{Sb}\int_0^{\frac{b}{2}} c_l c \sin(\Omega_y y) y\, dy. \tag{7.2.2}$$

(Keep in mind that c_l and c also still depend on y.) The rolling moment due to turbulence C_{l_g} is similar to the rolling moment due to a yawing motion $C_{l_{r_w}}$ caused by the wing. In fact, we can relate the two parameters through $C_{l_{u_g}}\left(\Omega_y \frac{b}{2}\right)$ according to

$$C_{l_{u_g}}\left(\Omega_y\frac{b}{2}\right) = -C_{l_{r_w}} h\left(\Omega_y\frac{b}{2}\right), \qquad \text{where} \qquad h\left(\Omega_y\frac{b}{2}\right) = \frac{b}{2}\frac{\int_0^{\frac{b}{2}} c_l c \sin(\Omega_y y) y\, dy}{\int_0^{\frac{b}{2}} c_l c y^2\, dy}. \tag{7.2.3}$$

Determining the yawing coefficient due to longitudinal gust C_{n_g} goes in a similar way. We now find that

$$C_{n_g} = C_{n_{u_g}}\left(\Omega_y\frac{b}{2}\right)\hat{u}_g, \qquad \text{where} \qquad C_{n_{u_g}}\left(\Omega_y\frac{b}{2}\right) = -C_{n_{r_w}} h\left(\Omega_y\frac{b}{2}\right). \tag{7.2.4}$$

The function $h\left(\Omega_y\frac{b}{2}\right)$ is exactly the same as earlier. Finally, the lateral forces due to longitudinal gusts $C_{Y_{u_g}}$ are assumed to be negligible. So, $C_{Y_{u_g}} = 0$.

7.2.2 Asymmetric forces and moments caused by lateral gusts v_g

Let's examine the asymmetric forces and moments caused by lateral gusts v_g. We can split up v_g in a similar way as u_g. However, v_g is an asymmetric velocity. So this time we need to use the symmetric part $v_{g_1}(x,y) = v_{g_{max}} \cos(\Omega_x x)\cos(\Omega_y y)$ in our calculations. Also, we assume that v_g is approximately constant along the wing. Thus, $\cos(\Omega_y y) \approx 1$. We now define the **gust angle of sideslip** β_g as

$$\beta_g = \frac{v_g}{V} = \frac{v_{g_{max}} \cos(\Omega_x x)}{V}. \tag{7.2.5}$$

We would like to find the coefficients C_{Y_g}, C_{l_g} and C_{n_g}. Using a derivation similar to the one used in the previous chapter, we can find that

$$C_{Y_g} = \left(C_{Y_{\beta_g}} + C_{Y_{\dot{\beta}_g}} D_b\right)\beta_g, \qquad C_{l_g} = \left(C_{l_{\beta_g}} + C_{l_{\dot{\beta}_g}} D_b\right)\beta_g \quad \text{and} \quad C_{n_g} = \left(C_{n_{\beta_g}} + C_{n_{\dot{\beta}_g}} D_b\right)\beta_g. \tag{7.2.6}$$

Also, like in the previous chapter, we have

$$C_{Y_{\beta_g}} = C_{Y_\beta}, \qquad C_{l_{\beta_g}} = C_{l_\beta} \qquad \text{and} \qquad C_{n_{\beta_g}} = C_{n_\beta}. \tag{7.2.7}$$

The other three coefficients can, also analagous to the previous chapter, be approximated using

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{\dot{\beta}}} + C_{Y_r}, \qquad C_{l_{\dot{\beta}_g}} = C_{l_{\dot{\beta}}} + C_{l_r} \qquad \text{and} \qquad C_{n_{\dot{\beta}_g}} = C_{n_{\dot{\beta}}} + C_{n_r}. \tag{7.2.8}$$

For aircraft with straight wings and a relatively small tailplane, these three derivatives are often negligible. So, for the sake of simplicity, we often simply use $C_{Y_{\dot{\beta}_g}} = C_{l_{\dot{\beta}_g}} = C_{n_{\dot{\beta}_g}} = 0$.

7.2.3 Asymmetric forces and moments caused by vertical gusts w_g

When examining vertical gusts, we use the symmetric part $w_{g_2}(x,y) = w_{g_{max}} \sin(\Omega_x x)\sin(\Omega_y y)$ of the vertical gust $w_g(x,y)$. The gust angle of attack is still defined as $\alpha_g(x,y) = w_g(x,y)/V$. The coefficients $C_{l_{\alpha_g}}$ and $C_{n_{\alpha_g}}$ are now very similar to the coefficients $C_{l_{u_g}}$ and $C_{n_{u_g}}$. In fact, we have

$$C_{l_g} = C_{l_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right)\alpha_g \qquad \text{and} \qquad C_{n_g} = C_{n_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right)\alpha_g. \tag{7.2.9}$$

Here, the functions $C_{l_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right)$ and $C_{n_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right)$ are similar to the functions $C_{l_{u_g}}\left(\Omega_y \frac{b}{2}\right)$ and $C_{n_{u_g}}\left(\Omega_y \frac{b}{2}\right)$, respectively. The above coefficients can also be related to the coefficients for a rolling motion, according to

$$C_{l_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right) = C_{l_{p_w}} h\left(\Omega_y \frac{b}{2}\right) \qquad \text{and} \qquad C_{n_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right) = C_{n_{p_w}} h\left(\Omega_y \frac{b}{2}\right). \tag{7.2.10}$$

Finally, we assume that the side force due to α_g is negligible. So, $C_{Y_{\alpha_g}} = 0$.

7.2.4 Alternative derivation of v_g coefficients

There is an alternative way to derive the coefficients $C_{Y_{\beta_g}}$, $C_{l_{\beta_g}}$, $C_{n_{\beta_g}}$, $C_{Y_{\dot{\beta}_g}}$, $C_{l_{\dot{\beta}_g}}$ and $C_{n_{\dot{\beta}_g}}$. In this method, we make use of the gust penetration effect, just like we did in the previous chapter. First, a gust β_g hits the wing and the fuselage. (We assume that the aircraft CG and the wing aerodynamic center coincide.) A time $\tau = l_v/V$ later, the gust hits the vertical tailplane. ($l_v \approx x_v - x_{cg}$ is the distance between the wing and the aircraft CG.) However, the gust at the vertical tailplane has a magnitude

$$\beta_{v_g} = \left(1 - \frac{\partial\sigma}{\partial\beta}\right)\beta_g, \tag{7.2.11}$$

with σ the **sidewash** caused by the wing/fuselage. We can now approximate

$$C_{Y_g} = C_{Y_{\beta_g}} \beta_g + C_{Y_{\dot{\beta}_g}} D_b \beta_g. \tag{7.2.12}$$

Based on the above data, the coefficients $C_{Y_{\beta_g}}$ and $C_{Y_{\dot{\beta}_g}}$ can be determined. They are

$$C_{Y_{\beta_g}} = C_{Y_{f_\beta}} - C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(1 - \frac{\partial \sigma}{\partial \beta}\right), \tag{7.2.13}$$

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(1 - \frac{\partial \sigma}{\partial \beta}\right). \tag{7.2.14}$$

By the way, $C_{Y_{f_\beta}}$ is the contribution of the fuselage to C_{Y_β}. It is used because the wing hardly effects the coefficient C_{Y_β}. A similar expression as the one above can be derived for C_{l_g} and C_{n_g}. However, for these two parameters, the coefficients are given by

$$C_{Y_{\beta_g}} = C_{l_{w_\beta}} - C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(\frac{z_v - z_{cg}}{b} \cos\alpha - \frac{x_v - x_{cg}}{\sin}\alpha\right) \left(1 - \frac{\partial \sigma}{\partial \beta}\right), \tag{7.2.15}$$

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(\frac{z_v - z_{cg}}{b} \cos\alpha - \frac{x_v - x_{cg}}{\sin}\alpha\right) \left(1 - \frac{\partial \sigma}{\partial \beta}\right), \tag{7.2.16}$$

$$C_{Y_{\beta_g}} = C_{n_{f_\beta}} - C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(\frac{x_v - x_{cg}}{b} \cos\alpha - \frac{z_v - z_{cg}}{\sin}\alpha\right) \left(1 - \frac{\partial \sigma}{\partial \beta}\right), \tag{7.2.17}$$

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(\frac{x_v - x_{cg}}{b} \cos\alpha - \frac{z_v - z_{cg}}{\sin}\alpha\right) \left(1 - \frac{\partial \sigma}{\partial \beta}\right). \tag{7.2.18}$$

7.3 The PSD function and the asymmetric equations of motion

7.3.1 The PSD function of force and moment coefficients

We now know how to find the force and moment coefficients that are acting on the aircraft. The next step is to find the PSD functions of them. The method for this is mostly the same for all coefficients. But we're going to demonstrate it on C_{l_g}. Equation (7.2.2) now implies that

$$S_{C_{l_g} C_{l_g}}(\Omega_x L_g, \Omega_y L_g, B) = C_{l_{u_g}}^2 \left(\Omega_y L_g B\right) S_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, \Omega_y L_g). \tag{7.3.1}$$

In the above equation, we have defined another dimensionless coefficient: $B = \frac{b}{2L_g}$. Usually, the coefficient B is known. So then the above equation is two-dimensional. It would, however, be preferable for the equation to be one-dimensional. We can make it one-dimensional using

$$S_{C_{l_g} C_{l_g}}(\Omega_x L_g, B) = C_{l_{rw}}^2 \int_0^\infty h^2 \left(\Omega_y L_g B\right) S_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, \Omega_y L_g) \, d(\Omega_y L_g) = C_{l_{rw}}^2 I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B), \tag{7.3.2}$$

where the **effective one-dimensional PSD function** $I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B)$ is defined as the integral in the above equation. The variance of the force coefficient C_{l_g} can now be found using

$$\mathrm{E}\left\{C_{l_g}^2\right\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{C_{l_g} C_{l_g}}(\Omega_x L_g, B) \, d(\Omega_x L_g) = \frac{1}{2\pi} C_{l_{rw}}^2 \int_{-\infty}^{+\infty} I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B) \, d(\Omega_x L_g). \tag{7.3.3}$$

7.3.2 Approximating the one-dimensional PSD function

The above method can be simplified. It can be assumed that the product $c_l c$ has a negligible influence on the value of $h\left(\Omega_y L_g B\right)$. In this case, $h\left(\Omega_y L_g B\right)$ can be solved analytically. We then have

$$h\left(\Omega_y L_g B\right) = \frac{b}{2} \frac{\int_0^{\frac{b}{2}} \sin(\Omega_y y) y \, dy}{\int_0^{\frac{b}{2}} y^2 \, dy} = 3 \frac{\sin(\Omega_y L_g B) - (\Omega_y L_g B)\cos(\Omega_y L_g B)}{(\Omega_y L_g B)^2}. \tag{7.3.4}$$

Based on this, the function $I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B)$, and similarly the function $I_{\alpha_g \alpha_g}(\Omega_x L_g, B)$ as well, can be approximated. This is done using the equations

$$I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B) = I_{\hat{u}_g \hat{u}_g}(0, B) \frac{1 + \tau_3^2 \Omega_x^2 L_g^2}{(1 + \tau_1^2 \Omega_x^2 L_g^2)(1 + \tau_2^2 \Omega_x^2 L_g^2)}, \tag{7.3.5}$$

$$I_{\alpha_g \alpha_g}(\Omega_x L_g, B) = I_{\alpha_g \alpha_g}(0, B) \frac{1 + \tau_6^2 \Omega_x^2 L_g^2}{(1 + \tau_4^2 \Omega_x^2 L_g^2)(1 + \tau_5^2 \Omega_x^2 L_g^2)}. \tag{7.3.6}$$

It is important to remember that the above equations are approximations. But they do prove to be quite acceptable approximations. The constants τ_1 to τ_6 in the above equation depend on B. Their values can be found in tables.

7.3.3 The asymmetric equations of motion for an aircraft in turbulence

Let's derive the asymmetric equations of motion of an aircraft, when turbulence is involved. Based on the assumptions that have been made, the relations that have been found and the coefficients that have been calculated, we can find that

$$\begin{bmatrix} C_{Y_\beta} - 2\mu_b D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2} D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} =$$

$$- \begin{bmatrix} 0 & C_{Y_{\delta_r}} & 0 & C_{Y_\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_{l_{\delta_a}} & C_{l_{\delta_r}} & -C_{l_{r_w}} & C_{l_\beta} & C_{l_{p_w}} \\ C_{n_{\delta_a}} & C_{n_{\delta_r}} & -C_{n_{r_w}} & C_{n_\beta} & C_{n_{p_w}} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \\ \hat{u}_g \\ \beta_g \\ \alpha_g \end{bmatrix}. \tag{7.3.7}$$

Just like in the previous chapter, the above equations of motion can be put in state space form. For that, you would have to use the definition $D_b = \frac{b}{V} \frac{d}{dt}$. Also, the equations can be combined with a state space form of the gust filters. But again, we won't discuss that here.