Atmospheric Flight Dynamics
Example Exam 2 – Solutions

1 Question

Given the autocovariance function,

$$C_{\bar{x}\bar{x}}(\tau) = \frac{1}{2}\cos(2\pi\tau) \tag{1.1}$$

of stochastic variable $\bar{x}$. Calculate the autospectrum $S_{\bar{x}\bar{x}}(\omega)$.

NOTE

Assume that,

$$cos(2\pi\tau) = \frac{e^{-j2\pi\tau} + e^{j2\pi\tau}}{2} \tag{1.2}$$

and,

$$\int_{-\infty}^{+\infty} e^{-j\omega\tau}\, d\tau = 2\pi\delta(\omega) \tag{1.3}$$

1 Solution

To find the autospectrum $S_{\bar{x}\bar{x}}(\omega)$, we simply take the Fourier transform of the autocovariance function $C_{\bar{x}\bar{x}}(\tau)$. So,

$$S_{\bar{x}\bar{x}}(\omega) = \mathcal{F}\left\{C_{\bar{x}\bar{x}}(\tau)\right\} = \mathcal{F}\left\{\frac{1}{2}\cos(2\pi\tau)\right\} = \frac{\pi}{2}\left(\delta(\omega - 2\pi) + \delta(\omega + 2\pi)\right). \tag{1.4}$$

You can do the last step if you know the Fourier transform of the cosine function by heart. If not, then you can also derive it. We then have

$$\mathcal{F}\left\{\cos(2\pi\tau)\right\} = \mathcal{F}\left\{\frac{e^{-j2\pi\tau} + e^{j2\pi\tau}}{2}\right\} = \int_{-\infty}^{+\infty} \frac{e^{-j2\pi\tau} + e^{j2\pi\tau}}{2} e^{-j\omega\tau}\, d\tau. \tag{1.5}$$

Rewriting the above equation gives

$$\mathcal{F}\left\{\cos(2\pi\tau)\right\} = \frac{1}{2}\left(\int_{-\infty}^{+\infty} e^{-j(\omega+2\pi)\tau}\, d\tau + \int_{-\infty}^{+\infty} e^{-j(\omega-2\pi)\tau}\, d\tau\right). \tag{1.6}$$

We can use the relation for $\delta(\omega)$ to simplify the above equation. We then get

$$\mathcal{F}\left\{\cos(2\pi\tau)\right\} = \frac{1}{2}\left(2\pi\delta(\omega - 2\pi) + 2\pi\delta(\omega + 2\pi)\right). \tag{1.7}$$

2 Question

Prove that,

(a) the variance of the "stochastic" variable $\bar{y} = c$ equals $\sigma_{\bar{y}}^2 = 0$

(b) $\mu_{\bar{x}}^2 = \mathrm{E}\left\{\bar{x}^2\right\} - \sigma_{\bar{x}}^2$

(c) if $\bar{y} = b\bar{x} + c$ then $\sigma_{\bar{y}}^2 = b^2\sigma_{\bar{x}}^2$

In the above mentioned questions b and c are constants.

2 Solution

(a) There are multiple ways to show this. One way is by using the expectation operator. We have

$$\sigma_{\bar{y}}^2 = \mathrm{E}\left\{(\bar{y} - \mu_{\bar{y}})^2\right\} = \mathrm{E}\left\{(c - c)^2\right\} = \mathrm{E}\{0\} = 0. \tag{2.1}$$

Note that we have used $\mu_{\bar{y}} = c$. Another way to show it is to use the probability density function. For a stochastic variable $\bar{y} = c$, the probability density function equals $f_{\bar{y}}(y) = \delta(y - c)$. The mean of $\bar{y}$ is now given by

$$\mu_{\bar{y}} = \int_{-\infty}^{+\infty} y f_{\bar{y}}(y)\, dy = \int_{-\infty}^{+\infty} y\delta(y - c)\, dy = c. \tag{2.2}$$

The variance of $\bar{y}$ can now be found using

$$\sigma_{\bar{y}}^2 = \int_{-\infty}^{+\infty} (y - \mu_{\bar{y}})^2 f_{\bar{y}}(y)\, dy = \int_{-\infty}^{+\infty} (y - c)^2 \delta(y - c)\, dy = (c - c)^2 = 0. \tag{2.3}$$

(b) The variance of $\bar{x}$ can be found according to

$$\sigma_{\bar{x}}^2 = \mathrm{E}\left\{(\bar{x} - \mu_{\bar{x}})^2\right\} = \mathrm{E}\left\{\bar{x}^2 - 2\bar{x}\mu_{\bar{x}} + \mu_{\bar{x}}^2\right\} = \mathrm{E}\left\{\bar{x}^2\right\} - 2\mu_{\bar{x}}\mathrm{E}\{\bar{x}\} + \mu_{\bar{x}}^2. \tag{2.4}$$

If we use the fact that $\mu_{\bar{x}} = \mathrm{E}\{x\}$, we can rewrite the above to

$$\sigma_{\bar{x}}^2 = \mathrm{E}\left\{\bar{x}^2\right\} - \mu_{\bar{x}}^2. \tag{2.5}$$

This is equivalent to the relation which we wanted to prove.

(c) First let's find the mean value of $\bar{y}$. It is equal to

$$\mu_{\bar{y}} = \mathrm{E}\{\bar{y}\} = \mathrm{E}\{b\bar{x} + c\} = b\mu_{\bar{x}} + c. \tag{2.6}$$

The variance of $\bar{y} = b\bar{x} + c$ can be found using

$$\sigma_{\bar{y}}^2 = \mathrm{E}\left\{(\bar{y} - \mu_{\bar{y}})^2\right\} = \mathrm{E}\left\{(b\bar{x} + c - b\mu_{\bar{x}} - c)^2\right\} = b^2\mathrm{E}\left\{(\bar{x} - \mu_{\bar{x}})^2\right\} = b^2\sigma_{\bar{x}}^2. \tag{2.7}$$

3 Question

In figure 1 the product function $R_{\bar{u}\bar{u}}(\tau)$ of the stationary stochastic process $\bar{u}$ is given. What can be said about the properties of the stochastic variable $\bar{u}$?

(a) It is white noise.

(b) It is noise with a small bandwidth.

(c) It is white noise plus a sinus.

(d) It is a sinus.

3 Solution

It can be noted that $R_{\bar{u}\bar{u}}(\tau)$ is a sinc function. If you transform a sinc function, you will get a block function. The PSD function $S_{\bar{u}\bar{u}}(\omega)$ is thus a block function. If this block function would be infinitely wide, then the PSD function would be constant. The result would thus be white noise. However, the block function is not infinitely wide. The bandwidth is thus limited. We therefore deal with white noise with a small bandwidth. The correct answer is (b).

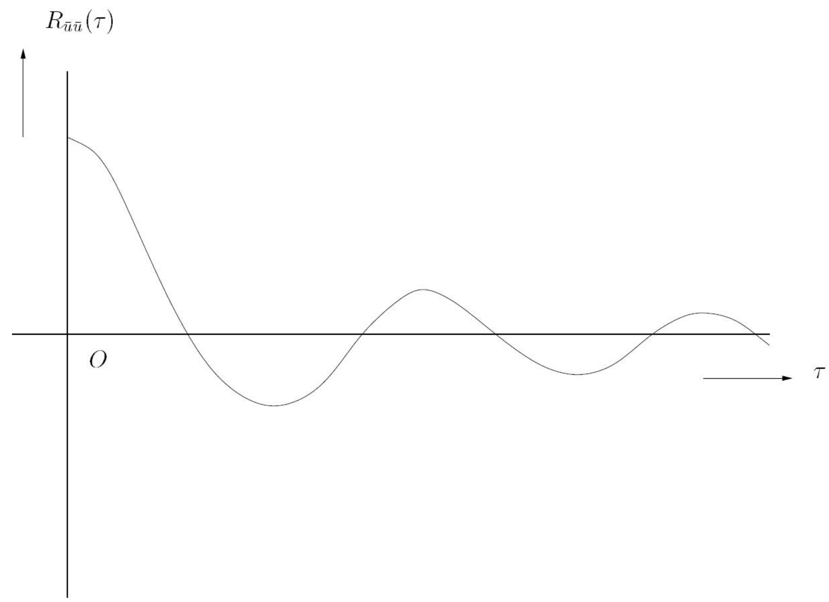

Figure 1: Product function $R_{\bar{u}\bar{u}}(\tau)$

4 Question

Given the probability density function of the stochastic variable $\bar{x}$ with parameter λ ($\lambda > 0$),

$$f_{\bar{x}}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{4.1}$$

Calculate the probability distribution function $F_{\bar{x}}(x)$, and prove that the mean value $\mu_{\bar{x}}$ and the variance $\sigma_{\bar{x}}^2$ are equal to,

$$\mu_{\bar{x}} = \frac{1}{\lambda} \qquad \text{and} \qquad \sigma_{\bar{x}}^2 = \frac{1}{\lambda^2}. \tag{4.2}$$

4 Solution

The probability distribution function $F_{\bar{x}}(x)$ can be found by integrating the probability density function $f_{\bar{x}}(x)$. This then gives

$$F_{\bar{x}}(x) = \int_{-\infty}^{x} f_{\bar{\tau}}(\tau)\, d\tau. \tag{4.3}$$

For $x < 0$, we thus simply have $F_{\bar{x}}(x) = 0$. If, however, $x \geq 0$, then

$$F_{\bar{x}}(x) = \int_{0}^{x} \lambda e^{-\lambda\tau}\, d\tau = \left[-e^{-\lambda\tau}\right]_0^x = 1 - e^{-\lambda x}. \tag{4.4}$$

Now we need to find the mean value $\mu_{\bar{x}}$. It is given by

$$\mu_{\bar{x}} = \int_{-\infty}^{+\infty} x f_{\bar{x}}(x)\, dx = \int_{0}^{+\infty} \lambda x e^{-\lambda x}\, dx. \tag{4.5}$$

Applying integration by parts now gives

$$\mu_{\bar{x}} = \left[-xe^{-\lambda x}\right]_0^{+\infty} + \int_0^{+\infty} e^{-\lambda x}\, dx = 0 + \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_0^{+\infty} = \frac{1}{\lambda}. \tag{4.6}$$

The variance $\sigma_{\bar{x}}^2$ can be found using

$$\sigma_{\bar{x}}^2 = \int_{-\infty}^{+\infty} (x - \mu_{\bar{x}})^2 f_{\bar{x}}(x)\, dx = \int_0^{+\infty} \lambda(x^2 - 2x/\lambda + 1/\lambda^2)e^{-\lambda x}\, dx. \tag{4.7}$$

Let's split the integral above up into three parts. Applying integration by parts on the part with x^2 gives

$$\int_0^{+\infty} \lambda x^2 e^{-\lambda x}\, dx = \left[-x^2 e^{-\lambda x}\right]_0^{+\infty} + \int_0^{+\infty} 2xe^{-\lambda x}\, dx = 0 + \int_0^{+\infty} 2xe^{-\lambda x}\, dx. \tag{4.8}$$

The part with x^2 thus cancels out with the part with $-2x/\lambda$. That saves some work. The variance now equals

$$\sigma_{\bar{x}}^2 = \int_0^{+\infty} 1/\lambda e^{-\lambda x}\, dx = \left[-\frac{1}{\lambda^2} e^{-\lambda x}\right]_0^{+\infty} = \frac{1}{\lambda^2}. \tag{4.9}$$

That finishes the proof.

5 Question

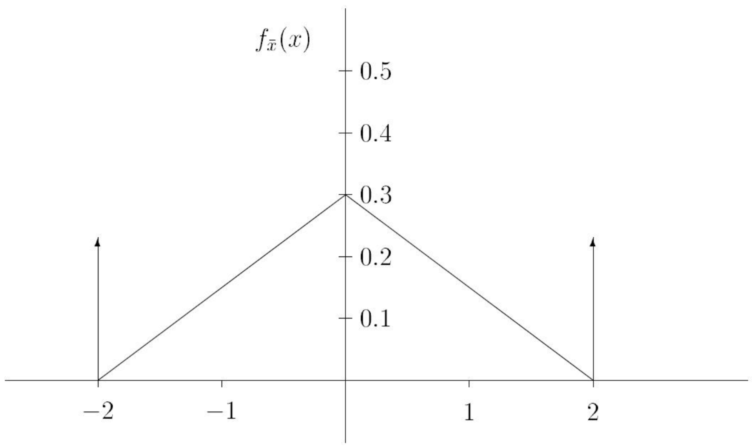

Figure 2: Probability density function $f_{\bar{x}}(x)$

The random variable $\bar{x}$ has a probability density function $f_{\bar{x}}(x)$ as depicted in figure 2.
What is the probability of $P(\bar{x} \geq -1)$?

(a) 0.125

(b) 0.275

(c) 0.725

(d) 0.750

(e) 0.875

(f) Not enough data available

5 Solution

We can note that the probability density function is symmetric about $x = 0$. Thus,

$$P(\bar{x} \leq 0) = P(\bar{x} \geq 0) = 0.5. \tag{5.1}$$

To find $P(-1 \leq \bar{x} \leq 0)$, we simply find the area under the function $f_{\bar{x}}(x)$ in this interval. This gives us

$$P(-1 \leq \bar{x} \leq 0) = 1 \cdot \frac{0.15 + 0.3}{2} = 0.225. \tag{5.2}$$

We thus have

$$P(\bar{x} \geq -1) = P(-1 \leq \bar{x} \leq 0) + P(\bar{x} \geq 0) = 0.725. \tag{5.3}$$

The correct answer is therefore (c).

6 Question

Prove that the Fourier transform of the signal $x(t - t_0)$ equals,

$$\mathcal{F}\left\{x(t - t_0)\right\} = \left(e^{-j\omega t_0}\right) X(\omega) \tag{6.1}$$

6 Solution

We first apply the definition of the Fourier transform. This gives

$$\mathcal{F}\left\{x(t - t_0)\right\} = \int_{-\infty}^{+\infty} x(t - t_0) e^{-j\omega t}\, dt. \tag{6.2}$$

Let's define $\tau = t - t_0$. We then have $t = \tau + t_0$ and $dt = d\tau$. So,

$$\mathcal{F}\left\{x(t - t_0)\right\} = \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega(\tau + t_0)}\, d\tau = e^{-j\omega t_0} \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau}\, d\tau = e^{-j\omega t_0} X(\omega). \tag{6.3}$$

This concludes the proof.

7 Question

Which of the following statements are true?

(a) $R_{\bar{x}\bar{y}}(\tau) = R_{\bar{y}\bar{x}}(\tau)$

(b) $C_{\bar{x}\bar{y}}(\tau) = C_{\bar{y}\bar{x}}(\tau)$

(c) $K_{\bar{x}\bar{x}}(\tau) = K_{\bar{x}\bar{x}}(-\tau)$

(d) $K_{\bar{x}\bar{x}}(0) = 1$

(e) $S_{\bar{x}\bar{y}}(\omega) = S_{\bar{y}\bar{x}}(\omega)$

(f) $S_{\bar{x}\bar{x}}(\omega) = S_{\bar{x}\bar{x}}(-\omega)$

7 Solution

(a) Let's examine $R_{\bar{x}\bar{y}}(\tau)$. We have

$$R_{\bar{x}\bar{y}}(\tau) = \mathrm{E}\left\{x(t)y(t+\tau)\right\} = \mathrm{E}\left\{y(t+\tau)x(t)\right\} = \mathrm{E}\left\{y(t)x(t-\tau)\right\} = R_{\bar{y}\bar{x}}(-\tau). \tag{7.1}$$

However, we don't generally have $R_{\bar{x}\bar{y}}(\tau) = R_{\bar{y}\bar{x}}(\tau)$. So, the first statement is false.

(b) We know that

$$C_{\bar{x}\bar{y}}(\tau) = R_{\bar{x}\bar{y}}(\tau) - \mu_{\bar{x}}\mu_{\bar{y}}. \tag{7.2}$$

So if the previous statement did not hold, then this one will not hold either. So, the second statement is false.

(c) For the function $K_{\bar{x}\bar{x}}(\tau)$, we have

$$K_{\bar{x}\bar{x}}(\tau) = \frac{\mathrm{E}\left\{x(t)x(t+\tau)\right\}}{\sigma_{\bar{x}}^2} = \frac{\mathrm{E}\left\{x(t+\tau)x(t)\right\}}{\sigma_{\bar{x}}^2} = \frac{\mathrm{E}\left\{x(t)x(t-\tau)\right\}}{\sigma_{\bar{x}}^2} = K_{\bar{x}\bar{x}}(-\tau). \tag{7.3}$$

So this statement holds. The third statement is thus true.

(d) We have

$$K_{\bar{x}\bar{x}}(0) = \frac{\mathrm{E}\left\{x(t)x(t)\right\}}{\sigma_{\bar{x}}^2} = \frac{\sigma_{\bar{x}}^2}{\sigma_{\bar{x}}^2} = 1. \tag{7.4}$$

This fourth statement is therefore true.

(e) In case of a zero-mean signal, the PSD function is the Fourier transform of the covariance function. So,

$$S_{\bar{x}\bar{y}}(\omega) = \int_{-\infty}^{+\infty} C_{\bar{x}\bar{y}}(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{+\infty} C_{\bar{y}\bar{x}}(-\tau)e^{-j\omega\tau}\,d\tau. \tag{7.5}$$

Let's substitute $t = -\tau$. Note that we have $dt = -d\tau$ and thus

$$S_{\bar{x}\bar{y}}(\omega) = -\int_{+\infty}^{-\infty} C_{\bar{y}\bar{x}}(t)e^{j\omega t}\,dt = \int_{-\infty}^{+\infty} C_{\bar{y}\bar{x}}(t)e^{j\omega t}\,dt = S_{\bar{y}\bar{x}}(\omega)^*. \tag{7.6}$$

Here, $S_{\bar{y}\bar{x}}(\omega)^*$ is the complex conjugate of $S_{\bar{y}\bar{x}}(\omega)$. This follows from the fact that $e^{j\omega t}$ is the complex conjugate of $e^{-j\omega t}$ and that $C_{\bar{y}\bar{x}}(t)$ is real. So we have $S_{\bar{x}\bar{y}}(\omega) = S_{\bar{y}\bar{x}}(\omega)^*$. But we don't in general have $S_{\bar{x}\bar{y}}(\omega) = S_{\bar{y}\bar{x}}(\omega)$. The fifth statement is thus false.

(f) Since $C_{\bar{x}\bar{x}}(\tau) = C_{\bar{x}\bar{x}}(-\tau)$, we have

$$S_{\bar{x}\bar{x}}(\omega) = \int_{-\infty}^{+\infty} C_{\bar{x}\bar{x}}(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{+\infty} C_{\bar{x}\bar{x}}(-\tau)e^{-j\omega\tau}\,d\tau. \tag{7.7}$$

If we again substitute $t = -\tau$, we get

$$S_{\bar{x}\bar{x}}(\omega) = \int_{-\infty}^{+\infty} C_{\bar{x}\bar{x}}(t)e^{-j(-\omega)t}\,d\tau = S_{\bar{x}\bar{x}}(-\omega). \tag{7.8}$$

The sixth statement thus holds.

8 Question

(a) The random variable $\bar{u}$ has a probability density function $f_{\bar{u}}(u)$ as depicted in figure 3. Calculate the probability $P(\bar{u} = 4)$.

(b) The random variable $\bar{u}$ has a probability density function $f_{\bar{u}}(u)$ as depicted in figure 4. Calculate the probability $P(\bar{u} = 1)$.

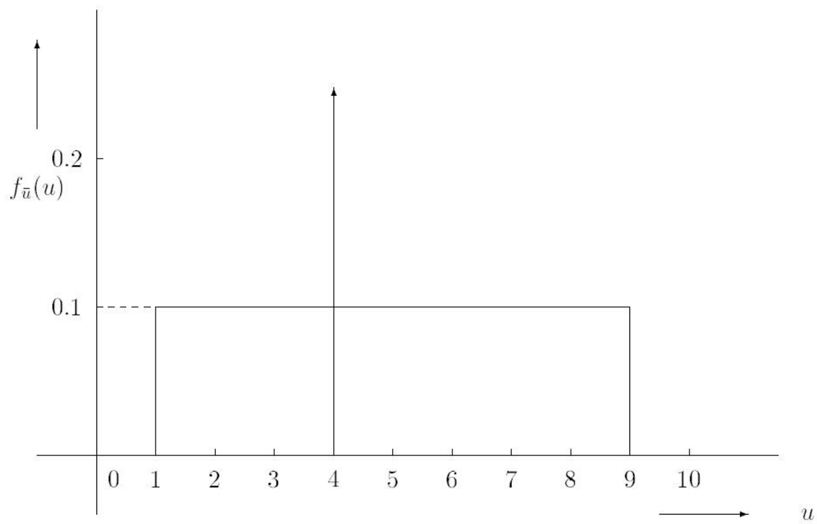

Figure 3: Probability density function $f_{\bar{u}}(u)$

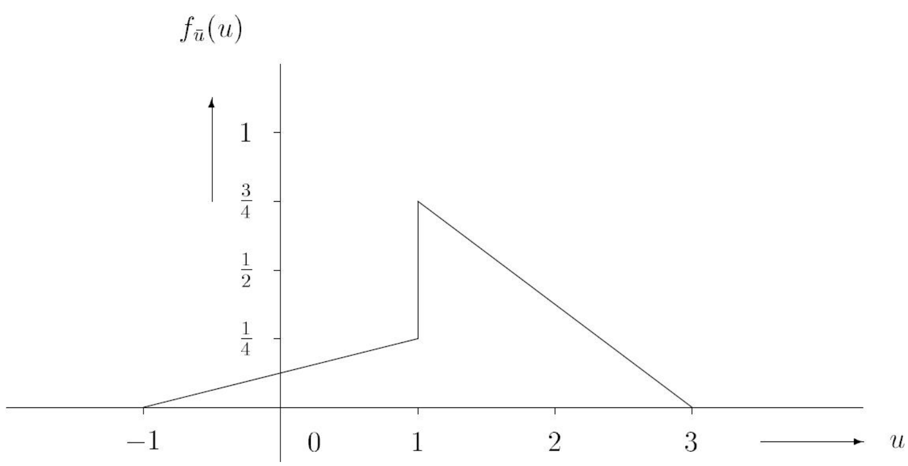

Figure 4: Probability density function $f_{\bar{u}}(u)$

8 Solution

(a) Normally, when a stochastic variable is continuous, then the chance that $\bar{u}$ is exactly a given value is simply equal to 0. However, this time there is a peak at $u = 4$. So, $P(\bar{u} = 4)$ now equals the magnitude of this peak.

The only problem is, we don't know the magnitude of the peak. However, we do know that the area under the graph of $f_{\bar{u}}(u)$ equals 1. Also, the area under the rectangle equals $0.1 \cdot 8 = 0.8$. The magnitude of the peak thus equals

$$P(\bar{u} = 4) = 1 - P(\bar{u} \neq 4) = 1 - 0.8 = 0.2. \tag{8.1}$$

(b) When a stochastic variable is continuous, and there is no delta-function-like peak, then the chance that $\bar{u}$ exactly equals a given value is simply 0. We see that this is the case here. So, $P(\bar{u} = 1) = 0$.

9 Question

Proof that the periodogram $I_{\bar{y}\bar{y}}[k]$ of the signal $y[n] = ax[n] + b$ equals,

$$I_{\bar{y}\bar{y}}[k] = a^2 I_{\bar{x}\bar{x}}[k] + (2a\mathrm{Re}\{X[k]\} + b)b\delta[k] \tag{9.1}$$

with,

$$I_{\bar{x}\bar{x}}[k] = X^*[k]X[k] \tag{9.2}$$

and $\mathrm{Re}\{X[k]\}$ the real part of the Fourier transform of $x[n]$.

Note: the Discrete Fourier Transform (FFT) of a constant b equals,

$$FFT\{b\} = \left(\frac{1}{N} \sum_{n=0}^{N-1} b e^{-j\frac{2\pi k}{N} n} \right) = b\delta[k] \tag{9.3}$$

with $\delta[k]$ the Kronecker delta function. Use the result $FFT\{b\} = b\delta[k]$ in your proof. Remember that $\delta[k]$ equals 0 for $k \neq 0$ and $\delta[k]$ equals 1 for $k = 0$.

9 Solution

First, we'll find an expression for $Y[k]$. We have

$$Y[k] = FFT\{y[n]\} = FFT\{ax[n] + b\} = a\,FFT\{x[n]\} + FFT\{b\} = aX[k] + b\delta[k]. \tag{9.4}$$

Since a and b are real constants, we also have

$$Y^*[k] = aX^*[k] + b\delta[k]. \tag{9.5}$$

The periodogram of $y[n]$ is now given by

$$I_{\bar{y}\bar{y}}[k] = Y^*[k]Y[k] = (aX^*[k] + b\delta[k])\,(aX[k] + b\delta[k])\,. \tag{9.6}$$

Working out brackets gives

$$I_{\bar{y}\bar{y}}[k] = a^2 X^*[k]X[k] + ab\,(X^*[k] + X[k])\,\delta[k] + b^2\delta[k]^2. \tag{9.7}$$

In the discrete domain, we have $\delta[k]^2 = \delta[k]$. Also, adding a complex number to its complex conjugate gives twice its real part. (That is, $X^*[k] + X[k] = 2\mathrm{Re}\{X[k]\}$.) And, if we also apply the definition for $I_{\bar{x}\bar{x}}[k]$, we will find that

$$I_{\bar{y}\bar{y}}[k] = a^2 I_{\bar{x}\bar{x}}[k] + (2a\mathrm{Re}\{X[k]\} + b)\,b\delta[k]. \tag{9.8}$$

And this is exactly what we needed to show.