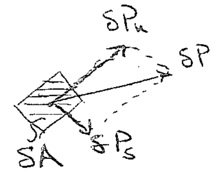
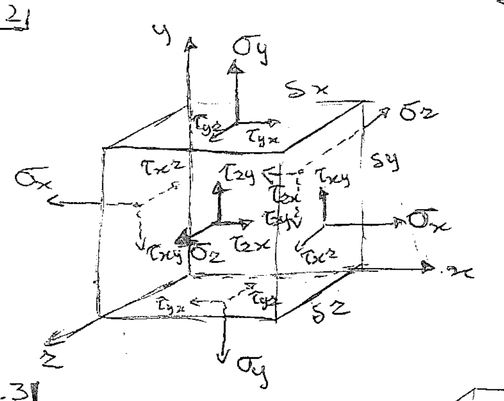
AE2-522: Lecture 1: 1.1-1.11, 1.15

1.1 σ, normal stress: $\sigma = \lim_{\delta A \to 0} \frac{\delta P_n}{\delta A}$

τ, shear stress: $\tau = \lim_{\delta A \to 0} \frac{\delta P_s}{\delta A}$



1.2

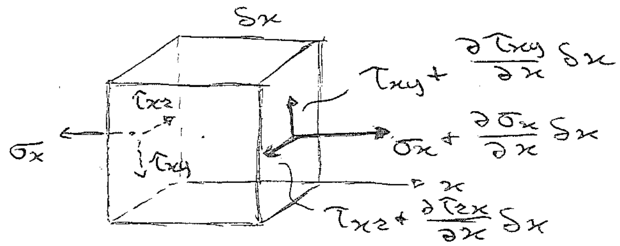


Two types of external forces:

- surface forces, distributed over surface area
- body forces, distributed over volume of body

1.3

Generally:



etc.

Taking moments about an axis through the centre of the element // to z-ax

$\tau_{xy}\,\delta y\,\delta z\,\frac{\delta x}{2} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\delta x\right)\delta y\,\delta z\,\frac{\delta x}{2} - \tau_{yx}\,\delta x\,\delta z\,\frac{\delta y}{2} - \left(\tau_{yx} + \frac{\delta \tau_{yx}}{\partial y}\delta y\right)\delta x\,\delta z\,\frac{\delta y}{2} =$

$\Rightarrow \tau_{xy}\,\delta y\,\delta z\,\delta x + \frac{\partial \tau_{xy}}{\partial x}\delta y\,\delta z\,\frac{(\delta x)^2}{2} - \tau_{yx}\,\delta x\,\delta z\,\delta y - \frac{\partial \tau_{yx}}{\partial y}\delta x\,\delta z\,\frac{(\delta y)^2}{2} = 0$

Divide by $\delta x\,\delta y\,\delta z$ and take limit as δx and δy go to zero:

$\Rightarrow \tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$.

Now take equilibrium of the element in the x-direction:

$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\delta x\right)\delta y\,\delta z - \sigma_x\,\delta y\,\delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\delta y\right)\delta x\,\delta z - \tau_{yx}\,\delta x\,\delta z + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\delta z\right)$

$\delta x\,\delta y - \tau_{zx}\,\delta x\,\delta y + X\,\delta x\,\delta y\,\delta z \quad \Rightarrow \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$

Similarly:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0$$
$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z = 0$$

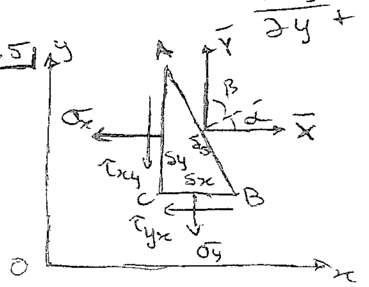
1.4

Plane stress, used when in z-axis is thin metal sheet, then $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

Thus:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + Y = 0$$

1.5



ΣF_x: $\bar{X}\,\delta s - \sigma_x\,\delta y - \tau_{yx}\,\delta x + X\,\tfrac{1}{2}\,\delta x\,\delta y = 0$ As $\delta x \to 0$:

$\bar{X} = \sigma_x \frac{dy}{ds} + \tau_{yx}\frac{dx}{ds}$ where $\frac{dy}{ds} = \cos\alpha = l$, $\frac{dx}{ds} = \cos\beta = m$

Thus: $\bar{X} = \sigma_x l + \tau_{yx} m$ and $\bar{Y} = \sigma_y m + \tau_{xy} l$

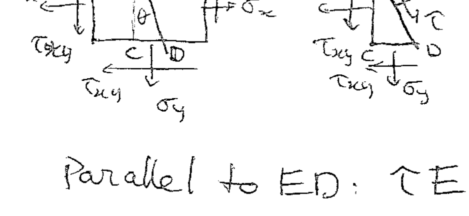
For a 3-D body this extends to:

$\bar{X} = \sigma_x l + \tau_{yx} m + \tau_{zx} n$

$\bar{Y} = \sigma_y m + \tau_{xy} l + \tau_{zy} n$

$\bar{Z} = \sigma_z n + \tau_{yz} m + \tau_{xz} l$

.61 To determine stress on plane ED: perpendicular to ED



$\sigma_n ED = \sigma_x ED \cos\theta + \sigma_y CD \sin\theta + \tau_{xy} EC \sin\theta + \tau_{xy} CD \cos\theta$. Divide by ED:

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \sin 2\theta \qquad (I)$$

Parallel to ED: $\tau ED = \sigma_x EC \sin\theta - \sigma_y CD \cos\theta - \tau_{xy} EC \cos\theta + \tau_{xy} CD \sin\theta$

Divide by ED and simplify: $\tau = \frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta - \tau_{xy} \cos 2\theta \qquad (II)$

1.71 σ_n varies with θ, with max/min when $\frac{d\sigma_n}{d\theta} = 0$: From (I)

$$\frac{d\sigma_n}{d\theta} = -2\sigma_x \cos\theta \sin\theta + 2\sigma_y \sin\theta \cos\theta + 2\tau_{xy} \cos 2\theta = 0$$

Hence: $-(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy} \cos 2\theta = 0$ or $\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$

Two solutions, θ and $\theta + \frac{\pi}{2}$, corresponding to no shear stress. The normal stresses in this case are called principle stresses.

$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$, $\cos 2\theta = \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$ and

$\sin 2(\theta + \frac{\pi}{2}) = \frac{-2\tau_{xy}}{- -}$, $\cos 2(\theta + \frac{\pi}{2}) = \frac{-(\sigma_x - \sigma_y)}{- - -}$

Rewrite (I): $\sigma_n = \frac{\sigma_x}{2}(1 + \cos 2\theta) + \frac{\sigma_y}{2}(1 - \cos 2\theta) + \tau_{xy} \sin 2\theta$. $\Rightarrow$

$\sigma_I = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$ max. principal stress

$\sigma_{II} = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$ min. principal stress.

Max. shear stress is determined similarly: From (II)

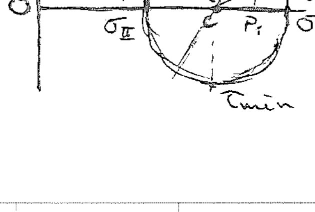
$\frac{d\tau}{d\theta} = (\sigma_x - \sigma_y)\cos 2\theta + 2\tau_{xy} \sin 2\theta = 0$, $\tan 2\theta = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$ $\Rightarrow$

$\sin 2\theta = \frac{-(\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$, $\cos 2\theta = \frac{2\tau_{xy}}{- - -}$

$\sin 2(\theta + \frac{\pi}{2}) = \frac{(\sigma_x - \sigma_y)}{- - -}$, $\cos 2(\theta + \frac{\pi}{2}) = \frac{-2\tau_{xy}}{- -}$ $\Rightarrow$ Substitute in (II):

$\tau_{max,min} = \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$ $\Rightarrow$ $\tau_{max} = \frac{\sigma_I - \sigma_{II}}{2}$

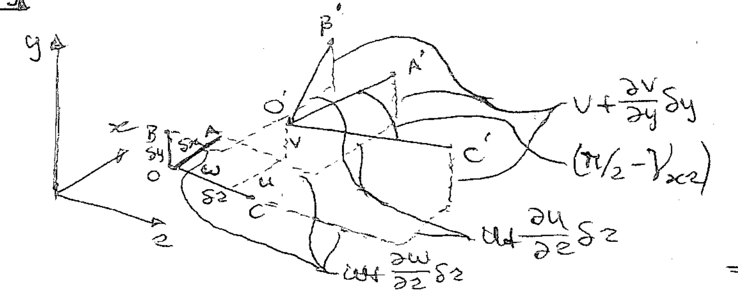
.81



$\sigma_I = OC + R = \frac{(\sigma_x + \sigma_y)}{2} + \sqrt{CP_1^2 + P_1Q_1^2} = \frac{(\sigma_x + \sigma_y)}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$

AE2-S22: Lecture 1: continued (2)

1.9



$$\varepsilon = \lim_{L \to 0} \frac{\Delta L}{L}$$

$$\varepsilon_x = \frac{O'A' - OA}{OA} = \frac{O'A' - \delta x}{\delta x}$$

$$(O'A')^2 = \left(\delta x + u + \frac{\partial u}{\partial x}\delta x - u\right)^2 + \left(v + \frac{\partial v}{\partial x}\delta x - v\right)^2 + \left(\omega + \frac{\partial w}{\partial x}\delta x\right.$$

$$\Rightarrow (O'A') = \delta x \sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}$$

Neglect second-order terms: $O'A' = \delta x\left(1 + 2\frac{\partial u}{\partial x}\right)^{1/2} \Rightarrow O'A' = \delta x\left(1 + \frac{\partial u}{\partial x}\right)$

We have: $\varepsilon_x = \frac{\partial u}{\partial x}$, $\varepsilon_y = \frac{\partial v}{\partial y}$, $\varepsilon_z = \frac{\partial w}{\partial z}$

$\cos A'O'C' = \cos\left(\frac{\pi}{2} - \gamma_{xz}\right) = \sin \gamma_{xz}$. $\gamma_{xz} << \Rightarrow \cos A'O'C' = \gamma_{xz} = \frac{(O'A')^2 + (O'C')^2 - (A'C')^2}{2(O'A')(O'C')}$.

$O'A' = \delta x\left(1 + \frac{\partial u}{\partial x}\right)$, $O'C' = \delta z\left(1 + \frac{\partial w}{\partial z}\right) \Rightarrow O'A' \simeq \delta x$, $O'C' \simeq \delta z$, $(A'C')^2 = \left(\delta z - \frac{\partial w}{\partial x}\delta x\right)^2 + \left(\delta x - \frac{\partial u}{\partial z}\delta z\right)^2$

$$\cos A'O'C' = \frac{(\delta x^2) + (\delta z)^2 - \left[\delta z - \left(\frac{\partial w}{\partial x}\right)\delta x\right]^2 - \left[\delta x - \left(\frac{\partial u}{\partial z}\right)\delta z\right]^2}{2\delta x \delta z}$$

$$\cos A'O'C' = \frac{2\left(\frac{\partial w}{\partial x}\right)\delta x \delta z + 2\left(\frac{\partial u}{\partial z}\right)\delta x \delta z}{2\delta x \delta z} \Rightarrow$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

1.10

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\frac{\partial v}{\partial x} + \frac{\partial^2}{\partial x \partial y}\frac{\partial u}{\partial y} = \frac{\partial^2}{\partial x^2}\frac{\partial v}{\partial y} + \frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial x} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2}$$

Similarly: $\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}$ and $\frac{\partial^2 \gamma_{xz}}{\partial x \partial z} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2}$

Now $\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} + \frac{\partial^2 \gamma_{xz}}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \Rightarrow$

$$\Rightarrow 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\Rightarrow 2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\Rightarrow 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

1.11

For plane strain $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \Rightarrow \varepsilon_x = \frac{\partial u}{\partial x}$, $\varepsilon_y = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$

and compatibility: $\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2}$

1.15

$$\varepsilon_x = \frac{\sigma_x}{E}, \quad \varepsilon_y = -\nu\frac{\sigma_x}{E}, \quad \varepsilon_z = -\nu\frac{\sigma_x}{E}$$

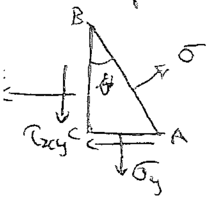
$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right], \quad \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right], \quad \varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

$\gamma = (\varepsilon_I - \varepsilon_{II})\sin 2\theta = \ldots = \frac{2(1+\nu)}{E}\tau$ where $\frac{E}{2(1+\nu)} = G$ and $\gamma = \frac{\tau}{G}$

For plane stress: $\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$, $\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$, $\varepsilon_z = \frac{-\nu}{E}(\sigma_x - \sigma_y)$

and $\gamma_{xy} = \frac{\tau_{xy}}{G}$

Example 1.1: $\sigma_x = 160\ N/mm^2$, $\sigma_y = -120\ N/mm^2$, $\sigma_{max} = 200\ N/mm^2$



What is τ_{xy}, σ_{min} and τ_{max} on AB?

Horizontal Equilibrium: $\sigma\, AB \cos\theta = \sigma_x BC + \tau_{xy} AC \Rightarrow \tau_{xy} \tan\theta = \sigma - \sigma_x$ (i)

Vertical Equilibrium: $\sigma\, AB \sin\theta = \sigma_y AC + \tau_{xy} BC \Rightarrow \tau_{xy} \cot\theta = \sigma - \sigma_y$ (ii)

Taking (i) and (ii): $\tau_{xy}^2 = (\sigma - \sigma_x)(\sigma - \sigma_y) \Rightarrow \tau_{xy} = \pm 113\ N/mm^2$

$$\Rightarrow \sigma^2 - \sigma(\sigma_x + \sigma_y) + \sigma_x \sigma_y - \tau_{xy}^2 = 0 \Rightarrow \sigma_{min} = -160\ N/mm^2$$

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{200 + 160}{2} = 180\ N/mm^2$$

Example 1.2: Rectangular element is subjected to tensile stresses of $83\ N/mm^2$ and $65\ N/mm^2$ on perpendicular planes. Determine in each direction ε, ε_{max}, τ_{max}. Take $E = 200000\ N/mm^2$, $\nu = 0.3$

Assume $\sigma_x = 83\ N/mm^2$, $\sigma_y = 65\ N/mm^2$, then:

$$\varepsilon_x = \frac{1}{200000}(83 - 0.3 \cdot 65) = 3.175 \cdot 10^{-4}$$

$$\varepsilon_y = \frac{1}{200000}(65 - 0.3 \cdot 83) = 2.005 \cdot 10^{-4}$$

$$\varepsilon_z = \frac{-0.3}{200000}(83 + 65) = -2.220 \cdot 10^{-4}$$

Since there are no shear stresses, σ_x & σ_y are principal stresses so that ε_x & ε_y are principal strains in direction of σ_x & σ_y.

$$\tau_{max} = \frac{83 - 65}{2} = 9\ N/mm^2 \text{ acting } 45^\circ \text{ to principal planes}$$

$$\gamma_{max} = \frac{2(1+\nu)}{E} \tau_{max} = \frac{2 \cdot (1 + 0.3) 9}{200\,000} = 1.17 \cdot 10^{-4}$$

Example 1.3: 2-D stress system. $\sigma_x = 60\ N/mm^2$, $\sigma_y = -40\ N/mm^2$, $\tau_{xy} = 50\ N/mm^2$

If $E = 2 \cdot 10^5\ N/mm^2$ and $\nu = 0.3$ calculate strain in x, y-dir. and γ_{xy}

Also calculate principal strains and their inclination.

$$\varepsilon_x = \frac{1}{200000}(60 + 0.3 \cdot 40) = 360 \cdot 10^{-6}$$

$$\varepsilon_y = \frac{1}{200\,000}(-40 - 0.3 \cdot 60) = -290 \cdot 10^{-6}$$

$$G = \frac{E}{2(1+\nu)} = \frac{200000}{2(1+0.3)} = 76923\ N/mm^2$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{50}{76923} = 650 \cdot 10^{-6}$$

$$\varepsilon_{max} = \varepsilon_I = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} = 10^{-6}\left[\frac{360 - 290}{2} + \frac{1}{2}\sqrt{(360+290)^2 + 650^2}\right] = 495 \cdot 10^{-6}$$

Similarly, $\varepsilon_{min} = \varepsilon_{II} = -425 \cdot 10^{-6}$

$$\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{650}{360 + 290} = 1 \Rightarrow 2\theta = 45^\circ \text{ or } 225^\circ \Rightarrow \theta = 22.5^\circ \text{ or } 112.5^\circ$$

AE2-522: Lecture 1: continued (4)

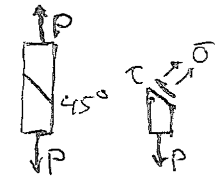
Lecture Notes:

How to draw Mohr's circle: 1. draw σ_x, τ_{xy}

2. draw $\sigma_y, -\tau_{xy}$

3. draw circle

Exercise 1.1: Calculate stresses in beam

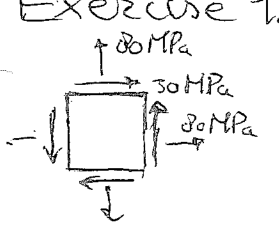


$$\Sigma F_v: 0 = -P + N\tfrac{1}{2}\sqrt{2} + T\tfrac{1}{2}\sqrt{2} \quad \Big\} \; N = T$$

$$\Sigma F_h: 0 = N\tfrac{1}{2}\sqrt{2} - T\tfrac{1}{2}\sqrt{2} \quad \Big\} \; N = \tfrac{1}{2}\sqrt{2}P$$

$$\sigma = \frac{\tfrac{1}{2}\sqrt{2}P}{\sqrt{2}A} = \tfrac{1}{2}\frac{P}{A} \qquad \tau = \tfrac{1}{2}\frac{P}{A}$$

Exercise 1.2: Mohr's circle, rotation of stress element

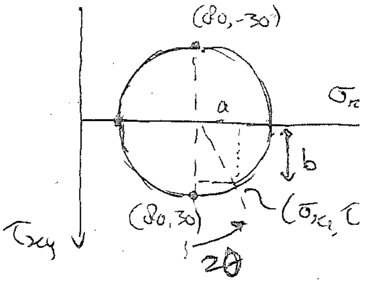


Rotate element over angle of 15° counterclockwise.

Use Mohr's circle to determine normal and shear stress

step 1: draw point $(\sigma_x, \tau_{xy}) = (80, 30)$

step 2: draw point $(\sigma_y, -\tau_{xy}) = (80, -30)$



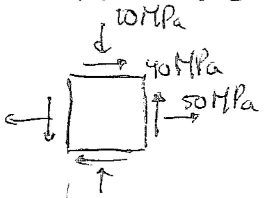
$a = 30 \sin 2\theta = 30 \sin 30 = 15$

$\Rightarrow \sigma_{x_1} = 80 + 15 = 95$ MPa

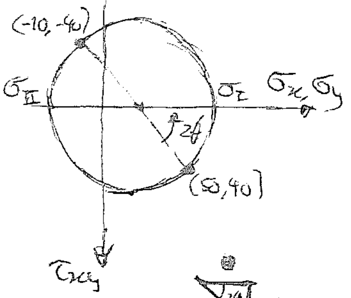
$\Rightarrow \sigma_{y_1} = 80 - 15 = 65$ MPa

$b = 30 \cos 2\theta = 30 \cos 30 = 15\sqrt{3} \Rightarrow \tau_{x_1 y_1} = 15\sqrt{3}$ MPa

Exercise 1.3: Mohr's circle



Rotate element over angle θ such that element only has principal stresses σ_I and σ_{II}

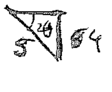


$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2} = 20 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{30^2 + 40^2} = 50 \text{ MPa}$$

Then, $\sigma_I = 20 + 50 = 70$ MPa,

$\sigma_{II} = 20 - 50 = -30$ MPa



$\Rightarrow \sin^{-1}\left(\frac{4}{5}\right) = 2\theta \Rightarrow \theta = 26.56°$

AE2-522: Lecture 2: 2.1–2.4, 2.6

2.1| Equilibrium Conditions for plane stress:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \qquad \varepsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial y} + Y = 0 \qquad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu \sigma_x)$$

and required stress-strain relationships: $\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$

$$\Rightarrow 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) + \frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y)$$

Substituting $\frac{\partial^2 \tau_{xy}}{\partial y \partial x} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial X}{\partial x} = -\frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial Y}{\partial y}$:

$$\Rightarrow -(1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} \quad \text{or}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)$$ Compatibility Eq. for plane stress

$\sigma_z = \nu(\sigma_x + \sigma_y) \Rightarrow \varepsilon_x = \frac{1}{E}\left((1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y\right)$ and $\varepsilon_y = \frac{1}{E}\left((1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x\right)$

also $\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \Rightarrow$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)$$ Compatibility Eq for plane strain

2.2| Consider 2-D case with no body forces:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} = 0 \quad \text{and} \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \quad (\mathrm{I})$$

Stress function ϕ: $\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$, $\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$, $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$

Substitution into (I): biharmonic equation $\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$

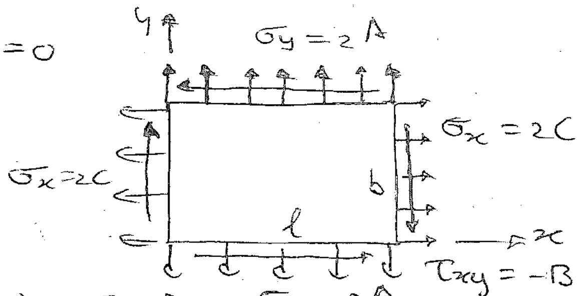
2.3| Inverse method: assume ϕ and determine loading conditions.

2.4| Principle of St. Venant: stresses at sections distant from surface of loading are essentially the same.

2.6|

Example 2.1: Consider the stress function $\phi = Ax^2 + Bxy + Cy^2$

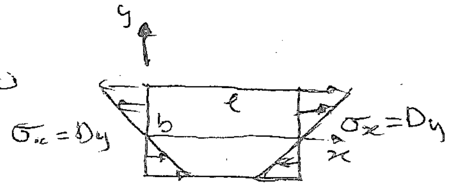
From biharmonic equation: $0=0$



$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 2C$$
$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 2A$$
$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -B$$

Example 2.2: $\phi = \frac{Ax^3}{6} + \frac{Bx^2y}{2} + \frac{Cxy^2}{2} + \frac{Dy^3}{6}$

From biharmonic equation: $0=0$



$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = Cx + Dy \qquad \text{If } A=B=C=0$$
$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = Ax + By$$
$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -Bx - Cy$$

$$\phi = \frac{Ax^4}{12} + \frac{Bx^3y}{6} + \frac{Cx^2y^2}{2} + \frac{Dxy^3}{6} + \frac{Ey^4}{12}$$

From biharmonic equation: $\frac{\partial^4\phi}{\partial x^4} = 2A$, $2\frac{\partial^4\phi}{\partial x^2\partial y^2} = 4C$, $\frac{\partial^4\phi}{\partial y^4} = 2E \Rightarrow$

$$E = -(2C+A)$$

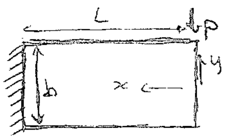
$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = Cx^2 + Dxy - (2C+A)y^2 \qquad \text{If } A=B=C=0:$$
$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = Ax^2 + Bxy + Cy^2 \qquad \sigma_x = Dxy,\ \sigma_y = 0,\ \tau_{xy} = -\frac{Dy^2}{2}$$
$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{Bx^2}{2} - 2Cxy - \frac{Dy^2}{2}$$

Lecture Notes:

Inverse Method: assume stress function, B.C., loading conditions

Semi-inverse Method: assume relation for σ, B.C., loading conditions.

Exercise 2.2: semi-inverse method



direct stress is directly proportional to the bending moment and height above the neutral axis. Assume for σ: $\sigma = Bxy$. Use $\sigma = \frac{\delta^2}{\delta ...}$ and the fact that $\tau = 0$ at $y = \pm\frac{b}{2}$ to find stress function ϕ.

$\sigma = Bxy = \frac{\delta^2\phi}{\delta y^2} \Rightarrow \frac{\delta\phi}{\delta y} = \frac{1}{2}Bxy^2 + f(x) + C \Rightarrow \phi = \frac{1}{6}Bxy^3 + yf(x) + Cy + g(x) + D.$

B.C.: $\tau = -\frac{\delta^2\phi}{\delta x\delta y} = -\frac{1}{2}By^2 - f'(x)$. At $y = b/2$: $0 = -\frac{1}{2}B\left(\frac{b}{2}\right)^2 - f'(x) \Rightarrow f'(x) = -B\frac{b^2}{8} \Rightarrow f(x) = -B\frac{b^2}{8}x$

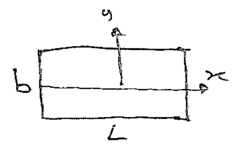
$$A = -B\frac{b^2}{8}$$

Therefore: $\phi = \frac{1}{6}Bxy^3 + Axy + Cy + g(x) + D$. Find $g(x)$ by using B.C. for σ_y:

$\frac{\delta\phi}{\delta x} = \frac{1}{6}By^3 + yA + g'(x) \Rightarrow \frac{\delta^2\phi}{\delta x^2} = g''(x) \Rightarrow \frac{\delta^2\phi}{\delta x^2} = Bxy \Rightarrow \frac{\delta^4\phi}{\delta x^4} = 0 \Rightarrow g^{IV}(x) = 0 \Rightarrow g(x) = \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C$

AE2-522: Lecture 2: continued (2)

Exercise 2-3: inverse method finding stress function



Given $\phi = \frac{Axy^3}{6} + Bxy$. Determine load case and stress distribution of the plate. Assume $y = \pm b/2$ do not carry any load

What are values of A and B ?

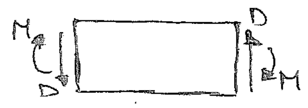
First check biharmonic equation: $\frac{\partial \phi}{\partial x} = \frac{Ay^3}{6} + By \quad \frac{\partial^2 \phi}{\partial x^2} = 0 \Rightarrow \frac{\partial^4 \phi}{\partial x^4} = 0 \Rightarrow \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0$

$\frac{\partial \phi}{\partial y} = \frac{Axy^2}{6} + Bx \quad \frac{\partial^2 \phi}{\partial y^2} = Axy, \ \frac{\partial^3 \phi}{\partial y^3} = Ax, \ \frac{\partial^4 \phi}{\partial y^4} = 0$

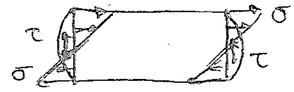
Stress components are: $\sigma_x = Axy, \ \sigma_y = 0, \ \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{Ay^2}{2} - B$

$\tau_{xy}(y = b/2) = 0 \Rightarrow B = -\frac{b^2}{8}A \Rightarrow \tau_{xy} = -\frac{Ay^2}{2} + \frac{b^2}{8}A$

Eq. Eq: $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$ and $\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$



Internal stress distribution looks like:



and the external load in a figure:

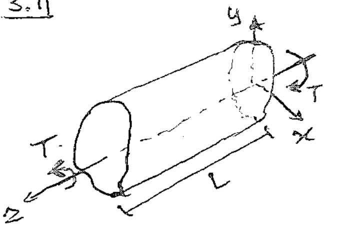
$\sigma = \frac{M}{I}y = Axy$. at $x = L/2$: $\frac{M}{I}y = A\frac{L}{2}y \Rightarrow A = \frac{2}{L}\frac{M}{I} = \frac{2M}{L}\frac{12}{b^3 t}$ where $I = \frac{1}{12}b^3 t$

Resulting shear force: $\int \tau_{xy} dA = \int\left(-\frac{Ay^2}{2} + \frac{b^2}{8}A\right) dA = \frac{1}{12}Ab^3 t = D = \frac{2M}{L}$

Moment equilibrium of applied forces and moments:

M_O: $0 = DL - M - M \Rightarrow \frac{2M}{L} \cdot L - 2M = 0$ ✓

3.1]



Assume that $\sigma_x = \sigma_y = \sigma_z = 0$ and $\tau_{xy} = 0 \Rightarrow \varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$

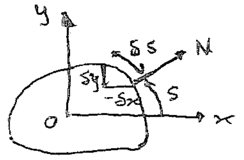
Eq. Eq: $\dfrac{\partial \tau_{xz}}{\partial z} = 0, \quad \dfrac{\partial \tau_{yz}}{\partial z} = 0, \quad \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{zy}}{\partial y} = 0$

Stress function: $\dfrac{\partial \phi}{\partial x} = -\tau_{zy}, \quad \dfrac{\partial \phi}{\partial y} = \tau_{zx}$

Comp Eq: $\dfrac{\partial}{\partial x}\left(-\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{xz}}{\partial y}\right) = 0 \Rightarrow \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right) = 0 \Rightarrow \dfrac{\partial}{\partial x}\nabla^2\phi = 0$

$\nabla^2 = \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$ $\qquad \dfrac{\partial}{\partial y}\left(\dfrac{\partial \gamma_{yz}}{\partial x} - \dfrac{\partial \gamma_{xz}}{\partial y}\right) = 0 \Rightarrow -\dfrac{\partial}{\partial y}\left(\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right) = 0 \Rightarrow -\dfrac{\partial}{\partial y}\nabla^2\phi = 0$

Thus, ϕ has to satisfy: $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = \text{constant}$ at all points within the bar.



ϕ must fulfil B. C.: $\tau_{yz} m + \tau_{xz} l = 0$ where $l = \dfrac{dy}{ds}, \; m = -\dfrac{dx}{ds}$ on surface

$\Rightarrow \dfrac{\partial \phi}{\partial x}\dfrac{dx}{ds} + \dfrac{\partial \phi}{\partial y}\dfrac{dy}{ds} = 0$ or $\dfrac{\partial \phi}{ds} = 0 \Rightarrow \phi = \text{constant} = 0$

On the ends of the bar: $l = m = 0, \; n = 1 \Rightarrow \bar{X} = \tau_{zx}, \; \bar{Y} = \tau_{zy}, \; \bar{Z} = 0$

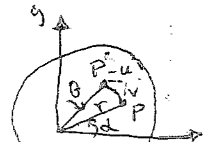
Thus, shear forces: $S_x = \iint \bar{X}\, dxdy = \iint \tau_{zx}\, dxdy = \iint \dfrac{\partial \phi}{\partial y} dxdy = \int dx \int \dfrac{\partial \phi}{\partial y} dy = 0$

$S_y = \iint \bar{Y}\, dxdy = \; - - - \; - - - - \; = -\int dy \int \dfrac{\partial \phi}{\partial x} dx = 0$

There is no resultant shear force, the forces represent a torque:

$T = \iint (\tau_{zy} x - \tau_{zx} y)\, dxdy = -\iint \dfrac{\partial \phi}{\partial x} x\, dxdy - \iint \dfrac{\partial \phi}{\partial y} y\, dxdy = 2\iint \phi\, dxdy$

Displacement of bar: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{\partial w}{\partial z} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = 0 \Rightarrow$ each cross-section rotates about a centre of rotation.



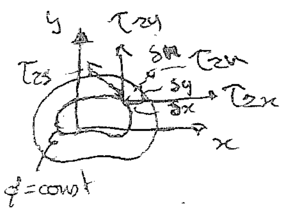
Point $P(r, \alpha)$ displaces to $P'(r, \alpha + \theta) \Rightarrow u = -r\theta \sin\alpha, \; v = r\theta\cos\alpha$

or $u = -\theta y, \; v = \theta x$. $\quad \gamma_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} = \dfrac{\tau_{zx}}{G}, \; \gamma_{zy} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} = \dfrac{\tau_{zy}}{G}$

$\dfrac{\partial w}{\partial x} = \dfrac{\tau_{zx}}{G} + \dfrac{d\theta}{dz} y, \quad \dfrac{\partial w}{\partial y} = \dfrac{\tau_{zy}}{G} - \dfrac{d\theta}{dz} x$

$0 = \dfrac{1}{G}\left(\dfrac{\partial \tau_{zx}}{\partial y} - \dfrac{\partial \tau_{zy}}{\partial x}\right) + 2\dfrac{d\theta}{dz} \Rightarrow \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = -2G\dfrac{d\theta}{dz} = \nabla^2\phi = \text{constant}$

Torsion constant J: $T = GJ\dfrac{d\theta}{dz}$ where $GJ = -\dfrac{4G}{\nabla^2\phi}\iint \phi\, dxdy$ (torsional rigidity)



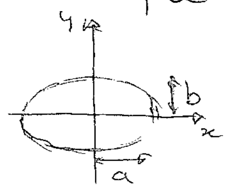
$\dfrac{\partial \phi}{\partial s} = \dfrac{\partial \phi}{\partial y}\dfrac{dy}{ds} + \dfrac{\partial \phi}{\partial x}\dfrac{dx}{ds} = 0 = \tau_{zx} l + \tau_{zy} m$

$\tau_{zn} = \tau_{zx} l + \tau_{zy} m$ ", $\quad \tau_{zs} = \tau_{zy} l - \tau_{zx} m$

Thus, resultant shear stress at any point is tangential to a ϕ-line

$\tau_{zs} = -\dfrac{\partial \phi}{\partial x} l - \dfrac{\partial \phi}{\partial y} m = -\dfrac{\partial \phi}{\partial x}\dfrac{dx}{dn} - \dfrac{\partial \phi}{\partial y}\dfrac{dy}{dn} = -\dfrac{\partial \phi}{\partial n}$

Example 3.1: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Choose stress function: $\phi = C\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$



then B. C. $\phi=0$ is satisfied and C can be computed:

$$2C\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = -2G\frac{d\theta}{dz} \quad \text{or} \quad C = -G\frac{d\theta}{dz}\frac{a^2b^2}{(a^2+b^2)} \Rightarrow \phi = -G\frac{d\theta}{dz}\frac{a^2b^2}{(a^2+b^2)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$$

Substitute in $T = 2\iint \phi\, dx\,dy$:

$$T = -2G\frac{d\theta}{dz}\frac{a^2b^2}{(a^2+b^2)}\left(\frac{1}{a^2}\underbrace{\iint x^2\,dx\,dy}_{I_{yy}=\pi a^3 b/4} + \frac{1}{b^2}\underbrace{\iint y^2\,dx\,dy}_{I_{xx}=\pi a b^3/4} - \underbrace{\iint dx\,dy}_{A=\pi ab}\right)$$

Thus:

$$T = G\frac{d\theta}{dz}\frac{\pi a^3b^3}{(a^2+b^2)} \Rightarrow J = \frac{\pi a^3 b^3}{(a^2+b^2)}$$

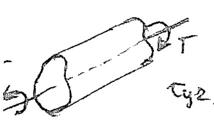
This gives $\tau_{zx} = -\frac{2Ty}{\pi a b^3}$, $\tau_{zy} = \frac{2Tx}{\pi a^3 b} \Rightarrow$

$$\frac{\partial w}{\partial x} = -\frac{2Ty}{\pi ab^3G} + \frac{T}{G}\frac{(a^2+b^2)}{\pi a^3b^3}y = \frac{T}{\pi a^3b^3G}(b^2-a^2)y; \quad \frac{\partial w}{\partial y} = \frac{2Tx}{\pi a^3bG} - \frac{T(a^2+b^2)}{G\pi a^3b^3}x = \frac{T}{\pi a^3b^3G}(b^2-a^2)x$$

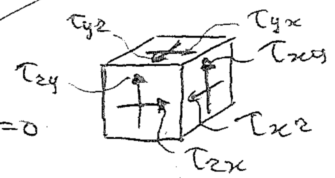
$$w = \frac{T(b^2-a^2)}{\pi a^3b^3G}yx + f_1(y) \quad \text{and} \quad w = \frac{T(b^2-a^2)}{\pi a^3b^3G}xy + f_2(x) \Rightarrow f_1(y) = f_2(x) = 0$$

Hence: $w = \frac{T(b^2-a^2)}{\pi a^3b^3G}xy$

Lecture Notes:



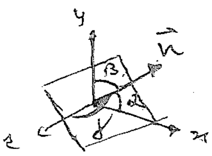
$$\frac{\partial \tau_{xz}}{\partial z}=0, \quad \frac{\partial \tau_{yz}}{\partial z}=0, \quad \frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}=0$$



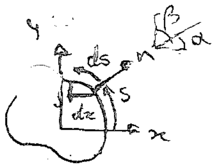
$\tau_{xy}=0$

Prandtl: $\frac{\partial\phi}{\partial x} = -\tau_{zy}$, $\frac{\partial\phi}{\partial y} = \tau_{zx}$, $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = \text{constant}$

$l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$



B.C: $\bar{X} = \bar{Y} = 0$



$l = \frac{dy}{ds}$

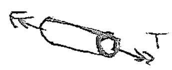
$m = \frac{-dx}{ds}$

$$\bar{Z} = \overset{0}{\sigma_z} n + \tau_{yz}m + \tau_{xz}l = 0$$

$$\frac{\partial\phi}{\partial x}\frac{dx}{ds}+\frac{\partial\phi}{\partial y}\frac{dy}{ds} = 0 \Rightarrow \frac{\partial\phi}{\partial s} = 0 \Rightarrow \phi = \text{constant}$$

On boundary $\phi = 0$.

Exercise 3-1: solid circular bar, radius a, loaded by torque T. Find ϕ such that $\phi=0$ and $\frac{\delta^2\phi}{\delta x^2}+\frac{\delta^2\phi}{\delta y^2} = \text{constant}$.



Try: $\phi = k(x^2+y^2-a^2)$. At border: $x^2+y^2=a^2 \Rightarrow \phi=0$ at boundary.

$\frac{\delta^2\phi}{\delta x^2}=2k$, $\frac{\delta^2\phi}{\delta y^2}=2k$. Thus, compatibility equations constant: $4k$

Exercise 3-3: Now calculate rate of twist and shear stress distributions.

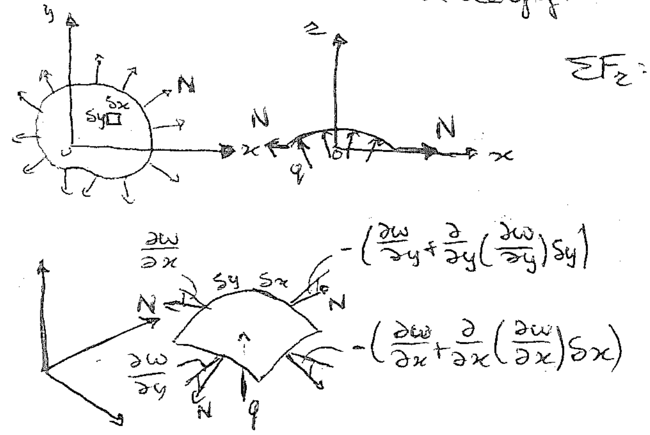
$$\frac{\delta^2\phi}{\delta x^2}+\frac{\delta^2\phi}{\delta y^2} = -2G\frac{d\theta}{dz} = 4k \Rightarrow k = -\tfrac{1}{2}G\frac{d\theta}{dz}. \quad T = 2\iint\phi\,dx\,dy = 2\iint k(x^2+y^2-a^2)\,dx\,dy. \quad A = \pi a^2$$

and $I_{xx} = I_{yy} = \frac{\pi a^4}{4}$. Then $T = G\frac{d\theta}{dz}\frac{\pi a^4}{2} \Rightarrow \frac{d\theta}{dz} = \frac{2T}{G\pi a^4} = \frac{T}{GI_p}$ = rate of twist.

$$\tau_{zy} = -\frac{\delta\phi}{\delta x} = -2kx = G\frac{d\theta}{dz} = \frac{Tx}{I_p}, \quad \tau_{zx} = \frac{\delta\phi}{\delta y} = 2ky = -G\frac{d\theta}{dz}y = -\frac{Ty}{I_p}$$

AE2-S22: Lecture 4: 3.3, 3.4, 9.6, 9.1.1-9.1.4

3.3| The membrane analogy:



$\Sigma F_z: -N\delta y\frac{\partial w}{\partial x}-N\delta y\left(-\frac{\partial w}{\partial x}-\frac{\partial^2 w}{\partial x^2}\delta x\right)-N\delta x\frac{\partial w}{\partial y}-$
$N\delta x\left(-\frac{\partial w}{\partial y}-\frac{\partial^2 w}{\partial y^2}\delta x\right)+q\,\delta x\,\delta y=0$

or $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\nabla^2 w=-\frac{q}{N}$ at all points within boundary and on boundary $w=0$.

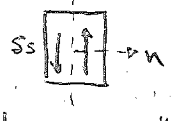
Thus, $w(x,y)=\phi(x,y)$ for constant q. $\Rightarrow$
$\frac{q}{N}=2G\frac{d\theta}{dz}$; $T=2\iint w\,dx\,dy$

3.4| Torsion of a narrow rectangular strip: same cross-sectional shape alon its length, so: $\frac{d^2\phi}{dx^2}=-2G\frac{d\theta}{dz}\rightarrow\phi=-G\frac{d\theta}{dz}x^2+Bx+C$. Substitute $\phi=0$ at $x=\pm t/2$: $\phi=-G\frac{d\theta}{dz}\left[x^2-\left(\frac{t}{2}\right)^2\right]$; $\tau_{zy}=2Gx\frac{d\theta}{dz}$, $\tau_{zx}=0$. and

$\tau_{zy,max}=\pm Gt\frac{d\theta}{dz}$; $J=\frac{st^3}{3}$ and $\tau_{zy,max}=\frac{3T}{st^3}$

$\frac{\partial w}{\partial x}=y\frac{d\theta}{dz}\Rightarrow w=xy\frac{d\theta}{dz}+C$ where $w=0$ at $x=y=0$, so $C=0$

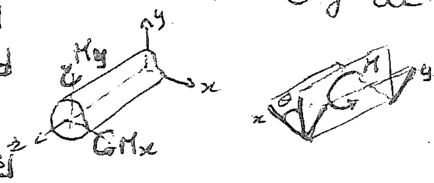
9.6|



$\tau_{zs}=2Gn\frac{d\theta}{dz}$, $\tau_{zn}=0$, $\tau_{zs,max}=\pm Gt\frac{d\theta}{dz}$, $J=\sum\frac{st^3}{3}$

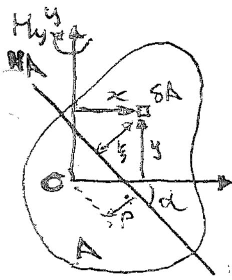
$T=GJ\frac{d\theta}{dz}\Rightarrow\tau_{zs}=\frac{2n}{J}T$, $\tau_{zs,max}=\pm\frac{tT}{J}$

9.1.1|
9.1.2|



$M_x=M\sin\theta$
$M_y=M\cos\theta$

9.1.3|



$\sigma_z=E\varepsilon_z$ on element of area δA at (x,y), distance ξ from neutral axis. If beam is bent to radius of curvature ρ about the neutral axis, then: $\varepsilon_z=\frac{\xi}{\rho}\Rightarrow\sigma_z=\frac{E\xi}{\rho}$

$\int_A\sigma_z\,dA=0\rightarrow\int_A\xi\,dA=0$. Thus, NA passes through C.

$\xi=x\sin\alpha+y\cos\alpha\Rightarrow\sigma_z=\frac{E}{\rho}(x\sin\alpha+y\cos\alpha)$

$M_x=\int_A\sigma_z y\,dA$, $M_y=\int_A\sigma_z x\,dA$, $I_{xx}=\int_A y^2dA$, $I_{yy}=\int_A x^2dA$, $I_{xy}=\int xy\,d$

$\Rightarrow M_x=\frac{E\sin\alpha}{\rho}I_{xy}+\frac{E\cos\alpha}{\rho}I_{xx}$, $M_y=\frac{E\sin\alpha}{\rho}I_{yy}+\frac{E\cos\alpha}{\rho}I_{xy}$

$\begin{Bmatrix}M_x\\M_y\end{Bmatrix}=\frac{E}{\rho}\begin{bmatrix}I_{xy}&I_{xx}\\I_{yy}&I_{xy}\end{bmatrix}\begin{Bmatrix}\sin\alpha\\\cos\alpha\end{Bmatrix}\Rightarrow\frac{E}{\rho}\begin{Bmatrix}\sin\alpha\\\cos\alpha\end{Bmatrix}=\begin{bmatrix}I_{xy}&I_{xx}\\I_{yy}&I_{xy}\end{bmatrix}^{-1}\begin{Bmatrix}M_x\\M_y\end{Bmatrix}\Rightarrow$

$\frac{E}{\rho}\begin{Bmatrix}\sin\alpha\\\cos\alpha\end{Bmatrix}=\frac{1}{I_{xx}I_{yy}-I_{xy}^2}\begin{bmatrix}-I_{xy}&I_{xx}\\I_{yy}&-I_{xy}\end{bmatrix}\begin{Bmatrix}M_x\\M_y\end{Bmatrix}$ so that: $\sigma_z=\left(\frac{M_yI_{xx}-M_xI_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\right)x+\left(\frac{M_xI_{yy}-M_y}{I_{xx}I_{yy}-I}\right.$

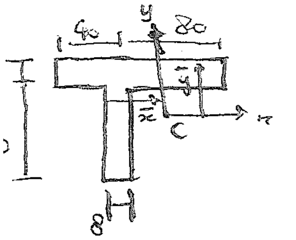
When beam cross-section is symmetric about either x- or y-axis; $I_{xy}=0$

$\Rightarrow\sigma_z=\frac{M_x}{I_{xx}}y+\frac{M_y}{I_{yy}}x$

9.1.4| Neutral axis always ~~passes~~ through centroid of a beam's cross-section and at all points on neutral axis is $\sigma = 0$: $0 = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) x_{NA} + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) y_{NA}$

$\frac{y_{NA}}{x_{NA}} = -\frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}}$, $\tan\alpha = -\frac{y_{NA}}{x_{NA}}$

Example 9.1: beam subjected to bending moment of 1500 N in vertical plane. Calculate max. stress due to bending and at which point it acts.



Position of centroid: $(120 \cdot 8 + 80 \cdot 8)\bar{y} = 120 \cdot 8 \cdot 4 + 80 \cdot 8 \cdot 48 \Rightarrow \bar{y} = 21.6$ mm

and $(120 \cdot 8 + 80 \cdot 8)\bar{x} = 80 \cdot 8 \cdot 4 + 120 \cdot 8 \cdot 24 \Rightarrow \bar{x} = 16$ mm

Next: $I_{xx} = \frac{120 \cdot 8^3}{12} + 120 \cdot 8 \cdot 17.6^2 + \frac{8 \cdot 80^3}{12} + 80 \cdot 8 \cdot 26.4^2 = 1.09 \cdot 10^6$ mm

$I_{yy} = \frac{8 \cdot 120^3}{12} + 120 \cdot 8 \cdot 8^2 + \frac{80 \cdot 8^3}{12} + 80 \cdot 8 \cdot 12^2 = 1.31 \cdot 10^6$ mm^4

$I_{xy} = 120 \cdot 8 \cdot 8 \cdot 17.6 + 80 \cdot 8 \cdot -12 \cdot -26.4 = 0.34 \cdot 10^6$ mm^4

$M = 1500$ N, $M_y = 0 \Rightarrow \sigma_z = 1.5y - 0.39x$. σ_x will be max. at $x = -8$ mm, and $y = -66.4$ mm. Thus $\sigma_{z,max} = -96$ N/mm^2

Lecture Notes:

If cross-section is made of two materials with E_1 and E_2: $dA^* = \frac{E}{E_1} dA$

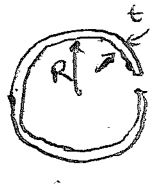
$I_{xx}^* = \int y^2 dA^*$, $I_{yy}^* = \int x^2 dA^*$, $I_{xy}^* = \int xy\, dA^* \Rightarrow M_x = \int \sigma_z y\, dA$, $M_y = \int \sigma_z x\, dA \Rightarrow$

$M_x = E_1 \int \frac{1}{\rho}(xy \sin\alpha + y^2 \cos\alpha)\frac{E}{E_1} dA = \frac{E_1}{\rho}(I_{xy}^* \sin\alpha + I_{xx}^* \cos\alpha)$

$M_y = E_1 \int \frac{1}{\rho}(x^2 \sin\alpha + xy \cos\alpha)\frac{E}{E_1} dA = \frac{E_1}{\rho}(I_{yy}^* \sin\alpha + I_{xy}^* \cos\alpha)$

$\sigma_z \quad \frac{E}{E_1}\left(\frac{M_y I_{xx}^* - M_x I_{xy}^*}{I_{xx}^* I_{yy}^* - I_{xy}^{*2}} x + \frac{M_x I_{yy}^* - M_y I_{xy}^*}{I_{xx}^* I_{yy}^* - I_{xy}^{*2}} y\right)$

Exercise 4-1: Torsion of a narrow strip: long tube with cut is loaded by torsion T. Calculate max. shear stress, rate of twist.

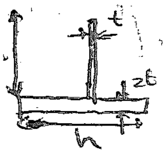


First calculate torsional constant J: $J = \frac{1}{3} s t^3 = \frac{1}{3} 2\pi R t^3$

max. shear stress is at outside of tube; where $x = t/2$. Then

$\tau_{max} = \frac{Tt}{J} = \frac{T}{\frac{2}{3}\pi R t^2} = \frac{3T}{2\pi R t^2}$. Rate of twist: $\frac{d\theta}{dz} = \frac{T}{GJ} = \frac{T}{G\frac{2}{3}\pi R t^3} = \frac{3T}{2G\pi R t^3}$

Exercise 4-2: Torsion of a narrow strip: thin walled open bar:

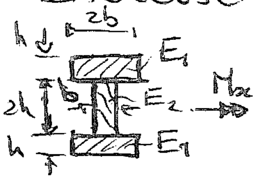


$J = \frac{1}{3} h t^3 + \frac{1}{3} h (2t)^3 = \frac{9}{3} h t^3 = 3ht^3$

$\tau_{max} = \frac{T 2t}{J} = \frac{2T}{3ht^2}$, $\frac{d\theta}{dz} = \frac{T}{GJ} = \frac{T}{G3ht^3}$

AE2-S22: Lecture 4: continued (2)

Exercise 4-3: Bending of bar consisting of two different materials.



Loaded by moment M_x about x-axis, use E_1 as reference; find $E_1 = 2E_2$. Calculate σ_{min} in E_1, and σ_{max} in E_2

1. Determine c.o.g., 2. Determine which moments of inertia are needed, 3. Calculate stress using engineering bending theory, 4. where are max. and min. stresses?

1). 2 symmetry lines, therefore c.o.g is exactly in the middle ($Q_{x_0} = \int y\,dx\,dy$) Ⓐ $Q_x = 0$

2) In case of symmetry, $I^*_{xy} = 0$. $\Rightarrow \sigma_z = \frac{E}{E_1}\left(\frac{M_y}{I^*_{yy}}x + \frac{M_x}{I^*_{xx}}y\right)$. Since only M_x is applied $\Rightarrow \sigma_z = \frac{E}{E_1}\left(\frac{M_x}{I^*_{xx}}y\right)$. Thus, only I^*_{xx} is needed: $I^*_{xx} = 29/3\, b h^3$

3) $\sigma_z = \frac{E}{E_1}\left(\frac{M_x}{29/3\, bh^3}y\right)$

4) For E_1: σ_{min} is at $y = h$: $\sigma_{z\,min} = \frac{3}{29}\frac{M_x}{bh^2}$

For E_2: σ_{max} is at $y = -h$: $\sigma_{z\,max} = \frac{3}{58}\frac{M_x}{bh^2}$

Ⓐ $Q_{x_0} = \int y\,dA^* = 2bh \cdot \frac{E_1}{E_1}\frac{7}{2}h + \frac{E_2}{E_1}\cdot b\cdot 2h \cdot 2h + \frac{E_1}{E_1}2bh\frac{h}{2} = 10\,bh^2$

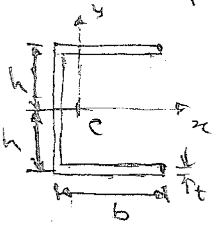
$Q_x = Q_{x_0} - y_0 A^*_{tot} = 10\,bh^2 - y_0\, 5bh = 0 \Rightarrow y_0 = 2h$

where $A^*_{tot} = \frac{E_1}{E_1}2bh + \frac{E_1}{E_1}2bh + \frac{E_2}{E_1}2bh = 5bh$

AE2-S22: Lecture 5: 9.1.8, 9.1.7

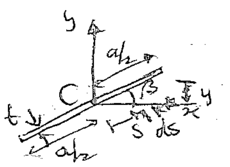
9.1.7 For thin-walled structures we can:

- assume stresses to be constant across the thickness
- neglect squares and higher powers of t.
- represent the section by the midline of its wall.



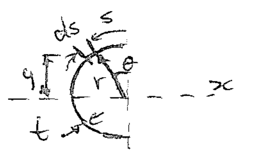
$I_{xx} = 2\cdot\left(\frac{1}{12}(b+t/2)t^3 + (b+t/2)th^2\right) + t\left(\frac{1}{12}(2(h-t/2))^3\right)$ which reduces to: $I_{xx} = 2bth^2 + t\frac{(2h)^3}{12}$

$I_{xy} = 0$



$I_{xx} = 2\int_0^{a/2} ty^2\,ds = 2\int_0^{a/2} t(s\sin\beta)^2\,ds \Rightarrow I_{xx} = \frac{a^3 t\sin^2\beta}{12}$, $I_{yy} = \frac{a^3 t\cos^2\beta}{12}$

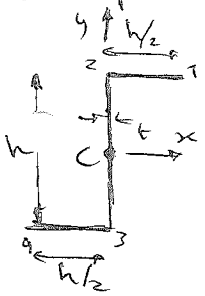
$I_{xy} = 2\int_0^{a/2} txy\,ds = 2\int_0^{a/2} t(s\cos\beta)(s\sin\beta)\,ds \Rightarrow I_{xy} = \frac{a^3 t\sin 2\beta}{24}$



$I_{xx} = \int_0^{\pi r} ty^2\,ds = \int_0^{\pi} t(r\cos\theta)^2 r\,d\theta = \frac{\pi r^3 t}{2}$

9.1.8 Assumptions are: uniform, homogeneous cross-section and plane sections remain plane after bending (only true if M_x and M_y are constant along the beam)

Example 9.3: Determine stress distribution in the thin-walled Z-section produced by a positive bending moment M_x



$\sigma_z = \frac{M_x(I_{yy}y - I_{xy}x)}{I_{xx}I_{yy} - I_{xy}^2}$

$I_{xx} = 2\frac{ht}{2}\left(\frac{h}{2}\right)^2 + \frac{th^3}{12} = \frac{h^3 t}{3}$ $\quad I_{yy} = 2\frac{t}{3}\left(\frac{h}{2}\right)^3 = \frac{h^3 t}{12}$

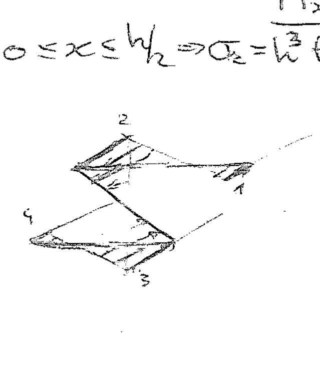
$I_{xy} = h/2\,t\cdot h/4\cdot h/2 + h/2\,t(-h/4)(-h/2) = h^3/8\,t$

$\Rightarrow \sigma_z = \frac{M_x}{h^3 t}(6.86y - 10.30x)$. On top flange: $y = h/2, 0 \le x \le h/2 \Rightarrow \sigma_z = \frac{M_x}{h^3 t}(3.43h - 10.3$

Hence: $\sigma_{z,1} = -\frac{1.72 M_x}{h^2 t}$ (C), $\sigma_{z,2} = \frac{3.43 M_x}{h^2 t}$ (T).

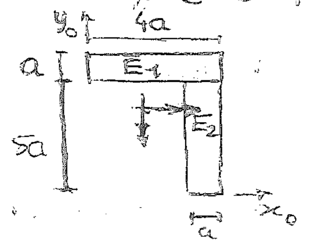
In the web: $h/2 \le y \le -h/2$, $x = 0$: $\sigma_z = \frac{M_x}{h^3 t}6.86y$

Hence. $\sigma_{z,2} = \frac{3.43 M_x}{h^2 t}$, $\sigma_{z,3} = -\frac{3.43 M_x}{h^2 t}$



Lower flange: antisymmetry.

Exercise 5-1: beam loaded by P and moments M_x and M_y, built up of two different materials: $E_2 = 2E_1$. Calculate σ_z.



Step 1: Calculate c.o.g:

First moment around x_0: $Q^*_{x_0} = 47a^3 = \left(\frac{E_1}{E_1} 4a^2 \cdot \frac{11}{2}a + \frac{E_2}{E_1} 5a^2 \frac{5}{2}a\right)$

using Steiner: $Q^*_{x_0} = y_{cc} \cdot A^*_{tot} = 0 \Rightarrow y_{cg} = \frac{47}{14}a$

First moment around y_0: $Q^*_{y_0} = 43a^3 \left(= \frac{E_1}{E_1} 4a^2 \cdot 2a + \frac{E_2}{E_1} 5a^2 \cdot \frac{7}{2}a\right)$ then

$Q^*_{y_0} = x_{CG} \cdot A^*_{tot} = 0 \Rightarrow x_{CG} = \frac{43}{14}a$

Step 2: Moments of inertia:

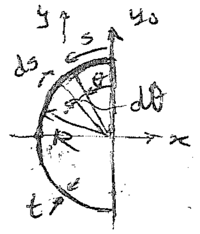
$$I^*_{xx} = \frac{E_1}{E_1} \frac{1}{12} 4a a^3 + A^*_1 \left(\frac{11}{2}a - \frac{47}{14}a\right)^2 + \frac{E_2}{E_1} \frac{1}{12} a (5a)^3 + A^*_2 \left(\frac{5}{2}a - \frac{47}{14}a\right)^2 = \frac{1969}{42}a^4$$

$$I^*_{yy} = \frac{E_1}{E_1} \frac{1}{12} a (4a)^3 + A^*_1 \left(2a - \frac{43}{14}a\right)^2 + \frac{E_2}{E_1} \frac{1}{12} 5a a^3 + A^*_2 \left(\frac{7}{2}a - \frac{43}{14}a\right)^2 = \frac{529}{42}a^4$$

$$I^*_{xy} = A^*_1 \left(2a - \frac{43}{14}a\right)\left(\frac{11}{2}a - \frac{47}{14}a\right) + A^*_2 \left(\frac{7}{2}a - \frac{43}{14}a\right)\left(\frac{5}{2}a - \frac{47}{14}a\right) = -\frac{90}{7}a^4$$

Step 3: $\sigma_z = \frac{E}{E_1}\left[\frac{P}{A^*} + \left(\frac{M_y I^*_{xx} - M_x I^*_{xy}}{I^*_{xx} I^*_{yy} - I^{*2}_{xy}}\right)x + \left(\frac{M_x I^*_{yy} - M_y I^*_{xy}}{I^*_{xx} I^*_{yy} - I^{*2}_{xy}}\right)y\right]$

Exercise 5-2: half-tube, calculate moments of inertia



~~Area~~: $A = \pi R t$.

First moment around y_0: $Q_{y_0} = \int x\,dx\,dy = \int x t\,ds = \int_0^\pi -R\sin\theta\, t R\,d\theta$

$= \ldots = -2tR^2$

$Q_{y_0} - x_{CG} A = 0 \Rightarrow x_{CG} = \frac{Q_{y_0}}{A} = -\frac{2R}{\pi}$

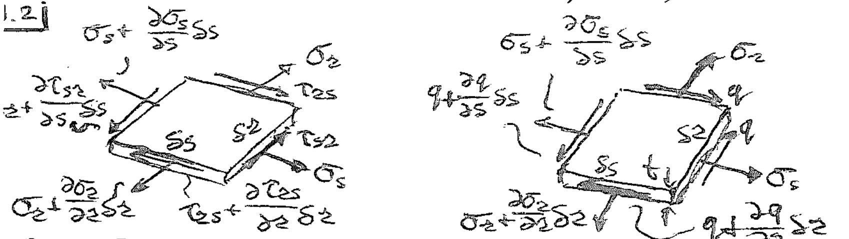
Second moment of inertia: $I_{xy} = 0$

$I_{xx} = \int y^2\,dx\,dy = \int y^2 t\,ds = \int_0^\pi (R\cos\theta)^2 t R\,d\theta = \ldots = \frac{1}{2}\pi t R^3$

$I_{yy} = \int x^2\,dx\,dy = \ldots = \left(\frac{\pi}{2} - \frac{4}{\pi}\right) t R^3$

Suppose tube loaded by moment M_x, then: $\sigma_z = \frac{M_x}{I_{xx}} y = \frac{M_x}{\frac{1}{2}\pi t R^3} y$ and maximum stress occurs at $y = R$: $\sigma_{z\,max} = \frac{2M_x}{\pi t R^2}$

1.2]

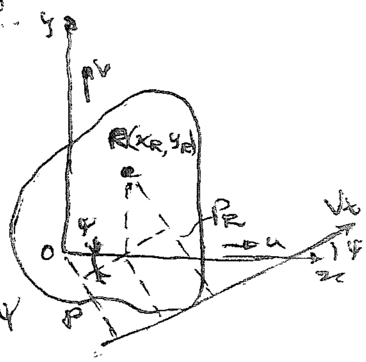


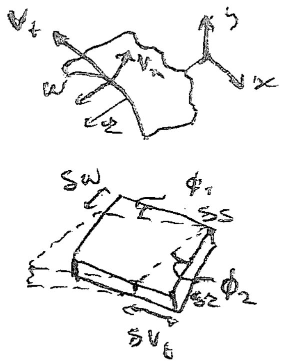
stresses on an element of a closed or open section beam.

$q = \tau t$

$$\left(\sigma_z + \frac{\partial \sigma_z}{\partial z}\delta z\right)t\,\delta s - \sigma_z t\,\delta s + \left(q + \frac{\partial q}{\partial s}\delta s\right)\delta z - q\,\delta z = 0 \Rightarrow \frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0 \text{ in } z \text{ direction}$$

Similarly, in s direction: $\frac{\partial q}{\partial z} + t\frac{\partial \sigma_s}{\partial s} = 0$





$$\varepsilon_z = \frac{\partial w}{\partial z}, \quad \varepsilon_s = \frac{\partial v_t}{\partial s} + \frac{v_n}{r}$$

$$\gamma = \phi_1 + \phi_2 = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z}$$

$$v_t = p\theta + u\cos\psi + v\sin\psi$$

$v_t = p_R\theta$ and $p_R = p - x_R\sin\psi + y_R\cos\psi$

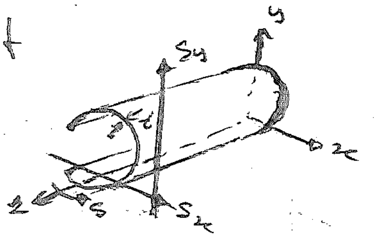
$$\Rightarrow v_t = p\theta - x_R\theta\sin\psi + y_R\theta\cos\psi$$

$$\Rightarrow \frac{\partial v_t}{\partial z} = p\frac{d\theta}{dz} - x_R\sin\psi\frac{d\theta}{dz} + y_R\cos\psi\frac{d\theta}{dz} = p\frac{d\theta}{dz} + \frac{du}{dz}\cos\psi + \frac{dv}{dz}\sin\psi$$

$$\Rightarrow x_R = -\frac{dv/dz}{d\theta/dz}, \quad y_R = \frac{du/dz}{d\theta/dz} \qquad R = \text{centre of twist}$$

3]

The open section beam supports shear loads S_x and S_y such that there is no twisting. The shear loads must pass through the shear centre. $\frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0$



$$\frac{\partial \sigma_z}{\partial z} = \frac{[(\partial M_y/\partial z)I_{xx} - (\partial M_x/\partial z)I_{xy}]}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{[(\partial M_x/\partial z)I_{yy} - (\partial M_y/\partial z)I_{xy}]}{I_{xx}I_{yy} - I_{xy}^2}y \qquad \frac{\partial M_x}{\partial z} = S_x, \frac{\partial M_y}{\partial z} = S_y$$

$$= \frac{(S_x I_{xx} - S_y I_{xy})}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{(S_y I_{yy} - S_x I_{xy})}{I_{xx}I_{yy} - I_{xy}^2}y$$

$$\Rightarrow \frac{\partial q}{\partial s} = -\frac{(S_x I_{xx} - S_y I_{xy})}{\cdots}tx + \frac{(S_y I_{yy} - S_x I_{xy})}{\cdots}ty$$

When integrating:

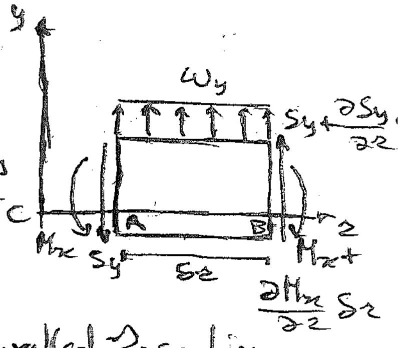
$$q_s = -\frac{S_x}{I_{yy}}\int_0^s tx\,ds - \frac{S_y}{I_{xx}}\int_0^s ty\,ds \quad \text{(for symmetric section)}$$

1.5]

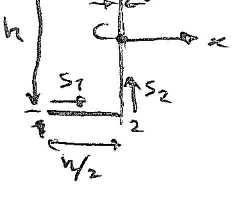
$$\left(S_y + \frac{\partial S_y}{\partial z}\delta z\right) + w_y\,\delta z - S_y = 0 \Rightarrow w_y = -\frac{\partial S_y}{\partial z}$$

Moment about A: $\left(M_x + \frac{\partial M_x}{\partial z}\delta z\right) - \left(S_y + \frac{\partial S_y}{\partial z}\delta z\right)\delta z - w_y\frac{(\delta z)^2}{2} - M_x = 0$

$$\Rightarrow S_y = \frac{\partial M_x}{\partial z}, \quad -w_y = \frac{\partial S_y}{\delta z} = \frac{\partial^2 M_x}{\partial z^2}. \text{ Likewise, } -w_x = \frac{\partial S_x}{\partial z} = \frac{\partial^2 M_y}{\partial z^2}$$



Example 9.4: determine shear flow distribution in thin-walled Z-section due to shear load S_y applied at shear centre.



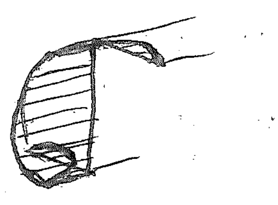
$$q_s = \frac{S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tx\,ds - \frac{S_y I_{yy}}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s ty\,ds$$

$$I_{xx} = \frac{h^3 t}{3}, \quad I_{yy} = \frac{h^3 t}{12}, \quad I_{xy} = \frac{h^3 t}{8} \Rightarrow q_s = \frac{S_y}{h^3}\int_0^s (10.32x - 6.84y)\,ds$$

On bottom flange: $y = -h/2$, $x = -h/2 + s_1$ where $0 \le s_1 \le h/2 \Rightarrow$

AE2-522: Lecture 6: continued

$q_{12} = \frac{S_y}{h^3}\int_0^{s_1}(10.32 s_1 - 1.74h)\,ds_1 = \frac{S_y}{h^3}(5.16 s_1^2 - 1.74 h s_1)$



At 1: $s_1 = 0$, $q_1 = 0$ At 2: $s_1 = h/2$, $q_2 = 0.42\frac{S_y}{h}$

In web; $y = -h/2 + s_2$, $0 \leq s_2 \leq h$ and $x = 0$. Thus:

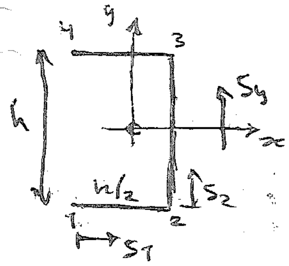
$q_{23} = \frac{S_y}{h^3}\int_0^{s_2}(3.42h - 6.84 s_2)\,ds_2 + q_2 = \frac{S_y}{h^3}(0.42h^2 + 3.42 h s_2 - 3.42 s_2^2)$

Lecture Notes:

$$\sigma_z = \frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} x + \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} y$$

$$S_y = \frac{\partial M_x}{\partial z}, \quad S_x = \frac{\partial M_y}{\partial z} \quad \therefore \quad \frac{\partial \sigma_z}{\partial z} = \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} x + \frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} y$$

Exercise 6-1: Determine shear flow distribution in thin-walled U-beam



1. I_{xx}, I_{yy}, I_{xy}:

Due to symmetry $I_{xy} = 0$, I_{yy} not needed

$I_{xx} = \frac{1}{3} t h^3$

2. Bending Theory formula: $q_s = -\frac{S_y}{I_{xx}}\int_0^s t y\, ds = -\frac{3S_y}{th^3}\int_0^s t y\, ds$

3. Calculate shear flow in part 1-2: $y = -h/2$:

$q_{12}(s_1) = -\frac{3S_y}{th^3}\int_0^s -t\frac{h}{2}\,ds_1 = \frac{3}{2}\frac{S_y}{h^2}s_1$. (linear)

At $s_1 = 0 \rightarrow q_{12} = 0$. At $s_1 = h/2 \rightarrow q_{12}(h/2) = \frac{3}{4}\frac{S_y}{h}$

In web 2-3: $y = -h/2 + s_2$: (parabolic)

$q_{23}(s_2) = \frac{-3S_y}{th^3}\int_0^s t(-\frac{h}{2} + s_2)\,ds_2 + q_{12}(h/2) = -\frac{3}{2}\frac{S_y}{h^3}(-hs_2 + s_2^2) + \frac{3}{4}\frac{S_y}{h}$

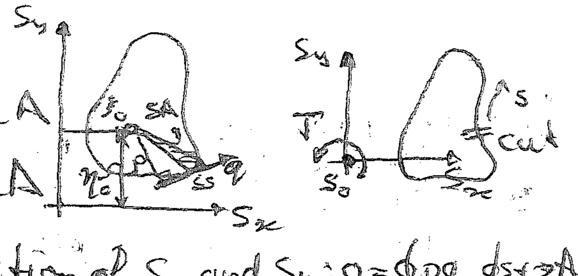
$q_{34}(0) = \frac{3}{4}\frac{S_y}{h}$. At $s_3 = h/2 \rightarrow q_{34}(h/2) = 0$

3.1 Shear centre is point in cross-section through which shear loads produce no twisting

3.4 For closed section with shear loads S_x and S_y and no body forces or hoop stresses.

$$\frac{\partial q}{\partial s}+t\frac{\partial \sigma_z}{\partial z}=0 \Rightarrow \int_0^s \frac{\partial q}{\partial s}ds = -\left(\frac{S_x I_{xx}-S_y I_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\right)\int_0^s tx\,ds-\left(\frac{S_y I_{yy}-S_x I_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\right)\int_0^s ty\,ds = q_s - q_{s,0}$$

where $q_{s,0}$ is shear flow in origin s. Thus $q_s = q_b + q_{s,0}$ where q_b is shear flow of open section. Thus, we cut closed beam.



$S_x\eta_0 - S_y\xi_0 = \oint pq\,ds$; $\delta A = \frac{1}{2}\delta s\cdot p \Rightarrow \oint p\,ds = 2A$

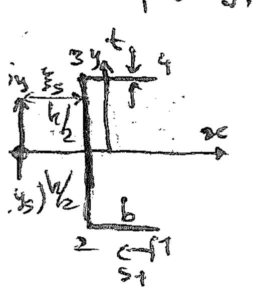
$S_x\eta_0 - S_y\xi_0 = \oint pq_b\,ds + q_{s,0}\oint p\,ds = \oint pq_b\,ds + q_{s,0}\,2A$

If moment centre coincides with lines of action of S_x and S_y: $0=\oint pq_{b}\,ds + 2Aq_{s,0}$

4.1 Rate of twist $\frac{d\theta}{dz}=\frac{1}{2A}\oint\frac{q_s}{Gt}ds$ At shear centre $\frac{d\theta}{dz}=0$

4.2

Example 9.5: Calculate position of shear centre of thin-walled channel



Shear centre lies at distance ξ_s from web. Apply shear load S_y, then shear flow distribution is given by: $q_s=-\left(\frac{S_x I_{xx}-S_y I_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\right)\int_0^s tx\,ds-\left(\frac{S_y I_{yy}-S_x I_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\right)\int_0^s$

Moment about any point in cross-section produced by these shear flows is equivalent to moment of applied shear load.

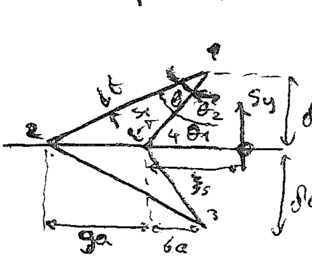
$I_{xy}=0,\ S_x=0$: $q_s=-\frac{S_y}{I_{xx}}\int_0^s ty\,ds$ where $I_{xx}=2bt\left(\frac{h}{2}\right)^2+\frac{th^3}{12}=\frac{h^3t}{12}\left(1+\frac{6b}{h}\right)$

$\Rightarrow q_s=\frac{-12S_y}{h^3(1+6b/h)}\int_0^s y\,ds$. On bottom flange: $y=-h/2$: $q_{12}=\frac{6S_y}{h^2(1+6b/h)}s_1$

Taking moments about mid-point of web: $S_y\xi_s = 2\int_0^b q_{12}\frac{h}{2}ds_1 = 2\int_0^b\frac{6S_y}{h^2(1+6b/h)}\frac{h}{2}s_1\,ds_1$

From which: $\xi_s=\frac{3b^2}{h(1+6b/h)}$

Example 9.6: Thin-walled closed section beam has singly symmetrical cross-section with flat walls and shear modulus G. Calculate distance of shear centre from point 4



Due to horizontal symmetry only apply S_y to determine $\xi_s \Rightarrow$

$I_{xy}=0,\ S_x=0 \Rightarrow q_s=-\frac{S_y}{I_{xx}}\int_0^s ty\,ds+q_{s,0}$ where

$I_{xx}=2\left[\int_0^{10a}t\left(\frac{8}{10}s_1\right)^2ds_1+\int_0^{17a}t\left(\frac{8}{17}s_2\right)^2ds_2\right]=1152a^3t$

$q_{b,41}=\frac{-S_y}{1152a^3t}\int_0^{s_1}t\left(\frac{8}{10}s_1\right)ds_1=\frac{-S_y}{1152a^3}\left(\frac{2}{5}s_1^2\right)$; $q_{b,12}=\frac{-S_y}{1152a^3}\left[\int_0^{s_2}(17a-s_2)\frac{8}{17}ds_2+40a^2\right]=$

$\frac{-S_y}{1152a^3}\left(-\frac{4}{17}s_2^2+8as_2+40a^2\right)\Rightarrow q_{s,0}=\frac{2S_y}{54a\cdot1152a^3}\left[\int_0^{10a}\frac{2}{5}s_1^2ds_1+\int_0^{17a}\left(-\frac{4}{17}s_2^2+8as_2+40a^2\right)ds_2\right]=$

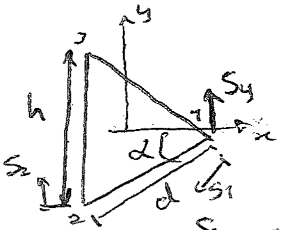
$\frac{S_y}{1152a^3}(58.7a^2)$. Take moment about point 2:

$S_y(\xi_s+9a)=2\int_0^{10a} q_{41}\,17a\sin\theta\, ds_1=\frac{S_y\,34a\sin\theta}{1152a^3}\int_0^{10a}\left(-\frac{2}{5}s_1^2+58.7a^2\right)ds_1$

$\sin\theta=\sin(\theta_1-\theta_2)=\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2$ where $\sin\theta_1=\frac{15}{17}$, $\cos_2=\frac{8}{10}$ etc

This gives: $\xi_s=-3.35a$

Exercise 7-1: shear flow in closed section beam



Thin walled, triangular cross-section loaded by shear forces

$I_{xy}=0$; $I_{xx}=\frac{1}{12}th^3+2\int_0^d(s_1\sin\alpha)^2\,t\,ds=\ldots=\frac{1}{12}th^2(h+2d)$

Make cut at point 1, so only q_{23} will appear in moment eq.

$q_{12}=-\frac{S_y}{I_{xx}}\int_0^{s_1}ty\,ds=-\frac{S_y}{I_{xx}}\int_0^{s_1}t(-s_1\sin\alpha)ds=\frac{3S_y}{hd(h+2d)}s_1^2$. At $s_1=d$: $q_{12}(d)=\frac{3S_y d}{h(h+2d)}$

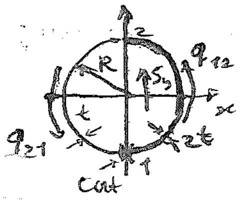
$q_{23}=q_{12}(d)-\frac{S_y}{I_{xx}}\int_0^{s_2}ty\,ds=q_{12}(d)-\frac{S_y}{I_{xx}}\int_0^{s_2}t(-h/2+s_2)ds=\ldots=\frac{3S_y d}{h(h+2d)}-\frac{6S_y}{h^2(h+2d)}(-hs_2+s$

After closing cut, take moment around point 1. Introduce circular shear flo

$M_1: 0=\int_0^h q_{23}\,d\cos\alpha\, ds_2-q_0 2A_{encl}\Rightarrow q_0=\frac{h+3d}{h(h+2d)}S_y\Rightarrow q_{12tot}=q_{12}-q_0$ and

$q_{23tot}=q_{23}-q_0$. To calculate pos. of shear centre we need rate of twist and set i to zero: $\frac{d\theta}{dz}=\frac{1}{2A_{encl}}\oint\frac{q_s}{Gt}ds$

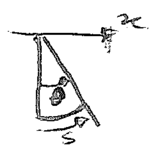
Exercise 7-2: Thin-walled tube. Shear centre must lie on x-axis due



symmetry, so only force in y-direction is needed. $I_{xy}=0$

$I_{xx}=\frac{1}{2}\pi tR^3+\frac{1}{2}\pi 2tR^3=\frac{3}{2}\pi tR^3$

Make cut at point 1, calculate q_{12}



$Q_x=\int y2t\,ds=\int_0^\theta -R\cos\theta\, 2t\, R\,d\theta=\ldots=-2tR^2\sin\theta$

$q_{12}(\theta)=-\frac{S_y}{I_{xx}}Q_x=\frac{4}{3}\frac{S_y\sin\theta}{\pi R}$ At $\theta=0$: $q_{12}(0)=0$, at $\theta=\pi$: $q_{12}(\pi)=0$

$q_{21}(\theta)=q_{12}(\pi)-\frac{S_y}{I_{xx}}Q_x=-\frac{2}{3}\frac{S_y\sin\theta}{\pi R}$ At $\theta=\pi$: $q_{21}(\pi)=0$ Ok!

Rate of twist: $\frac{d\theta}{dz}=\frac{1}{2GA_{encl}}\oint\frac{q\,ds}{t}=\ldots=F(q_t)\overset{*}{=}0$ where q_t is circular shear flo

Take moment about middle point: M_0: $S_y e=\int_0^\pi q_{12}(\theta)RR\,d\theta+\int_0 q_{21}(\theta)RR\,d\theta$

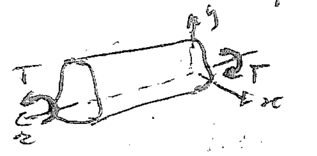
$\Rightarrow e=\frac{4}{3}\frac{R}{\pi}$

$* \quad \frac{d\theta}{dz}=\frac{1}{2G\pi R^2}\left(\int_0^\pi\frac{q_{12}(\theta)R\,d\theta}{2t}+\int_0^\pi\frac{q_{21}(\theta)R\,d\theta}{t}+\int_0^\pi\frac{q_0R\,d\theta}{2t}+\int_0^\pi\frac{q_0R\,d\theta}{t}\right)=0$

$\Rightarrow\left(q_0\frac{R}{2t}+q_0\frac{R}{t}\right)\int_0^\pi d\theta=0\Rightarrow q_0=0$

AE2-822 : Lecture 8: 9.4.1, 9.5, 9.7-9.7.3, 9.8, 9.9, 9.9.1

.5 Closed beam subjected to pure torque results in constant shear flow in the beam wall. $T = \oint pq\,ds = 2Aq$



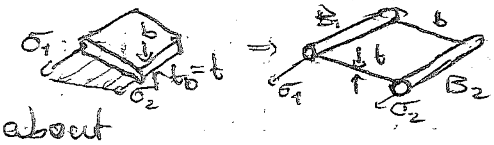
.7 Looking at a cross-section consisting of open and closed components:

7.1 Bending stays the same, no matter what the cross-section: $\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)x + \dots$

7.2 To determine shear stress distribution shear loads must be applied through shear centre of combined section

7.3 The closed component is dominant in torsion since its torsional stiffness is far greater than that of the attached open section

8 Idealization of structures: booms (constant stress along length) and skin (all shear stresses)



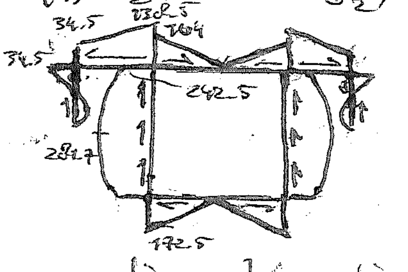
Taking moments about right-hand edge of each panel:

$\sigma_2 t_D \frac{b^2}{2} + \frac{1}{2}(\sigma_1 - \sigma_2) t_D b \frac{2}{3} b = \sigma_1 B_1 b \Rightarrow B_1 = \frac{t_D b}{6}\left(2 + \frac{\sigma_2}{\sigma_1}\right),\ B_2 = \frac{t_D b}{6}\left(2 + \frac{\sigma_1}{\sigma_2}\right)$

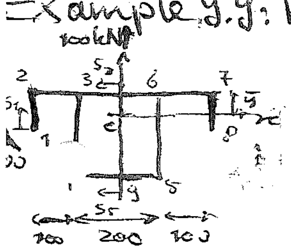
If $\sigma_1 = \sigma_2 \Rightarrow B_1 = B_2 = \frac{t_D b}{2}$ (axial load only), if $\sigma_1 = -\sigma_2 \Rightarrow B_1 = B_2 = \frac{t_D b}{6}$ (bending moment only)

.9 Position of neutral axis is derived from: $\int_A \sigma_z\,dA = 0$

9.1



Example 9.9: Determine shear flow distribution in beam section when subjected to shear load in vertical plane of symmetry, $t = 2\,mm$



Taking moments of area about upper surface:

$(4\cdot 100 \cdot 2 + 4\cdot 200\cdot 2)\bar{y} = 2\cdot 100\cdot 2\cdot 50 + 2\cdot 200\cdot 2\cdot 100 + 200\cdot 2\cdot 200 \Rightarrow \bar{y} = 75\,mm$

$I_{xx} = 2\left(\frac{2\cdot 100^3}{12} + 2\cdot 100\cdot 25^2\right) + 400\cdot 2\cdot 75^2 + 200\cdot 2\cdot 125^2 + 2\cdot\left(\frac{2\cdot 200^3}{12} + 2\cdot 200\cdot 25^3\right) = 14.5\cdot 10^6\,mm^4$

$I_{xy} = 0,\ S_x = 0 \Rightarrow q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,ds + q_{s,0}$ and in 123, 678: $q_{s,0} = 0$

$q_{12} = -\frac{100\cdot 10^3}{14.5\cdot 10^6}\int_0^{s_1} 2(-25+s_1)\,ds_1 = -69\cdot 10^{-4}(-50s_1 + s_1^2) \Rightarrow q_2 = -34.5\,N/mm$

$q_{23} = -69\cdot 10^{-4}\int_0^{s_2} 2\cdot 75\,ds_2 - 34.5 = -1.04s_2 - 34.5 \Rightarrow q_3 = -138.5\,N/mm$

Cut closed part, obtain q_0 for complete section and take moments. Due to symmetry 36 and 45: $q_s = q_{s,0} = 0$: $q_{03} = -69\cdot 10^{-4}\int_0^{s_3} 2\cdot 75\,ds_3 = -1.04s_3 \Rightarrow q_3 = -104\,N/mm$. For equil. at 3: $q_{34} = -138.5 - 104 = -242.5\,N/mm \Rightarrow q_{34} = -69\cdot 10^{-4}\int_0^{s_4} 2(75-s_4)\,ds_4 - 242.5 \Rightarrow$

$q_{34} = -104s_4 + 69\cdot 10^{-4}s_4^2 - 242.5 \Rightarrow q_4 = -172.5\,N/mm$. $q_{94} = -69\cdot 10^{-4}\int_0^{s_5} 2(-125)\,ds_2 \Rightarrow$

$q_{94} = 1.73\,S_5$