Vibrations Formula Overview

Free Vibrations

Basics

$$k = \frac{F}{\delta} \tag{1}$$

$$\omega_n = \sqrt{\frac{k}{m}} \tag{2}$$

$$f = \frac{\omega_n}{2\pi} \tag{3}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}} \tag{4}$$

Equation of motion

$$m\ddot{x} + kx = 0 \tag{5}$$

$$x(t) = A\sin\left(\omega_n t + \phi\right) \tag{6}$$

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} \tag{7}$$

$$\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) \tag{8}$$

Damped Motions

Definitions

$$c_{cr} = 2\sqrt{km} = 2m\omega_n \tag{9}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n} \tag{10}$$

Equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{11}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{12}$$

Underdamped motion

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{13}$$

$$x(t) = Ae^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right) \tag{14}$$

$$A = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2} \tag{15}$$

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right) \tag{16}$$

Overdamped motion

$$\omega_c = \omega_n\sqrt{\zeta^2 - 1} \tag{17}$$

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_c t} + a_2 e^{\omega_c t}\right) \tag{18}$$

$$a_1 = \frac{1}{2}x_0\left(1 - \frac{\zeta\omega_n}{\omega_c}\right) - \frac{v_0}{2\omega_c} \tag{19}$$

$$a_2 = \frac{1}{2}x_0\left(1 + \frac{\zeta\omega_n}{\omega_c}\right) + \frac{v_0}{2\omega_c} \tag{20}$$

Critically damped motion

$$x(t) = \left(a_1 + a_2 t\right)e^{-\omega_n t} \tag{21}$$

$$a_1 = x_0 \qquad \text{and} \qquad a_2 = v_0 + \omega_n x_0 \tag{22}$$

Harmonic Excitation

Free harmonic excitation

$$m\ddot{x} + kx = F_e(t) \tag{23}$$

$$r = \frac{\omega}{\omega_n} \tag{24}$$

$$F_e(t) = \hat{F}_e\cos\omega t \tag{25}$$

$$r \neq 1 \;\Rightarrow\; x_p(t) = \frac{\hat{F}_e}{m}\frac{1}{\left(\omega_n^2 - \omega^2\right)}\cos\omega t \tag{26}$$

$$r = 1 \;\Rightarrow\; x_p(t) = \frac{\hat{F}_e}{2m\omega_n}t\sin\omega_n t \tag{27}$$

Damped harmonic excitation

$$m\ddot{x} + c\dot{x} + kx = F_e(t) \tag{28}$$

$$F_e(x) = \hat{F}_e\cos\omega t \;\Rightarrow\; x_p(t) = X\cos\left(\omega t - \theta\right) \tag{29}$$

$$F_e(x) = \hat{F}_e\sin\omega t \;\Rightarrow\; x_p(t) = X\sin\left(\omega t - \theta\right) \tag{30}$$

$$X = \frac{\hat{F}_e}{k}\frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} \tag{31}$$

$$\theta = \arctan\left(\frac{2\zeta r}{1 - r^2}\right) \tag{32}$$

General Forced Vibration

The impulse function

$$m\ddot{x} + c\dot{x} + kx = F_e\delta(t-\tau) \tag{33}$$

$$x(t) = \hat{F}_e h(t-\tau) \tag{34}$$

$$h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{35}$$

The step function

$$m\ddot{x} + c\dot{x} + kx = \hat{F}_e u(t-\tau) \tag{36}$$

$$x(t) = \frac{\hat{F}_e}{k}\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\cos(\omega_d t - \theta)\right) \tag{37}$$

$$\theta = \arctan\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{38}$$

Fourier series forcing function

$$F_e(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(n\frac{2\pi}{T}t\right) + b_n \sin\left(n\frac{2\pi}{T}t\right)\right) \tag{39}$$

$$a_0 = \frac{2}{T}\int_0^T F_e(t)dt \tag{40}$$

$$a_n = \frac{2}{T}\int_0^T F_e(t)\cos\left(n\frac{2\pi}{T}t\right)dt \tag{41}$$

$$b_n = \frac{2}{T}\int_0^T F_e(T)\sin\left(n\frac{2\pi}{T}t\right)dt \tag{42}$$

Fourier series solution

$$x_p(t) = x_{a_0}(t) + \sum_{n=1}^{\infty}\left(x_{a_n}(t) + x_{b_n}(t)\right) \tag{43}$$

$$x_{a_0} = \frac{a_0}{2k} \tag{44}$$

$$x_{a_n} = \frac{a_n}{m} X \cos(n\omega_T t - \theta_n) \tag{45}$$

$$x_{b_n} = \frac{b_n}{m} X \sin(n\omega_T t - \theta_n) \tag{46}$$

$$X = \frac{1}{\sqrt{\left(\omega_n^2 - (n\omega_T)^2\right)^2 + (2\zeta n\omega_n\omega_T)^2}} \tag{47}$$

$$\theta_n = \arctan\left(\frac{2\zeta n\omega_n\omega_T}{\omega_n^2 - (n\omega_T)^2}\right) \tag{48}$$

Multiple-Degree-of-Freedom Systems

Equation of motion

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0} \tag{49}$$

Solution method 1

$$\mathbf{x}(t) = \mathbf{u}e^{i\omega t} \tag{50}$$

$$\left(K - \omega^2 M\right)\mathbf{u}e^{i\omega t} = \mathbf{0} \tag{51}$$

$$\mathbf{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\,\mathbf{u_1} + A_2 \sin(\omega_2 t + \phi_2)\,\mathbf{u_2} \tag{52}$$

Solution method 2

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = M^{-1/2}\mathbf{v}e^{i\omega t} \tag{53}$$

$$M^{-1/2}KM^{-1/2}\mathbf{v} = \widetilde{K}\mathbf{v} = \omega^2\mathbf{v} \tag{54}$$

$$\mathbf{u_1} = M^{-1/2}\mathbf{v_1} \quad \text{and} \quad \mathbf{u_2} = M^{-1/2}\mathbf{v_2} \tag{55}$$

$$\mathbf{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\,\mathbf{u_1} + A_2 \sin(\omega_2 t + \phi_2)\,\mathbf{u_2} \tag{56}$$

Modal analysis definitions

$$P = \begin{bmatrix}\mathbf{v_1} & \mathbf{v_2}\end{bmatrix} \tag{57}$$

$$S = M^{-1/2}P \quad \Leftrightarrow \quad S^{-1} = P^T M^{1/2} \tag{58}$$

$$\Lambda = P^T\widetilde{K}P = \begin{bmatrix}\omega_1^2 & 0 \\ 0 & \omega_2^2\end{bmatrix} \tag{59}$$

Modal analysis solution

$$\ddot{\mathbf{r}}(t) + \Lambda\mathbf{r}(t) = \mathbf{0} \tag{60}$$

$$\mathbf{r_0} = S^{-1}\mathbf{x_0} \quad \text{and} \quad \dot{\mathbf{r}}_\mathbf{0} = S^{-1}\dot{\mathbf{x}}_\mathbf{0} \tag{61}$$

$$\mathbf{x}(t) = S\mathbf{r}(t) \tag{62}$$