**Problems and Solutions Section 1.1 (1.1 through 1.19)**

1.1   The spring of Figure 1.2 is successively loaded with mass and the corresponding (static) displacement is recorded below. Plot the data and calculate the spring's stiffness. Note that the data contain some error. Also calculate the standard deviation.

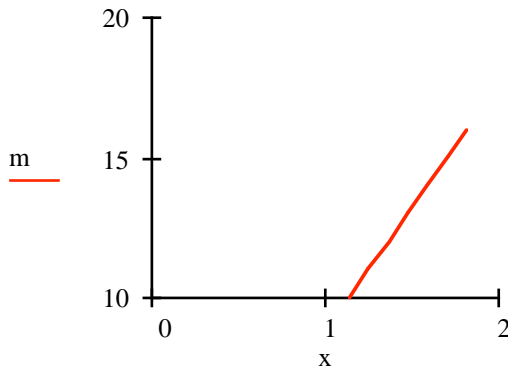| m(kg)	10	11	12	13	14	15	16
x(m)	1.14	1.25	1.37	1.48	1.59	1.71	1.82

Solution:

Free-body diagram:
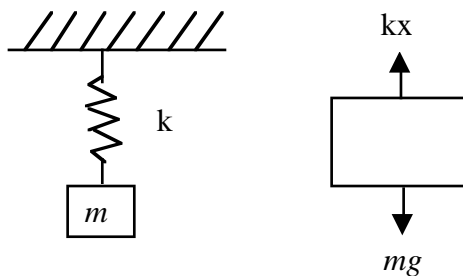
From the free-body diagram and static equilibrium:

$$kx = mg \quad (g = 9.81 m/s^2)$$
$$k = mg/x$$

$$\mu = \frac{\Sigma k_i}{n} = 86.164$$

Plot of mass in kg versus displacement in m

The sample standard deviation in computed stiffness is:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(k_i - \mu)^2}{n-1}} = \mathbf{0.164}$$

Computation of slope from mg/x

m(kg)	x(m)	k(N/m)
10	1.14	86.05
11	1.25	86.33
12	1.37	85.93
13	1.48	86.17
14	1.59	86.38
15	1.71	86.05
16	1.82	86.24

1.2   Derive the solution of $m\ddot{x} + kx = 0$ and plot the result for at least two periods for the case with $\omega_n = 2$ rad/s, $x_0 = 1$ mm, and $v_0 = \sqrt{5}$ mm/s.

Solution:

Given:

$$m\ddot{x} + kx = 0 \tag{1}$$

Assume: $x(t) = ae^{rt}$. Then: $\dot{x} = are^{rt}$ and $\ddot{x} = ar^2e^{rt}$. Substitute into equation (1) to get:

$$mar^2e^{rt} + kae^{rt} = 0$$

$$mr^2 + k = 0$$

$$r = \pm\sqrt{\frac{k}{m}}\,i$$

Thus there are two solutions:

$$x_1 = c_1 e^{\left(\sqrt{\frac{k}{m}}i\right)t}, \quad \text{and} \quad x_2 = c_2 e^{\left(-\sqrt{\frac{k}{m}}i\right)t}$$

$$\text{where } \omega_n = \sqrt{\frac{k}{m}} = 2 \text{ rad/s}$$

The sum of x_1 and x_2 is also a solution so that the total solution is:

$$x = x_1 + x_2 = c_1 e^{2it} + c_2 e^{-2it}$$

Substitute initial conditions: $x_0 = 1$ mm, $v_0 = \sqrt{5}$ mm/s

$$x(0) = c_1 + c_2 = x_0 = 1 \Rightarrow \underline{c_2 = 1 - c_1}, \text{ and } v(0) = \dot{x}(0) = 2ic_1 - 2ic_2 = v_0 = \sqrt{5} \text{ mm/s}$$

$$\Rightarrow \underline{-2c_1 + 2c_2 = \sqrt{5}\,i}. \quad \text{Combining the two underlined expressions (2 eqs in 2 unkowns):}$$

$$-2c_1 + 2 - 2c_1 = \sqrt{5}\,i \Rightarrow \underline{\underline{c_1 = \frac{1}{2} - \frac{\sqrt{5}}{4}i}}, \text{ and } \underline{\underline{c_2 = \frac{1}{2} + \frac{\sqrt{5}}{4}i}}$$

Therefore the solution is:

$$x = \left(\frac{1}{2} - \frac{\sqrt{5}}{4}i\right)e^{2it} + \left(\frac{1}{2} + \frac{\sqrt{5}}{4}i\right)e^{-2it}$$
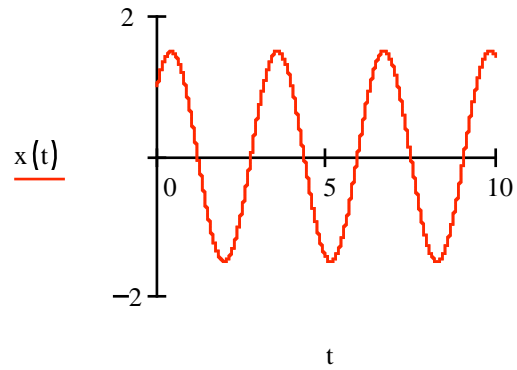
Using the Euler formula to evaluate the exponential terms yields:

$$x = \left(\frac{1}{2} - \frac{\sqrt{5}}{4}i\right)(\cos 2t + i\sin 2t) + \left(\frac{1}{2} + \frac{\sqrt{5}}{4}i\right)(\cos 2t - i\sin 2t)$$

$$\Rightarrow x(t) = \cos 2t + \frac{\sqrt{5}}{2}\sin 2t = \sqrt{\frac{3}{2}}\sin(2t + 0.7297)$$

Using Mathcad the plot is:

$$x(t) := \cos(2 \cdot t) + \frac{\sqrt{5}}{2} \cdot \sin(2 \cdot t)$$

1.3   Solve $m\ddot{x} + kx = 0$ for $k = 4$ N/m, $m = 1$ kg, $x_0 = 1$ mm, and $v_0 = 0$. Plot the solution.

Solution:

This is identical to problem 2, except $v_0 = 0$. $\left(\omega_n = \sqrt{\dfrac{k}{m}} = 2 \text{ rad/s} \right)$. Calculating the

initial conditions:

$$x(0) = c_1 + c_2 = x_0 = 1 \Rightarrow c_2 = 1 - c_1$$

$$v(0) = \dot{x}(0) = 2ic_1 - 2ic_2 = v_0 = 0 \Rightarrow c_2 = c_1$$
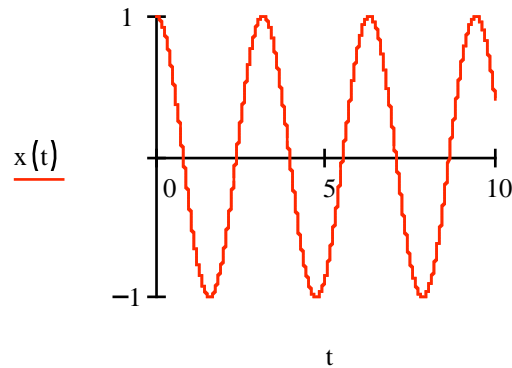
$$c_2 = c_1 = 0.5$$

$$x(t) = \frac{1}{2}e^{2it} + \frac{1}{2}e^{-2it} = \frac{1}{2}(\cos 2t + i\sin 2t) + \frac{1}{2}(\cos 2t - i\sin 2t)$$

$$\underline{x(t) = \cos(2t)}$$

The following plot is from Mathcad:

$$x(t) := \cos(2 \cdot t)$$

Alternately students may use equation (1.10) directly to get

$$x(t) = \frac{\sqrt{2^2(1)^2 + 0^2}}{2}\sin(2t + \tan^{-1}[\frac{2 \cdot 1}{0}])$$

$$= 1\sin(2t + \frac{\pi}{2}) = \cos 2t$$

1.4 The amplitude of vibration of an undamped system is measured to be 1 mm. The phase shift from $t = 0$ is measured to be 2 rad and the frequency is found to be 5 rad/s. Calculate the initial conditions that caused this vibration to occur. Assume the response is of the form $x(t) = A\sin(\omega_n t + \phi)$.

Solution:

Given: $A = 1\,\text{mm}$, $\phi = 2\,\text{rad}$, $\omega = 5\,\text{rad/s}$. For an *undamped* system:
$$x(t) = A\sin(\omega_n t + \phi) = 1\sin(5t + 2) \quad \text{and}$$
$$v(t) = \dot{x}(t) = A\omega_n \cos(\omega_n t + \phi) = 5\cos(5t + 2)$$
Setting $t = 0$ in these expressions yields:
$$x(0) = 1\sin(2) = \underline{0.9093\ \text{mm}}$$
$$v(0) = 5\cos(2) = \underline{-\ 2.081\ \text{mm/s}}$$

1.5 Find the equation of motion for the hanging spring-mass system of Figure P1.5, and compute the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency.

Figure P1.5

Solution:
The free-body diagram of problem system in (a) for the static case and in (b) for the dynamic case, where x is now measured from the static equilibrium position.

(a)   (b)
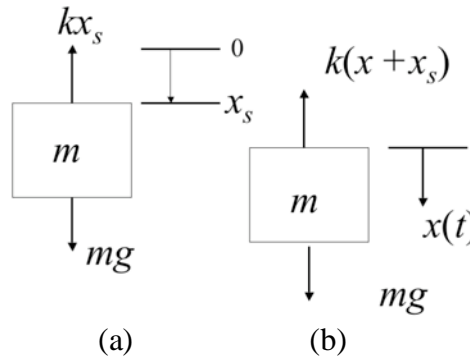
From a force balance in the static case (a): $mg = kx_s$, where x_s is the static deflection of the spring. Next let the spring experience a dynamic deflection $x(t)$ governed by summing the forces in (b) to get

$$m\ddot{x}(t) = mg - k(x(t) + x_s) \Rightarrow m\ddot{x}(t) + kx(t) = mg - kx_s$$

$$\Rightarrow \underline{m\ddot{x}(t) + kx(t) = 0} \Rightarrow \underline{\omega_n} = \sqrt{\frac{k}{m}}$$

since $mg = kx_s$ from static equilibrium.

1.6 Find the equation of motion for the system of Figure P1.6, and find the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency. Assume the block slides without friction.
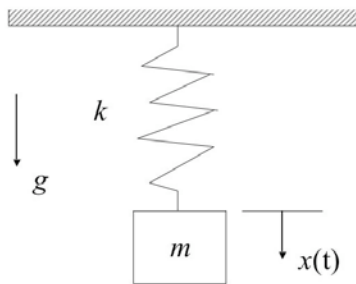

Figure P1.6

Solution:
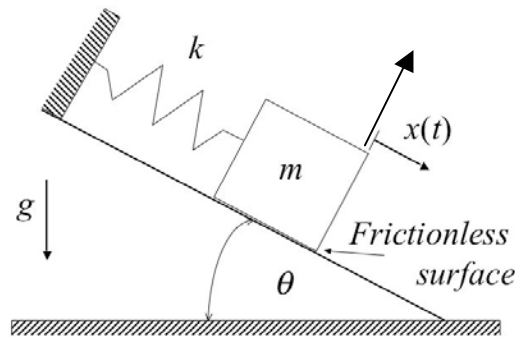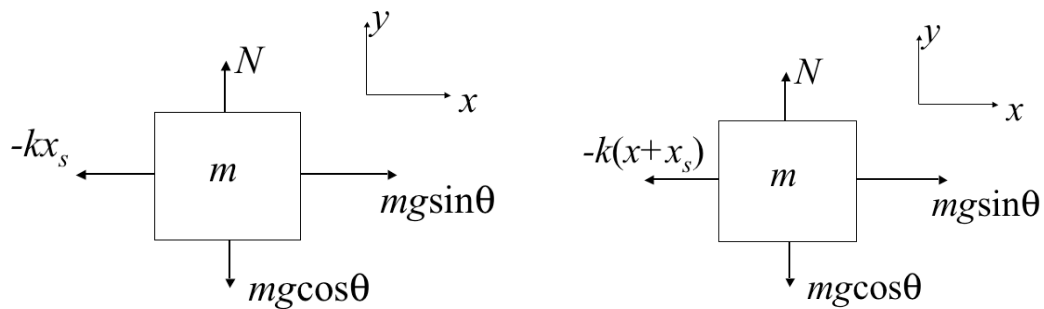Choosing a coordinate system along the plane with positive down the plane, the free-body diagram of the system for the static case is given and (a) and for the dynamic case in (b):

In the figures, N is the normal force and the components of gravity are determined by the angle θ as indicated. From the static equilibrium: $-kx_s + mg\sin\theta = 0$. Summing forces in (b) yields:

$$\sum F_i = m\ddot{x}(t) \Rightarrow m\ddot{x}(t) = -k(x + x_s) + mg\sin\theta$$

$$\Rightarrow m\ddot{x}(t) + kx = -kx_s + mg\sin\theta = 0$$

$$\Rightarrow \underline{m\ddot{x}(t) + kx = 0}$$

$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k}{m}}} \text{ rad/s}$$

1.7 An undamped system vibrates with a frequency of 10 Hz and amplitude 1 mm. Calculate the maximum amplitude of the system's velocity and acceleration.

Solution:

Given: First convert Hertz to rad/s: $\omega_n = 2\pi f_n = 2\pi(10) = 20\pi$ rad/s. We also have that $A = 1$ mm.

For an undamped system:

$$x(t) = A\sin(\omega_n t + \phi)$$

and differentiating yields the velocity: $v(t) = A\omega_n \cos(\omega_n t + \phi)$. Realizing that both the sin and cos functions have maximum values of 1 yields:

$$v_{max} = A\omega_n = 1(20\pi) = \textbf{62.8 mm/s}$$

Likewise for the acceleration: $a(t) = -A\omega_n^2 \sin(\omega_n t + \phi)$

$$a_{max} = A\omega_n^2 = 1(20\pi)^2 = \textbf{3948 mm/s}^2$$

1.8   Show by calculation that $A \sin(\omega_n t + \phi)$ can be represented as $B\sin\omega_n t + C\cos\omega_n t$ and calculate C and B in terms of A and ϕ.

Solution:

This trig identity is useful: $\sin(a + b) = \sin a \cos b + \cos a \sin b$

Given: $A\sin(\omega_n t + \phi) = A\sin(\omega_n t)\cos(\phi) + A\cos(\omega_n t)\sin(\phi)$

$$= B\sin\omega_n t + C\cos\omega_n t$$
$$\text{where } B = A\cos\phi \quad and \quad C = A\sin\phi$$

1.9   Using the solution of equation (1.2) in the form $x(t) = B\sin\omega_n t + C\cos\omega_n t$ calculate the values of B and C in terms of the initial conditions x_0 and v_0.

Solution:
Using the solution of equation (1.2) in the form

$$x(t) = B\sin\omega_n t + C\cos\omega_n t$$

and differentiate to get:
$$\dot{x}(t) = \omega_n B\cos(\omega_n t) - \omega_n C\sin(\omega_n t)$$

Now substitute the initial conditions into these expressions for the position and velocity to get:
$$x_0 = x(0) = B\sin(0) + C\cos(0) = C$$
$$v_0 = \dot{x}(0) = \omega_n B\cos(0) - \omega_n C\sin(0)$$
$$= \omega_n B(1) - \omega_n C(0) = \omega_n B$$

Solving for B and C yields:

$$B = \frac{v_0}{\omega_n}, \text{ and } C = x_0$$

Thus $x(t) = \dfrac{v_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t$

1.10 Using Figure 1.6, verify that equation (1.10) satisfies the initial velocity condition.

Solution: Following the lead given in Example 1.1.2, write down the general expression of the velocity by differentiating equation (1.10):

$$x(t) = A\sin(\omega_n t + \phi) \Rightarrow \dot{x}(t) = A\omega_n \cos(\omega_n t + \phi)$$
$$\Rightarrow v(0) = A\omega_n \cos(\omega_n 0 + \phi) = A\omega_n \cos(\phi)$$

From the figure:

Figure 1.6

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}, \quad \cos\phi = \frac{v_0/\omega_n}{\sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}}$$

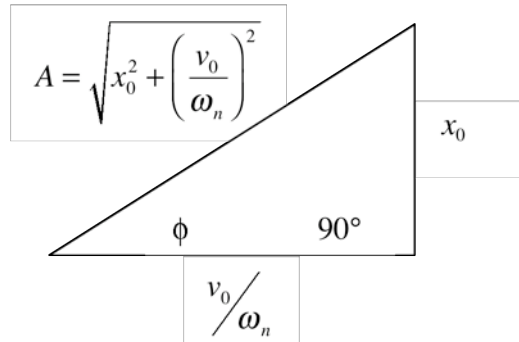Substitution of these values into the expression for $v(0)$ yields

$$v(0) = A\omega_n \cos\phi = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}\,(\omega_n)\frac{\dfrac{v_0}{\omega_n}}{\sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}} = v_0$$

verifying the agreement between the figure and the initial velocity condition.

1.11 (a)A 0.5 kg mass is attached to a linear spring of stiffness 0.1 N/m. Determine the natural frequency of the system in hertz. b) Repeat this calculation for a mass of 50 kg and a stiffness of 10 N/m. Compare your result to that of part a.

Solution: From the definition of frequency and equation (1.12)

$$(a) \qquad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{.5}{.1}} = 0.447 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{2.236}{2\pi} = \underline{0.071 \text{ Hz}}$$

$$(b) \qquad \omega_n = \sqrt{\frac{50}{10}} = 0.447 \text{rad/s}, f_n = \frac{\omega_n}{2\pi} = \underline{0.071 \text{ Hz}}$$

Part (b) is the same as part (a) thus very different systems can have same natural frequencies.

1.12 Derive the solution of the single degree of freedom system of Figure 1.4 by writing Newton's law, *ma = -kx*, in differential form using *adx = vdv* and integrating twice.

Solution: Substitute $a = vdv/dx$ into the equation of motion *ma = -kx*, to get *mvdv = -kxdx*. Integrating yields:

$$\frac{v^2}{2} = -\omega_n^2 \frac{x^2}{2} + c^2, \quad \text{where } c \text{ is a constant}$$

$$\text{or} \quad v^2 = -\omega_n^2 x^2 + c^2 \Rightarrow$$

$$v = \frac{dx}{dt} = \sqrt{-\omega_n^2 x^2 + c^2} \Rightarrow$$

$$dt = \frac{dx}{\sqrt{-\omega_n^2 x^2 + c^2}}, \quad \text{write } u = \omega_n x \text{ to get:}$$

$$t - 0 = \frac{1}{\omega_n} \int \frac{du}{\sqrt{c^2 - u^2}} = \frac{1}{\omega_n} \sin^{-1}\left(\frac{u}{c}\right) + c_2$$

Here c_2 is a second constant of integration that is convenient to write as $c_2 = -\phi/\omega_n$. Rearranging yields

$$\omega_n t + \phi = \sin^{-1}\left(\frac{\omega_n x}{c}\right) \Rightarrow$$

$$\frac{\omega_n x}{c} = \sin(\omega_n t + \phi) \Rightarrow$$

$$\underline{x(t) = A\sin(\omega_n t + \phi), \quad A = \frac{c}{\omega_n}}$$

in agreement with equation (1.19).

1.13   Determine the natural frequency of the two systems illustrated.

(a)   (b)
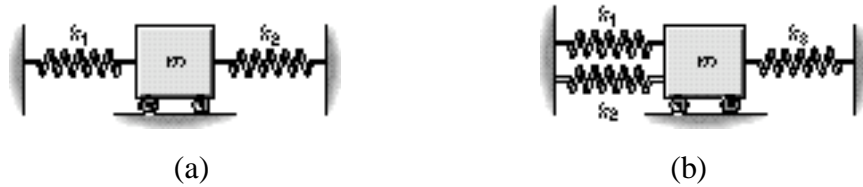
Figure P1.13

Solution:

(a) Summing forces from the free-body diagram in the x direction yields:

$$m\ddot{x} = -k_1 x - k_2 x \Rightarrow$$

$$m\ddot{x} + k_1 x + k_2 x = 0$$

$$m\ddot{x} + x(k_1 + k_2) = 0, \text{ dividing by } m \text{ yields:}$$

$$\ddot{x} + \left(\frac{k_1 + k_2}{m}\right)x = 0$$

$\leftarrow$ $+x$

$-k_1 x$ $-k_2 x$

Free-body diagram for part a

Examining the coefficient of x yields:

$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}}$$
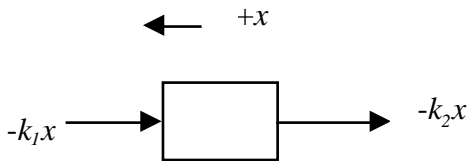
(b) Summing forces from the free-body diagram in the x direction yields:

$\leftarrow$ $+x$

$-k_1 x$

$-k_3 x$

$-k_2 x$
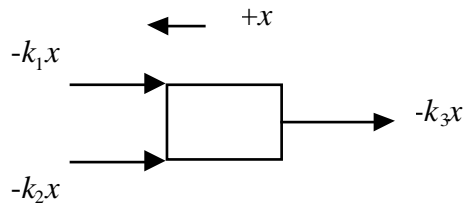
Free-body diagram for part b

$$m\ddot{x} = -k_1 x - k_2 x - k_3 x, \Rightarrow$$

$$m\ddot{x} + k_1 x + k_2 x + k_3 x = 0 \Rightarrow$$

$$m\ddot{x} + (k_1 + k_2 + k_3)x = 0 \Rightarrow \ddot{x} + \frac{(k_1 + k_2 + k_3)}{m}x = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{k_1 + k_2 + k_3}{m}}$$

1.14* Plot the solution given by equation (1.10) for the case $k = 1000$ N/m and $m = 10$ kg for two complete periods for each of the following sets of initial conditions: a) $x_0 = 0$ m, $v_0 = 1$ m/s, b) $x_0 = 0.01$ m, $v_0 = 0$ m/s, and c) $x_0 = 0.01$ m, $v_0 = 1$ m/s.

Solution: Here we use Mathcad:
a) all units in m, kg, s

$m := 10$ $k := 1000$

$$\omega_n := \sqrt{\frac{k}{m}}$$

$x0 := 0.0$

$v0 := 1$

$$T := \frac{2 \cdot \pi}{\omega n}$$

$$fn := \frac{\omega n}{2 \cdot \pi}$$

$$\phi := \operatorname{atan}\left(\frac{\omega n \cdot x0}{v0}\right)$$

$$x(t) := A \cdot \sin\left(\omega n \cdot t + \phi\right)$$

parts b and c are plotted in the above by simply changing the initial conditions as appropriate

$$A := \frac{1}{\omega n} \cdot \sqrt{x0^2 \cdot \omega n^2 + v0^2}$$

$x(t)$

$xb(t)$

$xc(t)$
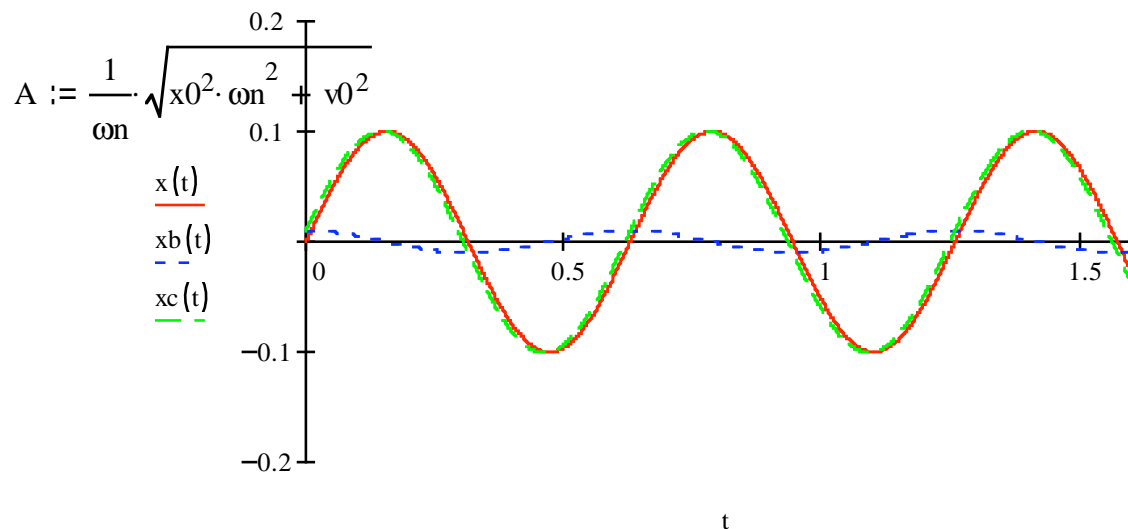
1.15* Make a three dimensional surface plot of the amplitude *A* of an undamped oscillator given by equation (1.9) versus x_0 and v_0 for the range of initial conditions given by $-0.1 \le x_0 \le 0.1$ m and $-1 \le v_0 \le 1$ m/s, for a system with natural frequency of 10 rad/s.

Solution: Working in Mathcad the solution is generated as follows:

$\omega n := 10$

$N := 25$ $\qquad i := 0 .. N$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad j := 0 .. N$

$v0_j := -1 + \left(\dfrac{2}{N} \cdot j\right)$ $\qquad\qquad\qquad x0_i := -0.1 + \left(\dfrac{0.2}{N}\right) \cdot i$

$$A(x0, v0) := \dfrac{1}{\omega n} \cdot \sqrt{\omega n^2 \cdot (x0)^2 + (v0)^2}$$

$M_{i,j} := A\left(x0_i, v0_j\right)$
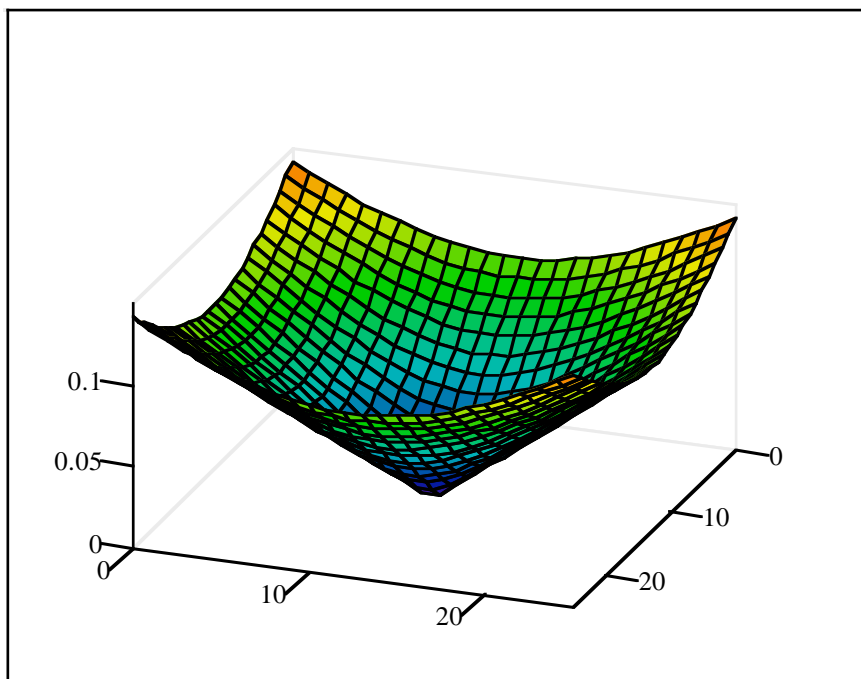
M

Amplitude vs initial conditions

1.16 A machine part is modeled as a pendulum connected to a spring as illustrated in Figure P1.16. Ignore the mass of pendulum's rod and derive the equation of motion. Then following the procedure used in Example 1.1.1, linearize the equation of motion and compute the formula for the natural frequency. Assume that the rotation is small enough so that the spring only deflects horizontally.
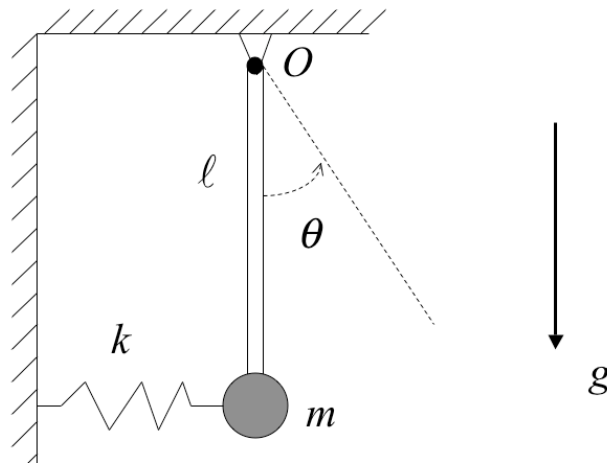
Figure P1.16

Solution: Consider the free body diagram of the mass displaced from equilibrium:
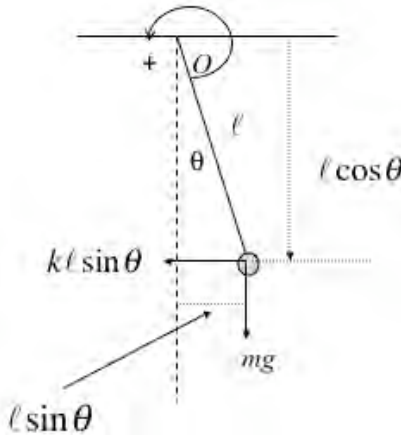
There are two forces acting on the system to consider, if we take moments about point O (then we can ignore any forces at O). This yields

$$\sum M_O = J_O \alpha \Rightarrow m\ell^2 \ddot{\theta} = -mg\ell \sin\theta - k\ell \sin\theta \cdot \ell \cos\theta$$
$$\Rightarrow \underline{m\ell^2 \ddot{\theta} + mg\ell \sin\theta + k\ell^2 \sin\theta \cos\theta = 0}$$

Next consider the small θ approximations to that $\sin\theta \sim \theta$ and $\cos\theta=1$. Then the linearized equation of motion becomes:

$$\ddot{\theta}(t) + \left(\frac{mg + k\ell}{m\ell}\right)\theta(t) = 0$$

Thus the natural frequency is

$$\omega_n = \sqrt{\frac{mg + k\ell}{m\ell}} \text{ rad/s}$$

1.17 A pendulum has length of 250 mm. What is the system's natural frequency in Hertz?

Solution:

Given: l = 250 mm

Assumptions: small angle approximation of sin

From Window 1.1, the equation of motion for the pendulum is as follows:

$$I_O\ddot{\theta} + mg\theta = 0, \quad \text{where } I_O = ml^2 \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

The coefficient of θ yields the natural frequency as:

$$\omega_n = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.8 \text{ m/s}^2}{0.25 \text{ m}}} = 6.26 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 0.996 \text{ Hz}$$

1.18 The pendulum in Example 1.1.1 is required to oscillate once every second. What length should it be?

Solution:

Given: $f = 1 \ Hz$ (one cycle per second)

$$\omega_n = 2\pi f = \sqrt{\frac{g}{l}}$$

$$\therefore l = \frac{g}{(2\pi f)^2} = \frac{9.81}{4\pi^2} = 0.248 \ m$$

1.19   The approximation of sin θ = θ, is reasonable for θ less than 10°. If a pendulum of length

0.5 m, has an initial position of θ(0) = 0, what is the maximum value of the initial angular

velocity that can be given to the pendulum with out violating this small angle

approximation? (be sure to work in radians)

Solution: From Window 1.1, the linear equation of the pendulum is

$$\ddot{\theta}(t) + \frac{g}{\ell}\theta(t) = 0$$

For zero initial position, the solution is given in equation (1.10) by

$$\theta(t) = \frac{v_0\sqrt{\ell}}{\sqrt{g}}\sin(\sqrt{\frac{g}{\ell}}t) \Rightarrow |\theta| \le \frac{v_0\sqrt{\ell}}{\sqrt{g}}$$

since sin is always less then one. Thus if we need θ < 10°= 0.175 rad, then we need to

solve:

$$\frac{v_0\sqrt{0.5}}{\sqrt{9.81}} = 0.175$$

for v_0 which yields:

$$v_0 \le \mathbf{0.773 \ \ rad/s.}$$

Problems and Solutions for Section 1.2 and Section 1.3 (1.20 to 1.51)

Problems and Solutions Section 1.2   (Numbers 1.20 through 1.30)

1.20* Plot the solution of a linear, spring and mass system with frequency $\omega_n = 2$ rad/s, $x_0 = 1$ mm and $v_0 = 2.34$ mm/s, for at least two periods.

Solution: From Window 1.18, the plot can be formed by computing:

$$A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2} = 1.54 \text{ mm}, \quad \phi = \tan^{-1}(\frac{\omega_n x_0}{v_0}) = 40.52°$$

$$x(t) = A\sin(\omega_n t + \phi)$$

This can be plotted in any of the codes mentioned in the text. In Mathcad the program looks like.
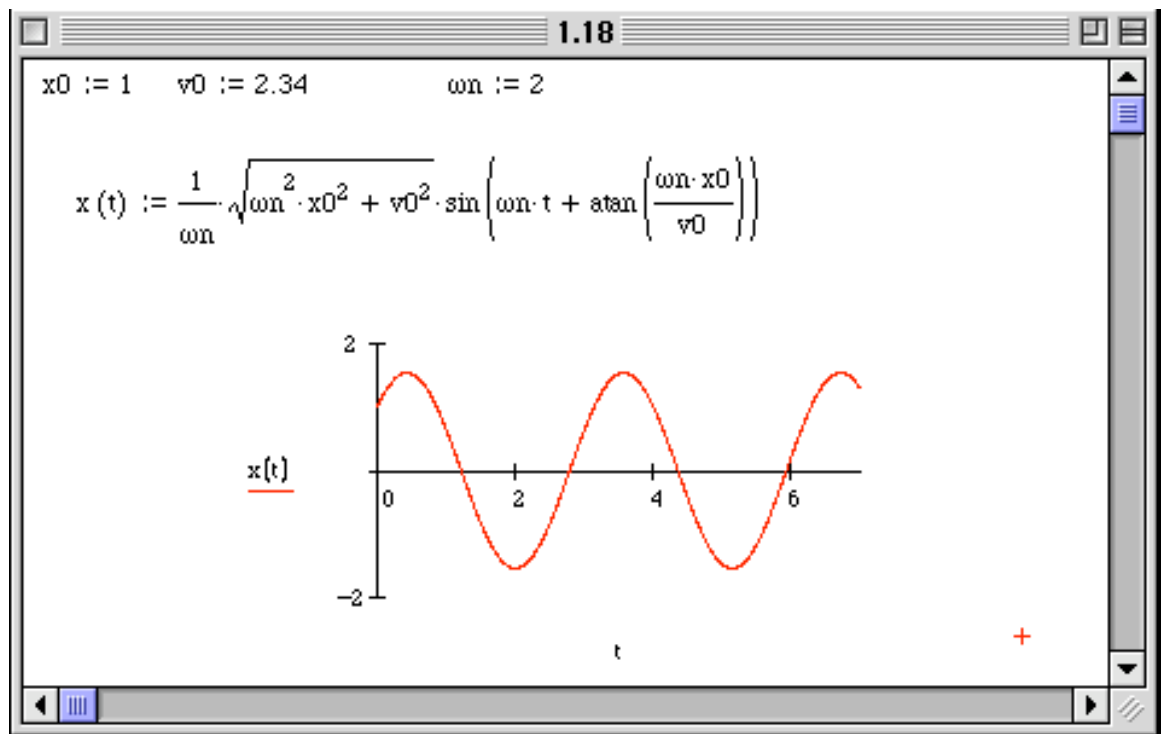
In this plot the units are in mm rather than meters.

1.21* Compute the natural frequency and plot the solution of a spring-mass system with mass of 1 kg and stiffness of 4 N/m, and initial conditions of $x_0 = 1$ mm and $v_0 = 0$ mm/s, for at least two periods.

Solution: Working entirely in Mathcad, and using the units of mm yields:
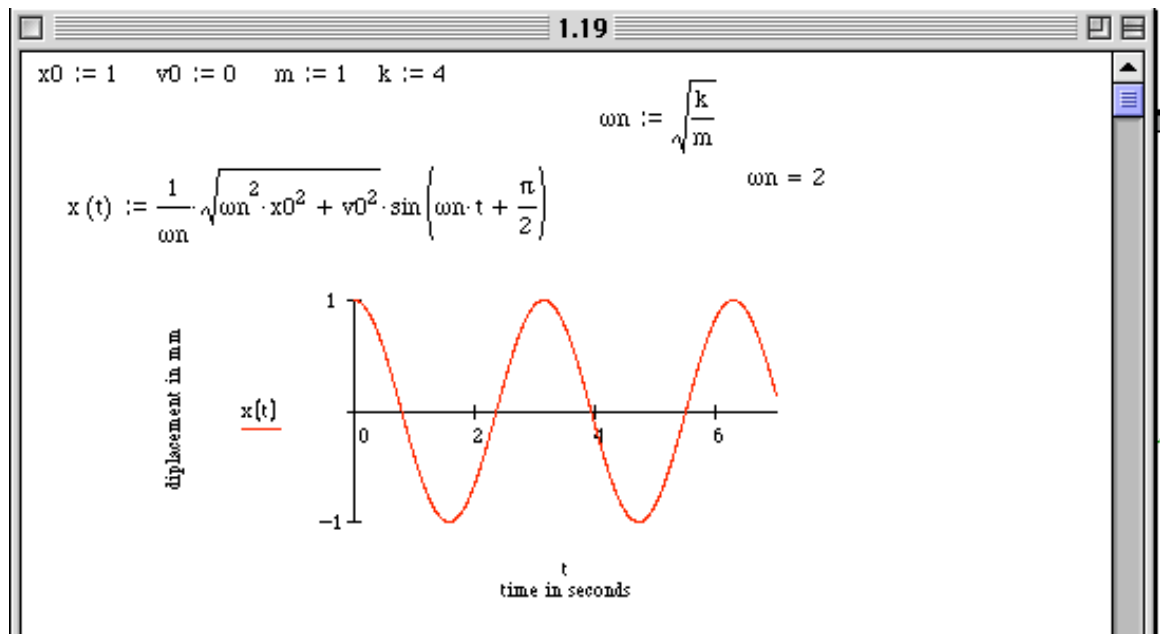
Any of the other codes can be used as well.

1.22   To design a linear, spring-mass system it is often a matter of choosing a spring constant such that the resulting natural frequency has a specified value. Suppose that the mass of a system is 4 kg and the stiffness is 100 N/m. How much must the spring stiffness be changed in order to increase the natural frequency by 10%?

Solution: Given $m = 4$ kg and $k = 100$ N/m the natural frequency is

$$\omega_n = \sqrt{\frac{100}{4}} = 5 \text{ rad/s}$$

Increasing this value by 10% requires the new frequency to be 5 x 1.1 = 5.5 rad/s. Solving for k given m and ω_n yields:

$$5.5 = \sqrt{\frac{k}{4}} \Rightarrow k = (5.5)^2(4) = 121 \text{ N/m}$$

Thus the stiffness k must be increased by about 20%.

1.23   Referring to Figure 1.8, if the maximum peak velocity of a vibrating system is 200 mm/s at 4 Hz and the maximum allowable peak acceleration is 5000 mm/s^2, what will the peak displacement be?

$v = 200 \, \text{mm/sec}$

$x \, (\text{mm})$

$a = 5000 \, \text{mm/sec}^2$

$f = 4 \, Hz$
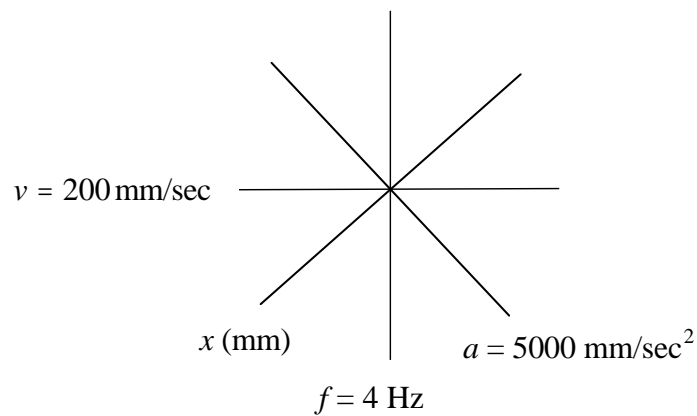
Solution:

Given: $v_{max} = 200$ mm/s   @ $4 \, Hz$

$\quad\quad\quad a_{max} = 5000$ mm/s @ $4 \, Hz$

$x_{max} = A$

$v_{max} = A\omega_n$

$a_{max} = A\omega_n{}^2$

$\therefore x_{max} = \dfrac{v_{max}}{\omega_n} = \dfrac{v_{max}}{2\pi f} = \dfrac{200}{8\pi} = 7.95 \; mm$

At the center point, the peak displacement will be **x = 7.95 mm**

1.24 Show that lines of constant displacement and acceleration in Figure 1.8 have slopes of +1 and –1, respectively. If rms values instead of peak values are used, how does this affect the slope?

Solution: Let

$$x = x_{max} \sin \omega_n t$$
$$\dot{x} = x_{max}\omega_n \cos \omega_n t$$
$$\ddot{x} = -x_{max}\omega_n^2 \sin \omega_n t$$

Peak values:

$$\dot{x}_{max} = x_{max}\omega_n = 2\pi f x_{max}$$
$$\ddot{x}_{max} = x_{max}\omega_n^2 = (2\pi f)^2 x_{max}$$

Location:

$$\ln \dot{x}_{max} = \ln x_{max} + \ln 2\pi f$$
$$\ln \dot{x}_{max} = \ln \ddot{x}_{max} - \ln 2\pi f$$

Since x_{max} is constant, the plot of $\ln \dot{x}_{max}$ versus $\ln 2\pi f$ is a straight line of slope +1. If $\ln \ddot{x}_{max}$ is constant, the plot of $\ln \dot{x}_{max}$ versus $\ln 2\pi f$ is a straight line of slope –1. Calculate RMS values

Let

$$x(t) = A\sin \omega_n t$$
$$\dot{x}(t) = A\omega_n \cos \omega_n t$$
$$\ddot{x}(t) = -A\omega_n^2 \sin \omega_n t$$

Mean Square Value: $\bar{x}^2 = \lim\limits_{T \to \infty} \dfrac{1}{T}\int\limits_0^T x^2(t)\, dt$

$\bar{x}^2 = \lim\limits_{T \to \infty} \dfrac{1}{T}\int\limits_0^T A^2 \sin^2 \omega_n t \, dt = \lim\limits_{T \to \infty} \dfrac{A^2}{T}\int\limits_0^T (1 - \cos 2\omega_n t)\, dt = \dfrac{A^2}{2}$

$\overline{\dot{x}}^2 = \lim\limits_{T \to \infty} \dfrac{1}{T}\int\limits_0^T A^2 \omega_n^2 \cos^2 \omega_n t \, dt = \lim\limits_{T \to \infty} \dfrac{A^2 \omega_n^2}{T}\int\limits_0^T \dfrac{1}{2}(1 + \cos 2\omega_n t)\, dt = \dfrac{A^2 \omega_n^2}{2}$

$\overline{\ddot{x}}^2 = \lim\limits_{T \to \infty} \dfrac{1}{T}\int\limits_0^T A^2 \omega_n^4 \sin^2 \omega_n t \, dt = \lim\limits_{T \to \infty} \dfrac{A^2 \omega_n^4}{T}\int\limits_0^T \dfrac{1}{2}(1 + \cos 2\omega_n t)\, dt = \dfrac{A^2 \omega_n^4}{2}$

Therefore,

$$x_{rms} = \sqrt{\bar{x}^2} = \dfrac{\sqrt{2}}{2}\, A$$

$$\dot{x}_{rms} = \sqrt{\overline{\dot{x}}^2} = \dfrac{\sqrt{2}}{2}\, A\omega_n$$

$$\ddot{x}_{rms} = \sqrt{\overline{\ddot{x}}^2} = \dfrac{\sqrt{2}}{2}\, A\omega_n^2$$

The last two equations can be rewritten as:

$$\dot{x}_{rms} = x_{rms}\,\omega = 2\pi f\, x_{rms}$$

$$\ddot{x}_{rms} = x_{rms}\,\omega^2 = 2\pi f\, \dot{x}_{rms}$$

The logarithms are:

$$\ln \dot{x}_{max} = \ln x_{max} + \ln 2\pi f$$

$$\ln \ddot{x}_{max} = \ln \dot{x}_{max} + \ln 2\pi f$$

The plots of $\ln \dot{x}_{rms}$ versus $\ln 2\pi f$ is a straight line of slope $+1$ when x_{rms} is constant, and -1 when $\ddot{x}_{rms}$ is constant. Therefore **the slopes are unchanged.**

1.25   A foot pedal mechanism for a machine is crudely modeled as a pendulum connected to a spring as illustrated in Figure P1.25. The purpose of the spring is to keep the pedal roughly vertical. Compute the spring stiffness needed to keep the pendulum at 1° from the horizontal and then compute the corresponding natural frequency. Assume that the angular deflections are small, such that the spring deflection can be approximated by the arc length, that the pedal may be treated as a point mass and that pendulum rod has negligible mass. The values in the figure are $m = 0.5$ kg, $g = 9.8$ m/s^2, $\ell_1 = 0.2$ m and $\ell_2 = 0.3$ m.
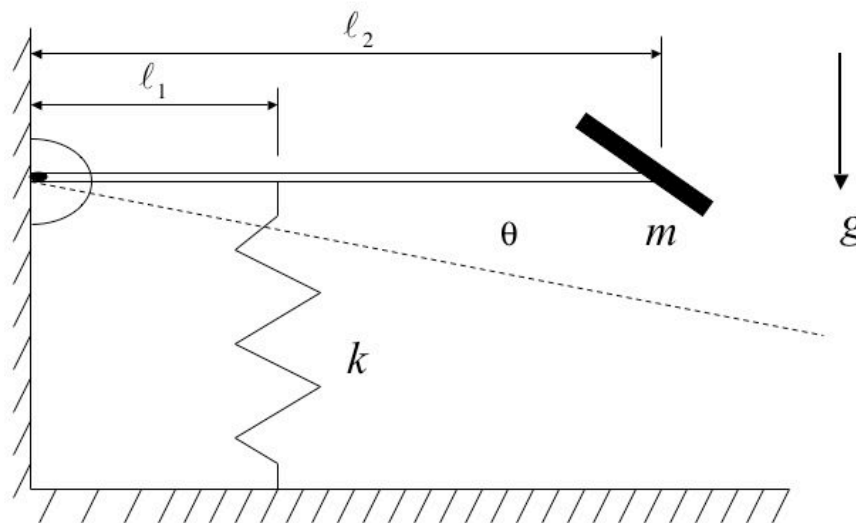
Figure P1.25

Solution: You may want to note to your students, that many systems with springs are often designed based on static deflections, to hold parts in specific positions as in this case, and yet allow some motion. The free-body diagram for the system is given in the figure.
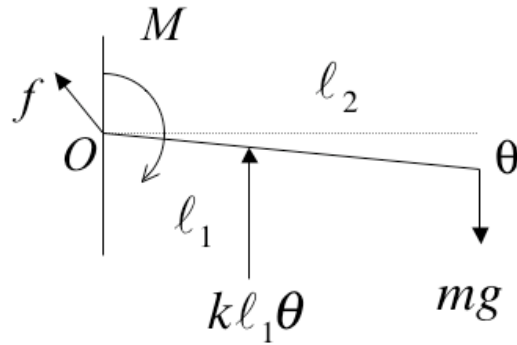
For static equilibrium the sum of moments about point O yields (θ_1 is the static deflection):

$$\sum M_0 = -\ell_1\theta_1(\ell_1)k + mg\ell_2 = 0$$
$$\Rightarrow \ell_1^2\theta_1 k = mg\ell_2 \qquad\qquad (1)$$
$$\Rightarrow k = \frac{mg\ell_2}{\ell_1^2\theta_1} = \frac{0.5\cdot 0.3}{(0.2)^2\,\dfrac{\pi}{180}} = \underline{2106\ \text{N/m}}$$

Again take moments about point O to get the dynamic equation of motion:

$$\sum M_O = J\ddot\theta = m\ell_2^2\ddot\theta = -\ell_1^2 k(\theta + \theta_1) + mg\ell_2 = -\ell_1^2 k\theta + \ell_1^2 k\theta_1 - mg\ell_2\theta$$

Next using equation (1) above for the static deflection yields:

$$m\ell_2^2\ddot\theta + \ell_1^2 k\theta = 0$$

$$\Rightarrow \ddot\theta + \left(\frac{\ell_1^2 k}{m\ell_2^2}\right)\theta = 0$$

$$\Rightarrow \omega_n = \frac{\ell_1}{\ell_2}\sqrt{\frac{k}{m}} = \frac{0.2}{0.3}\sqrt{\frac{2106}{0.5}} = \underline{43.27\ \text{rad/s}}$$

1.26   An automobile is modeled as a 1000-kg mass supported by a spring of stiffness $k = 400{,}000$ N/m. When it oscillates it does so with a maximum deflection of 10 cm. When loaded with passengers, the mass increases to as much as 1300 kg. Calculate the change in frequency, velocity amplitude, and acceleration amplitude if the maximum deflection remains 10 cm.

Solution:

Given: $m_1 = 1000\ kg$

$m_2 = 1300\ kg$

$k = 400{,}000\ N/m$

$$x_{max} = A = 10 \text{ cm}$$

$$\omega_{n1} = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{400{,}000}{1000}} = 20 \ rad/s$$

$$\omega_{n2} = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{400{,}000}{1300}} = 17.54 \ rad/s$$

$$\Delta\omega = 17.54 - 20 = -2.46 \ rad/s$$

$$\Delta f = \frac{\Delta\omega}{2\pi} = \left| \frac{-2.46}{2\pi} \right| = 0.392 \ Hz$$

$$v_1 = A\omega_{n1} = 10 \text{ cm} \times 20 \ rad/s = 200 \ cm/s$$

$$v_2 = A\omega_{n2} = 10 \text{ cm} \times 17.54 \text{ rad/s} = 175.4 \ cm/s$$

$$\Delta v = 175.4 - 200 = \textbf{-24.6 cm/s}$$

$$a_1 = A\omega_{n1}{}^2 = 10 \text{ cm} \times (20 \ rad/s)^2 = 4000 \ cm/s^2$$

$$a_2 = A\omega_{n2}{}^2 = 10 \text{ cm} \times (17.54 \text{ rad/s})^2 = 3077 \ cm/s^2$$

$$\Delta a = 3077 - 4000 = \textbf{-923 cm/s}^2$$

1.27 The front suspension of some cars contains a torsion rod as illustrated in Figure P1.27 to improve the car's handling. (a) Compute the frequency of vibration of the wheel assembly given that the torsional stiffness is 2000 N m/rad and the wheel assembly has a mass of 38 kg. Take the distance $x = 0.26$ m. (b) Sometimes owners put different wheels and tires on a car to enhance the appearance or performance. Suppose a thinner tire is put on with a larger wheel raising the mass to 45 kg. What effect does this have on the frequency?
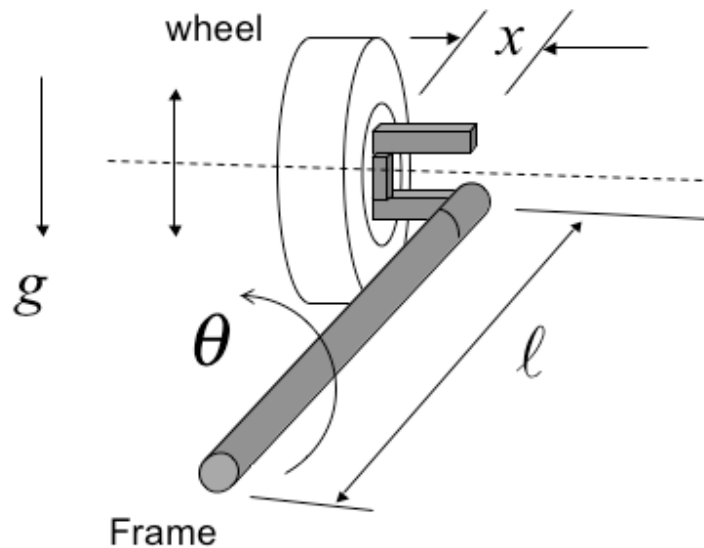
Figure P1.27

Solution: (a) Ignoring the moment of inertial of the rod, and computing the moment of inertia of the wheel as mx^2, the frequency of the shaft mass system is

$$\omega_n = \sqrt{\frac{k}{mx^2}} = \sqrt{\frac{2000 \text{ N} \cdot \text{m}}{38 \cdot \text{kg} \, (0.26 \text{ m})^2}} = \underline{27.9 \text{ rad/s}}$$

(b) The same calculation with 45 kg will *reduce* the frequency to

$$\omega_n = \sqrt{\frac{k}{mx^2}} = \sqrt{\frac{2000 \text{ N} \cdot \text{m}}{45 \cdot \text{kg} \, (0.26 \text{ m})^2}} = \underline{25.6 \text{ rad/s}}$$

This corresponds to about an 8% change in unsprung frequency and could influence wheel hop etc. You could also ask students to examine the effect of increasing x, as commonly done on some trucks to extend the wheels out for appearance sake.

1.28 A machine oscillates in simple harmonic motion and appears to be well modeled by an undamped single-degree-of-freedom oscillation. Its acceleration is measured to have an amplitude of 10,000 mm/s² at 8 Hz. What is the machine's maximum displacement?

Solution:

Given: $a_{max} = 10,000$ mm/s² @ 8 *Hz*

The equations of motion for position and acceleration are:

$$x = A\sin(\omega_n t + \phi) \qquad (1.3)$$
$$\ddot{x} = -A\omega_n^2 \sin(\omega_n t + \phi) \qquad (1.5)$$

The amplitude of acceleration is $A\omega_n^2 = 10,000$ mm/s² and $\omega_n = 2\pi f = 2\pi(8) = 16\pi$ rad/s, from equation (1.12).

The machine's displacement is $A = \dfrac{10,000}{\omega_n^2} = \dfrac{10,000}{(16\pi)^2}$

A = 3.96 mm

1.29 A simple undamped spring-mass system is set into motion from rest by giving it an initial velocity of 100 mm/s. It oscillates with a maximum amplitude of 10 mm. What is its natural frequency?

Solution:

Given: $x_0 = 0$, $v_0 = 100$ mm/s, $A = 10$ mm

From equation (1.9), $A = \dfrac{v_0}{\omega_n}$ or $\omega_n = \dfrac{v_0}{A} = \dfrac{100}{10}$, so that: **$\omega_n$= 10 rad/s**

1.30 An automobile exhibits a vertical oscillating displacement of maximum amplitude 5 cm and a measured maximum acceleration of 2000 cm/s². Assuming that the automobile can be modeled as a single-degree-of-freedom system in the vertical direction, calculate the natural frequency of the automobile.

Solution:

Given: $A = 5$ cm. From equation (1.15)

$$|\ddot{x}| = A\omega_n^2 = 2000 \text{ cm/s}$$

Solving for ω_n yields:

$$\omega_n = \sqrt{\frac{2000}{A}} = \sqrt{\frac{2000}{5}}$$

$$\omega_n = \mathbf{20\,rad/s}$$

Problems Section 1.3 (Numbers 1.31 through 1.46)

1.31   Solve $\ddot{x} + 4\dot{x} + x = 0$ for $x_0 = 1$ mm, $v_0 = 0$ mm/s. Sketch your results and determine which root dominates.

Solution:

Given $\ddot{x} + 4\dot{x} + x = 0$ where $x_0 = 1\,\text{mm}, v_0 = 0$

Let
$$x = ae^{rt} \Rightarrow \dot{x} = are^{rt} \Rightarrow \ddot{x} = ar^2 e^{rt}$$
Substitute these into the equation of motion to get:
$$ar^2 e^{rt} + 4are^{rt} + ae^{rt} = 0$$
$$\Rightarrow r^2 + 4r + 1 = 0 \Rightarrow r_{1,2} = -2 \pm \sqrt{3}$$

So
$$x = a_1 e^{\left(-2+\sqrt{3}\right)t} + a_2 e^{\left(-2-\sqrt{3}\right)t}$$
$$\dot{x} = \left(-2+\sqrt{3}\right)a_1 e^{\left(-2+\sqrt{3}\right)t} + \left(-2-\sqrt{3}\right)a_2 e^{\left(-2-\sqrt{3}\right)t}$$
Applying initial conditions yields,
$$x_0 = a_1 + a_2 \quad \Rightarrow \quad x_0 - a_2 = a_1 \qquad\qquad (1)$$
$$v_0 = \left(-2+\sqrt{3}\right)a_1 + \left(-2-\sqrt{3}\right)a_2 \qquad\qquad (2)$$
Substitute equation (1) into (2)
$$v_0 = \left(-2+\sqrt{3}\right)(x_0 - a_2) + \left(-2-\sqrt{3}\right)a_2$$
$$v_0 = \left(-2+\sqrt{3}\right)x_0 - 2\sqrt{3}\,a_2$$
Solve for a_2
$$a_2 = \frac{-v_0 + \left(-2+\sqrt{3}\right)x_0}{2\sqrt{3}}$$

Substituting the value of a_2 into equation (1), and solving for a_1 yields,

$$a_1 = \frac{v_0 + \left(2+\sqrt{3}\right)x_0}{2\sqrt{3}}$$

$$\therefore x(t) = \frac{v_0 + \left(2+\sqrt{3}\right)x_0}{2\sqrt{3}} e^{\left(-2+\sqrt{3}\right)t} + \frac{-v_0 + \left(-2+\sqrt{3}\right)x_0}{2\sqrt{3}} e^{\left(-2-\sqrt{3}\right)t}$$

The response is dominated by the root: $\underline{-2 + \sqrt{3}}$ as the other root dies off very fast.

1.32 Solve $\ddot{x} + 2\dot{x} + 2x = 0$ for $x_0 = 0$ mm, $v_0 = 1$ mm/s and sketch the response. You may wish to sketch $x(t) = e^{-t}$ and $x(t) = -e^{-t}$ first.

Solution:

Given $\ddot{x} + 2\dot{x} + x = 0$ where $x_0 = 0$, $v_0 = 1$ mm/s

Let: $x = ae^{rt} \Rightarrow \dot{x} = are^{rt} \Rightarrow \ddot{x} = ar^2 e^{rt}$

Substitute into the equation of motion to get

$$ar^2 e^{rt} + 2are^{rt} + ae^{rt} = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow r_{1,2} = -1 \pm i$$

So

$$x = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t} \Rightarrow \dot{x} = (-1+i)c_1 e^{(-1+i)t} + (-1-i)c_2 e^{(-1-i)t}$$

Initial conditions:

$$x_0 = x(0) = c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1 \qquad (1)$$
$$v_0 = \dot{x}(0) = (-1+i)c_1 + (-1-i)c_2 = 1 \qquad (2)$$

Substituting equation (1) into (2)

$$v_0 = (-1+i)c_1 - (-1-i)c_1 = 1$$
$$c_1 = -\frac{1}{2}i, \quad c_2 = \frac{1}{2}i$$
$$x(t) = -\frac{1}{2}ie^{(-1+i)t} + \frac{1}{2}ie^{(-1-i)t} = -\frac{1}{2}ie^{-t}\left(e^{it} - e^{-it}\right)$$

Applying Euler's formula

$$x(t) = -\frac{1}{2}ie^{-t}\left(\cos t + i\sin t - (\cos t - i\sin t)\right)$$
$$\underline{x(t) = e^{-t}\sin t}$$

Alternately use equations (1.36) and (1.38). The plot is similar to figure 1.11.

1.33 Derive the form of λ_1 and λ_2 given by equation (1.31) from equation (1.28) and the definition of the damping ratio.

Solution:

Equation (1.28): $\lambda_{1,2} = -\dfrac{c}{2m} \pm \dfrac{1}{2m}\sqrt{c^2 - 4km}$

Rewrite, $\lambda_{1,2} = -\left(\dfrac{c}{2\sqrt{m}\sqrt{m}}\right)\left(\dfrac{\sqrt{k}}{\sqrt{k}}\right) \pm \dfrac{1}{2\sqrt{m}\sqrt{m}}\left(\dfrac{\sqrt{k}}{\sqrt{k}}\right)\left(\dfrac{c}{c}\right)\sqrt{c^2 - \left(2\sqrt{km}^{-2}\right)\left(\dfrac{c}{c}\right)^2}$

Rearrange, $\lambda_{1,2} = -\left(\dfrac{c}{2\sqrt{km}}\right)\left(\dfrac{\sqrt{k}}{\sqrt{m}}\right) \pm \dfrac{c}{2\sqrt{km}}\left(\dfrac{\sqrt{k}}{\sqrt{m}}\right)\left(\dfrac{1}{c}\right)\sqrt{c^2\left[1 - \left(\dfrac{2\sqrt{km}}{c}\right)^2\right]}$

Substitute:

$$\omega_n = \sqrt{\dfrac{k}{m}} \text{ and } \zeta = \dfrac{c}{2\sqrt{km}} \Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \zeta\omega_n\left(\dfrac{1}{c}\right)c\sqrt{1 - \left(\dfrac{1}{\zeta^2}\right)}$$

$$\Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2\left[1 - \left(\dfrac{1}{\zeta^2}\right)\right]}$$

$$\Rightarrow \underline{\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$$

1.34 Use the Euler formulas to derive equation (1.36) from equation (1.35) and to determine the relationships listed in Window 1.4.

Solution:

Equation (1.35): $x(t) = e^{-\zeta\omega_n t}(a_1 e)^{j\omega_n\sqrt{1-\zeta^2}\,t} - a_2 e^{-j\omega_n\sqrt{1-\zeta^2}\,t}$

From Euler,

$$x(t) = e^{-\zeta\omega_n t}(a_1 \cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + a_1 j\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)$$

$$+ a_2 \cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) - a_2 j\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right))$$

$$= e^{-\zeta\omega_n t}(a_1 + a_2)\cos\omega_d t + j(a_1 - a_2)\sin\omega_d t$$

Let: $A_1 = (a_1 + a_2)$, $A_2 = (a_1 - a_2)$, then this last expression becomes

$$x(t) = e^{-\zeta\omega_n t} A_1 \cos\omega_d t + A_2 \sin\omega_d t$$

Next use the trig identity:

$$A = \sqrt{A_1 + A_2}, \qquad \phi = \tan^{-1}\frac{A_1}{A_2}$$

to get: $\underline{x(t) = e^{-\zeta\omega_n t} A \sin(\omega_d t + \phi)}$

1.35   Using equation (1.35) as the form of the solution of the underdamped system, calculate the values for the constants a_1 and a_2 in terms of the initial conditions x_0 and v_0.

Solution:

Equation (1.35):

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{j\omega_n\sqrt{1-\zeta^2}\,t} + a_2 e^{-j\omega_n\sqrt{1-\zeta^2}\,t}\right)$$

$$\dot{x}(t) = (-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\,)a_1 e^{\left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)t} + (-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\,)a_2\, e^{\left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)t}$$

Initial conditions

$$x_0 = x(0) = a_1 + a_2 \Rightarrow a_1 = x_0 - a_2 \tag{1}$$

$$v_0 = \dot{x}(0) = (-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\,)a_1 + (-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\,)a_2 \tag{2}$$

Substitute equation (1) into equation (2) and solve for a_2

$$v_0 = \left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\,\right)(x_0 - a_2) + \left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\,\right)a_2$$

$$v_0 = \left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\,\right)x_0 - 2j\omega_n\sqrt{1-\zeta^2}\,a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 - \zeta\omega_n x_0 + j\omega_n\sqrt{1-\zeta^2}\,x_0}{2j\omega_n\sqrt{1-\zeta^2}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + \zeta\omega_n x_0 + j\omega_n\sqrt{1-\zeta^2}\,x_0}{2j\omega_n\sqrt{1-\zeta^2}}$$

1.36   Calculate the constants A and ϕ in terms of the initial conditions and thus verify equation (1.38) for the underdamped case.

Solution:

From Equation (1.36),

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

Applying initial conditions ($t = 0$) yields,

$$x_0 = A\sin\phi \qquad\qquad\qquad\qquad (1)$$

$$v_0 = \dot{x}_0 = -\zeta\omega_n A\sin\phi + \omega_d A\cos\phi \qquad\qquad\qquad (2)$$

Next solve these two simultaneous equations for the two unknowns A and ϕ.

From (1),

$$A = \frac{x_0}{\sin\phi} \qquad\qquad\qquad\qquad (3)$$

Substituting (3) into (1) yields

$$v_0 = -\zeta\omega_n x_0 + \frac{\omega_d x_0}{\tan\phi} \quad\Rightarrow\quad \tan\phi = \frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} \ .$$

Hence,

$$\phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] \qquad\qquad\qquad\qquad (4)$$

From (3), $\sin\phi = \dfrac{x_0}{A}$ (5)

and From (4), $\cos\phi = \dfrac{v_0 + \zeta\omega_n x_0}{(x_0\omega_d)^2 + (v_0 + \zeta\omega_n x_0)^2}$ (6)

Substituting (5) and (6) into (2) yields,

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$$

which are the same as equation (1.38)

1.37 Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equations (1.42) and (1.43) for the overdamped case.

Solution: From Equation (1.41)

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{\omega_n\sqrt{\zeta^2-1}\,t} + a_2 e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

taking the time derivative yields:

$$\dot{x}(t) = (-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})a_1 e^{\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t} + (-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})a_2\, e^{\left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}$$

Applying initial conditions yields,

$$x_0 = x(0) = a_1 + a_2 \quad \Rightarrow \quad x_0 - a_2 = a_1 \qquad (1)$$

$$v_0 = \dot{x}(0) = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)a_1 + \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)a_2 \qquad (2)$$

Substitute equation (1) into equation (2) and solve for a_2

$$v_0 = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)(x_0 - a_2) + \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)a_2$$

$$v_0 = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)x_0 - 2\omega_n\sqrt{\zeta^2-1}\,a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 - \zeta\omega_n x_0 + \omega_n\sqrt{\zeta^2-1}\,x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + \zeta\omega_n x_0 + \omega_n\sqrt{\zeta^2-1}\,x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

1.38 Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equation (1.46) for the critically damped case.

Solution:

From Equation (1.45),

$$x(t) = (a_1 + a_2 t)e^{-\omega_n t}$$

$$\Rightarrow \dot{x}_0 = -\omega_n a_1 e^{-\omega_n t} - \omega_n a_2 t e^{-\omega_n t} + a_2 e^{-\omega_n t}$$

Applying the initial conditions yields:

$$x_0 = a_1 \qquad\qquad\qquad (1)$$

and

$$v_0 = \dot{x}(0) = a_2 - \omega_n a_1 \qquad (2)$$

solving these two simultaneous equations for the two unknowns a_1 and a_2.

Substituting (1) into (2) yields,

$$a_1 = x_0$$

$$a_2 = v_0 + \omega_n x_0$$

which are the same as equation (1.46).

1.39 Using the definition of the damping ratio and the undamped natural frequency, derive equitation (1.48) from (1.47).

Solution:

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{thus,} \quad \frac{k}{m} = \omega_n^2$$

$$\zeta = \frac{c}{2\sqrt{km}} \quad \text{thus,} \quad \frac{c}{m} = \frac{2\zeta\sqrt{km}}{m} = 2\zeta\omega_n$$

Therefore, $\ddot{x} + \dfrac{c}{m}\dot{x} + \dfrac{k}{m}x = 0$

becomes,

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$$

1.40 For a damped system, m, c, and k are known to be $m = 1$ kg, $c = 2$ kg/s, $k = 10$ N/m. Calculate the value of ζ and ω_n. Is the system overdamped, underdamped, or critically damped?

Solution:

Given: $m = 1$ kg, $c = 2$ kg/s, $k = 10$ N/m

Natural frequency: $\quad \omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{10}{1}} = 3.16 \; rad/s$

Damping ratio: $\quad \zeta = \dfrac{c}{2\omega_n m} = \dfrac{2}{2(3.16)(1)} = 0.316$

Damped natural frequency: $\omega_d = \sqrt{10}\sqrt{1 - \left(\dfrac{1}{\sqrt{10}}\right)^2} = 3.0 \; rad/s$

Since $0 < \zeta < 1$, the system is **underdamped**.

1.41 Plot $x(t)$ for a damped system of natural frequency $\omega_n = 2$ rad/s and initial conditions $x_0 = 1$ mm, $v_0 = 1$ mm, for the following values of the damping ratio: $\zeta = 0.01$, $\zeta = 0.2$, $\zeta = 0.1$, $\zeta = 0.4$, and $\zeta = 0.8$.

Solution:

Given: $\omega_n = 2$ rad/s, $x_0 = 1$ mm, $v_0 = 1$ mm, $\zeta_i = [0.01;\ 0.2;\ 0.1;\ 0.4;\ 0.8]$
Underdamped cases:

$$\therefore \omega_{di} = \omega_n \sqrt{1 - \zeta_i^2}$$

From equation 1.38,

$$A_i = \sqrt{\frac{\left(v_0 + \zeta_i \omega_n x_0\right)^2 + \left(x_0 \omega_{di}\right)^2}{\omega_{di}^2}} \qquad\qquad \phi_i = \tan^{-1}\frac{x_0 \omega_{di}}{v_0 + \zeta_i \omega_n x_0}$$

The response is plotted for each value of the damping ratio in the following using Matlab:
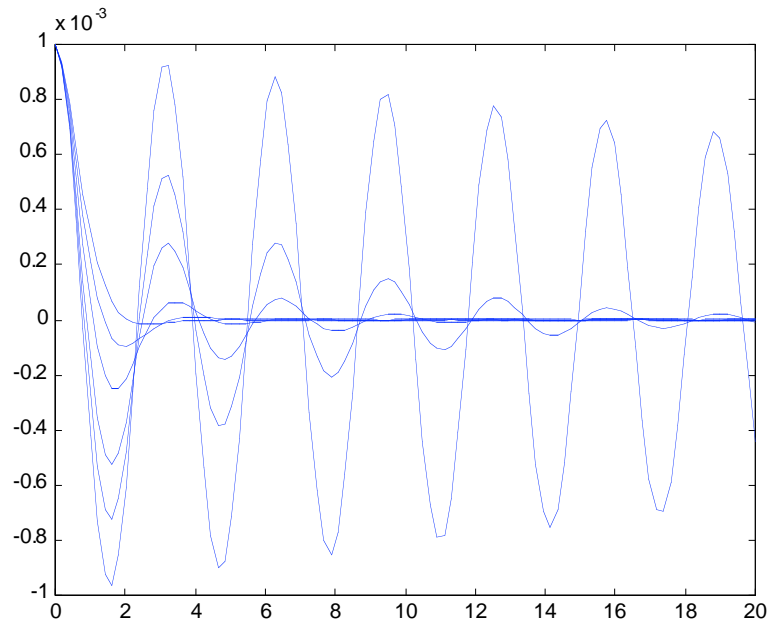
1.42 Plot the response $x(t)$ of an underdamped system with $\omega_n = 2$ rad/s, $\zeta = 0.1$, and $v_0 = 0$ for the following initial displacements: $x_0 = 10$ mm and $x_0 = 100$ mm.

Solution:

Given: $\omega_n = 2$ rad/s, $\zeta = 0.1$, $v_0 = 0$, $x_0 = 10$ mm and $x_0 = 100$ mm.

Underdamped case:

$$\therefore \omega_d = \omega_n \sqrt{1 - \zeta_i^2} = 2\sqrt{1 - 0.1^2} = 1.99 \text{ rad/s}$$

$$A = \sqrt{\frac{\left(v_0 + \zeta\omega_n x_0\right)^2 + \left(x_0\omega_d\right)^2}{\omega_d^2}} = 1.01\, x_0$$

$$\phi = \tan^{-1} \frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} = 1.47 \text{ rad}$$

where

$$x(t) = Ae^{-\zeta\omega_n t} \sin\left(\omega_d t + \phi\right)$$

The following is a plot from Matlab.
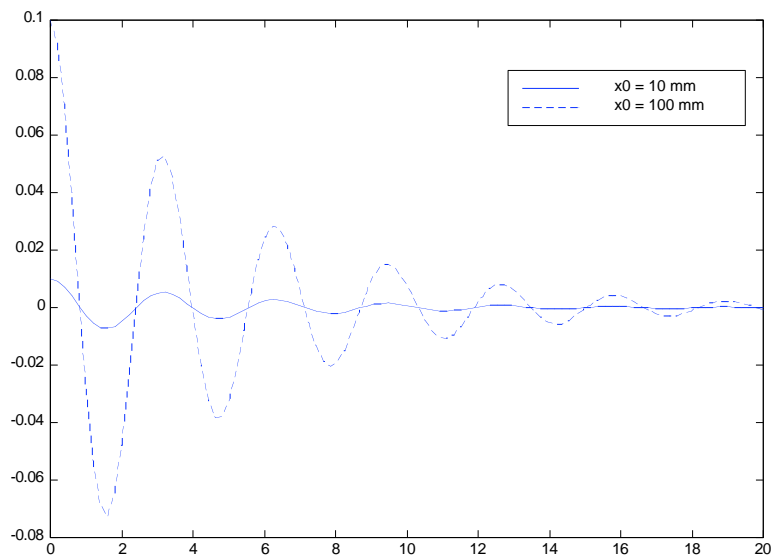
1.43   Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the response.

Solution: This is a problem with negative damping which can be used to tie into Section 1.8 on stability, or can be used to practice the method for deriving the solution using the method suggested following equation (1.13) and eluded to at the start of the section on damping.   To this end let $x(t) = Ae^{\lambda t}$ the equation of motion to get:

$$(\lambda^2 - \lambda + 1)e^{\lambda t} = 0$$

This yields the characteristic equation:

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}j, \quad \text{where } j = \sqrt{-1}$$

There are thus two solutions as expected and these combine to form

$$x(t) = e^{0.5t}(Ae^{\frac{\sqrt{3}}{2}jt} + Be^{-\frac{\sqrt{3}}{2}jt})$$

Using the Euler relationship for the term in parenthesis as given in Window 1.4, this can be written as

$$x(t) = e^{0.5t}(A_1 \cos\frac{\sqrt{3}}{2}t + A_2 \sin\frac{\sqrt{3}}{2}t)$$

Next apply the initial conditions to determine the two constants of integration:
$$x(0) = 1 = A_1(1) + A_2(0) \Rightarrow A_1 = 1$$

Differentiate the solution to get the velocity and then apply the initial velocity condition to get
$$\dot{x}(t) =$$

$$\frac{1}{2}e^0(A_1 \cos\frac{\sqrt{3}}{2}0 + A_2 \sin\frac{\sqrt{3}}{2}0) + e^0\frac{\sqrt{3}}{2}(-A_1 \sin\frac{\sqrt{3}}{2}0 + A_2 \cos\frac{\sqrt{3}}{2}0) = 0$$

$$\Rightarrow A_1 + \sqrt{3}(A_2) = 0 \Rightarrow A_2 = -\frac{1}{\sqrt{3}},$$

$$\Rightarrow x(t) = e^{0.5t}(\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)$$

This function oscillates with increasing amplitude as shown in the following plot which shows the increasing amplitude. This type of response is referred to as a flutter instability. This plot is from Mathcad.

$$x(t) := e^{0.5 \cdot t} \left[\cos\left(\frac{\sqrt{3}}{2} \cdot t\right) - \frac{1}{\sqrt{3}} \cdot \sin\left(\frac{\sqrt{3}}{2} \cdot t\right) \right]$$
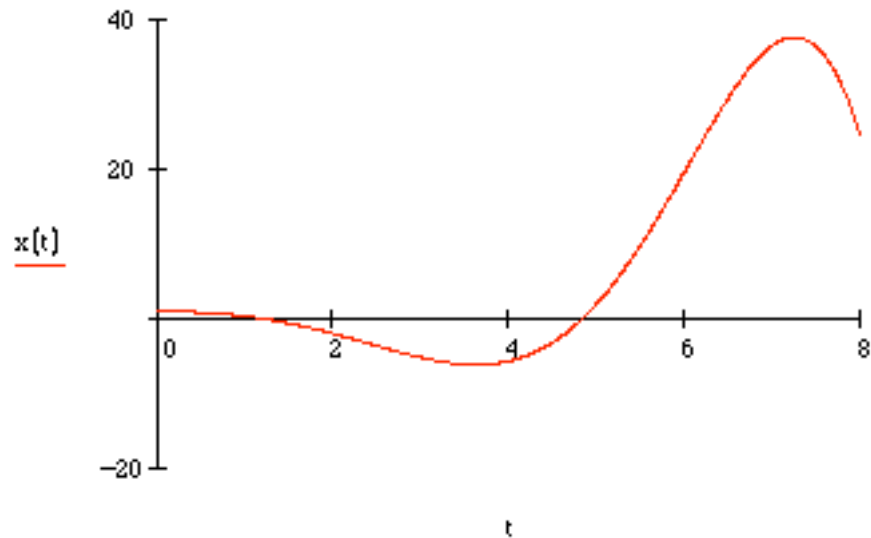
1.44 A spring-mass-damper system has mass of 100 kg, stiffness of 3000 N/m and damping coefficient of 300 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Does the solution oscillate?

Solution: Working straight from the definitions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3000 \text{ N/m}}{100 \text{ kg}}} = 5.477 \text{ rad/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{300}{2\sqrt{km}} = \frac{300}{2\sqrt{(3000)(100)}} = 0.274$$

Since ζ is less then 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n \sqrt{1 - \zeta^2} = 5.27 \text{ rad/s}$.

1.45 A sketch of a valve and rocker arm system for an internal combustion engine is give in Figure P1.45. Model the system as a pendulum attached to a spring and a mass and assume the oil provides viscous damping in the range of $\zeta = 0.01$. Determine the equations of motion and calculate an expression for the natural frequency and the damped natural frequency. Here J is the rotational inertia of the rocker arm about its pivot point, k is the stiffness of the valve spring and m is the mass of the valve and stem. Ignore the mass of the spring.
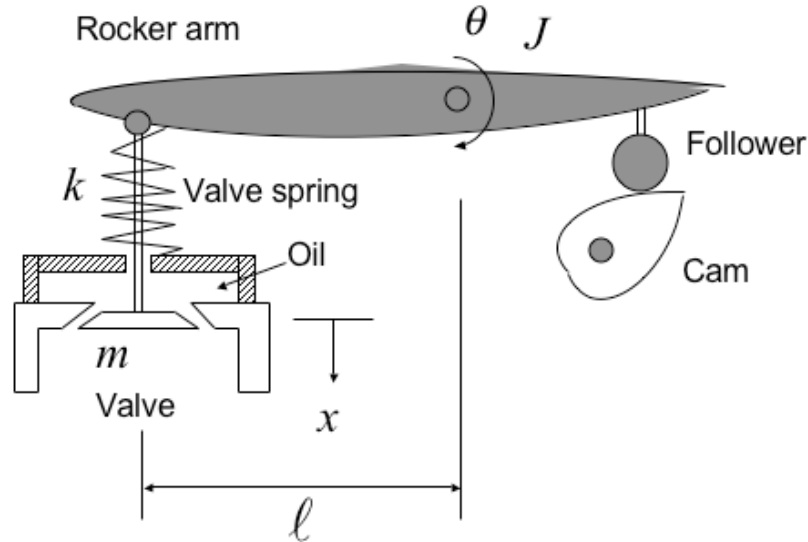
Figure P1.45

Solution: The model is of the form given in the figure. You may wish to give this figure as a hint as it may not be obvious to all students.
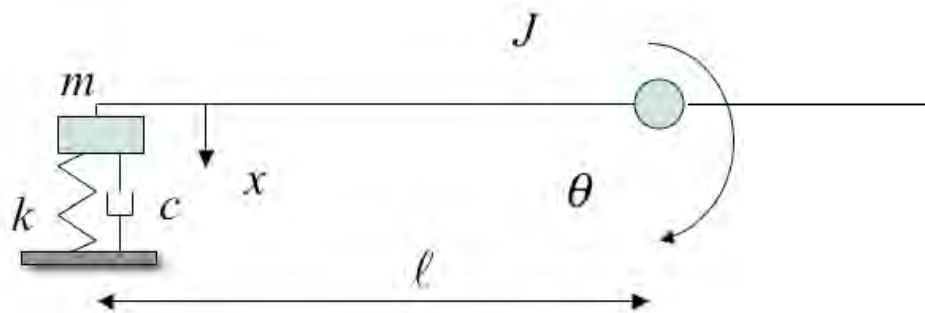
Taking moments about the pivot point yields:

$$(J + m\ell^2)\ddot{\theta}(t) = -kx\ell - c\dot{x}\ell = -k\ell^2\theta - c\ell^2\dot{\theta}$$

$$\Rightarrow (J + m\ell^2)\ddot{\theta}(t) + c\ell^2\dot{\theta} + k\ell^2\theta = 0$$

Next divide by the leading coefficient to get;

$$\ddot{\theta}(t) + \left(\frac{c\ell^2}{J + m\ell^2}\right)\dot{\theta}(t) + \frac{k\ell^2}{J + m\ell^2}\theta(t) = 0$$

From the coefficient of q, the undamped natural frequency is

$$\omega_n = \sqrt{\frac{k\ell^2}{J + m\ell^2}} \text{ rad/s}$$

From equation (1.37), the damped natural frequency becomes

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 0.99995\sqrt{\frac{k\ell^2}{J + m\ell^2}} \sim \sqrt{\frac{k\ell^2}{J + m\ell^2}}$$

This is effectively the same as the undamped frequency for any reasonable accuracy. However, it is important to point out that the resulting response will still decay, even though the frequency of oscillation is unchanged. So even though the numerical value seems to have a negligible effect on the frequency of oscillation, the small value of damping still makes a substantial difference in the response.

1.46 A spring-mass-damper system has mass of 150 kg, stiffness of 1500 N/m and damping coefficient of 200 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Is the system overdamped, underdamped or critically damped? Does the solution oscillate?

Solution: Working straight from the definitions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500 \text{ N/m}}{150 \text{ kg}}} = 3.162 \text{ rad/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{200}{2\sqrt{km}} = \frac{200}{2\sqrt{(1500)(150)}} = 0.211$$

This last expression follows from the equation following equation (1.29). Since ζ is less then 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.091 \text{ rad/s}$, which follows from equation (1.37).

1.47* The system of Problem 1.44 is given a zero initial velocity and an initial displacement of 0.1 m. Calculate the form of the response and plot it for as long as it takes to die out.

Solution: Working from equation (1.38) and using Mathcad the solution is:

1.42

$m := 100 \quad k := 3000 \quad c := 300$

$$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}} \qquad \omega d := \omega n \cdot \sqrt{1 - \zeta^2}$$

$v0 := 0 \quad x0 := 0.1$

$$A := \frac{1}{\omega d} \cdot \sqrt{\left(v0 + \zeta \cdot x0 \cdot \omega n\right)^2 + \left(x0 \cdot \omega d\right)^2}$$

$$\phi := \operatorname{atan}\left(\frac{\omega d \cdot x0}{v0 + \zeta \cdot \omega n \cdot x0}\right)$$

$A = 0.104 \qquad \phi = 1.293 \qquad \omega n = 5.477$

$$\omega d = 5.268$$

$\zeta = 0.274$

$$x(t) := A \cdot \sin\left(\omega d \cdot t + \phi\right) \cdot e^{-\zeta \cdot \omega n \cdot t}$$
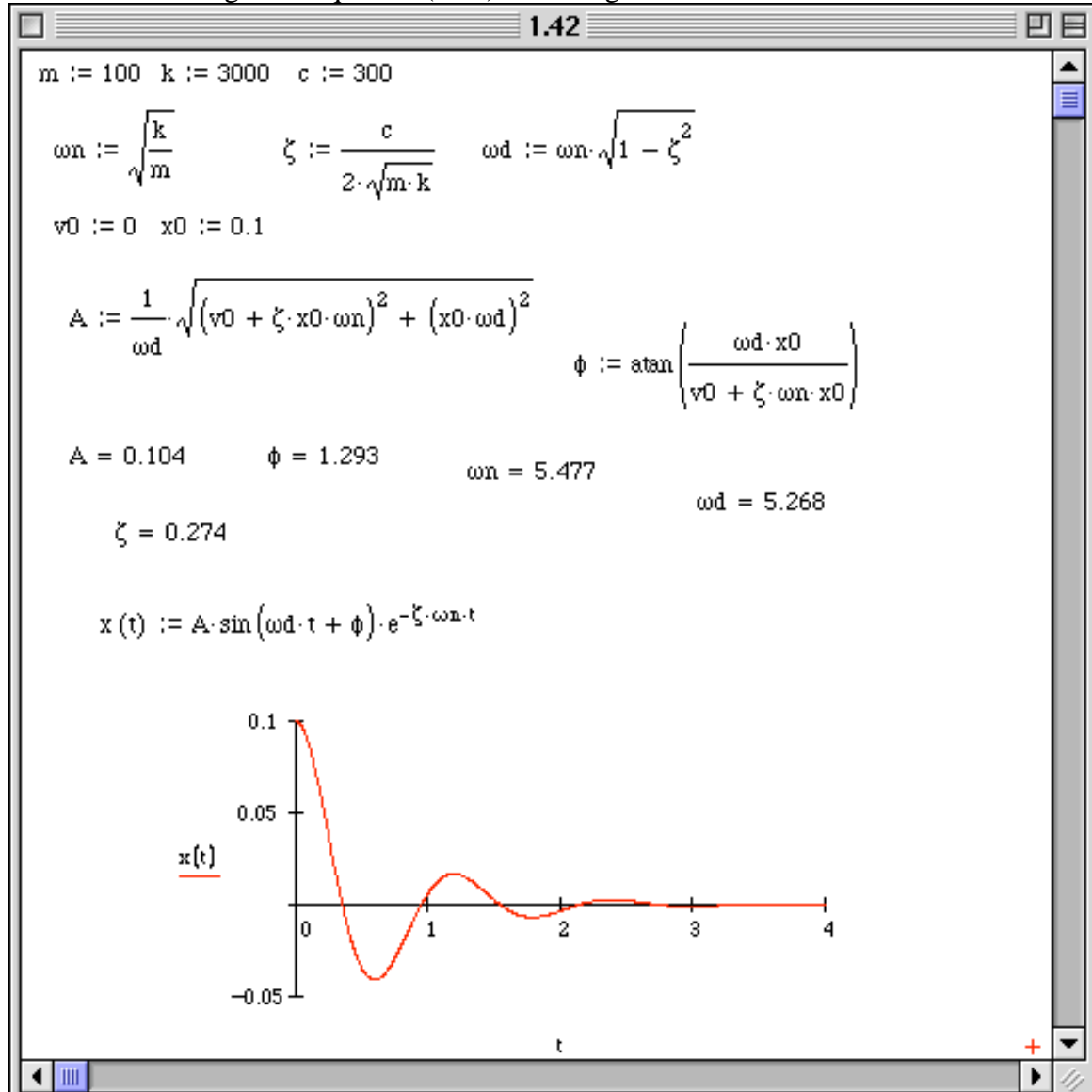
1.48* The system of Problem 1.46 is given an initial velocity of 10 mm/s and an initial displacement of -5 mm. Calculate the form of the response and plot it for as long as it takes to die out. How long does it take to die out?

Solution: Working from equation (1.38), the form of the response is programmed in Mathcad and is given by:
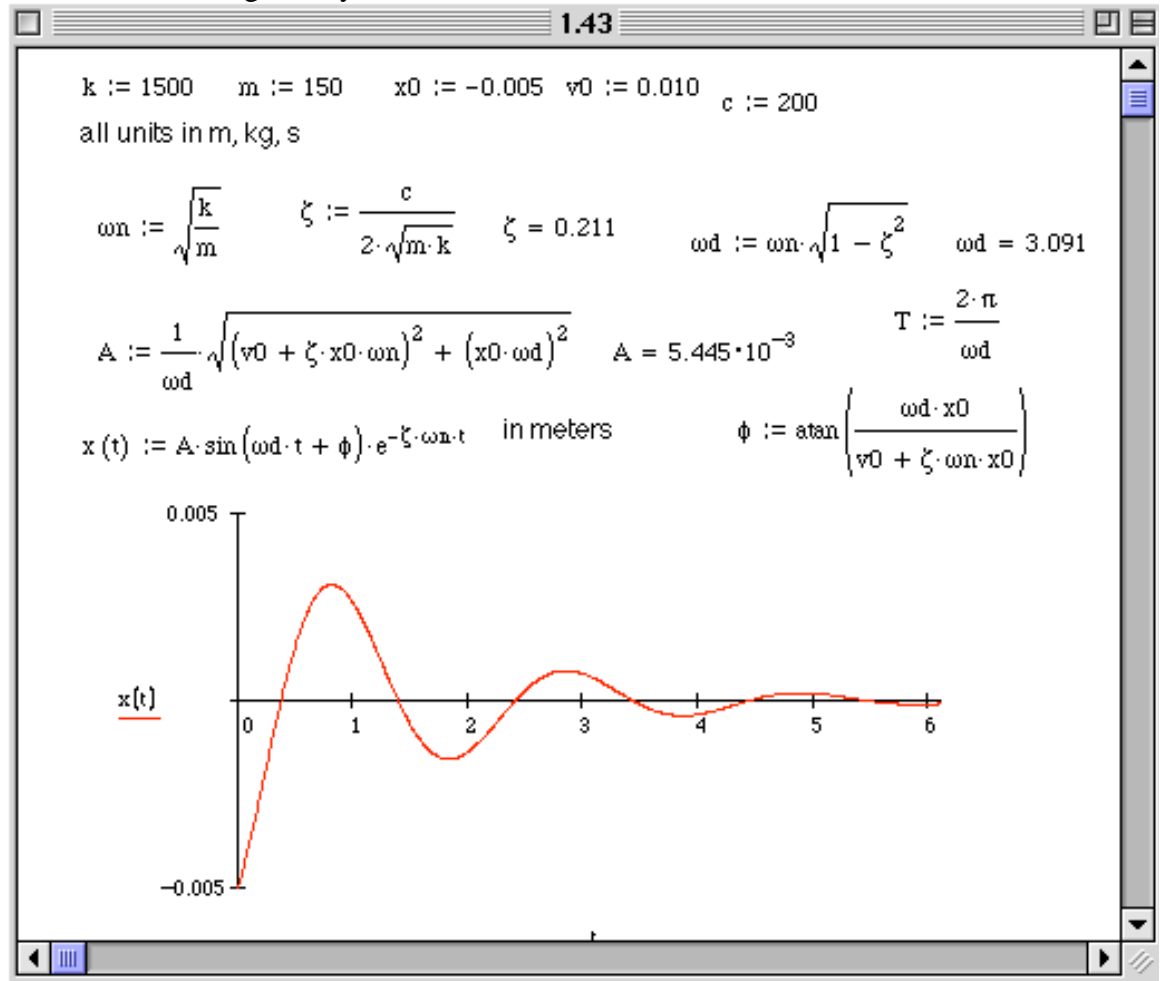
It appears to take a little over 6 to 8 seconds to die out. This can also be plotted in Matlab, Mathematica or by using the toolbox.

1.49* Choose the damping coefficient of a spring-mass-damper system with mass of 150 kg and stiffness of 2000 N/m such that it's response will die out after about 2 s, given a zero initial position and an initial velocity of 10 mm/s.

Solution: Working in Mathcad, the response is plotted and the value of c is changed until the desired decay rate is meet:

$$c := 800 \qquad\qquad k := 2000 \qquad v0 := 0.010 \qquad x0 := 0$$

$$m := 150$$

$$\zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}}$$

$$\omega n := \sqrt{\frac{k}{m}} \qquad\qquad \omega d := \omega n \cdot \sqrt{1 - \zeta^2}$$

$$x(t) := A \cdot \sin(\omega n \cdot t + \phi) \cdot e^{-\zeta \cdot \omega n \cdot t}$$

$$\phi := \operatorname{atan}\left(\frac{\omega d \cdot x0}{v0 + \zeta \cdot \omega n \cdot x0}\right)$$

In this case $\zeta = 0.73$ which is very large!
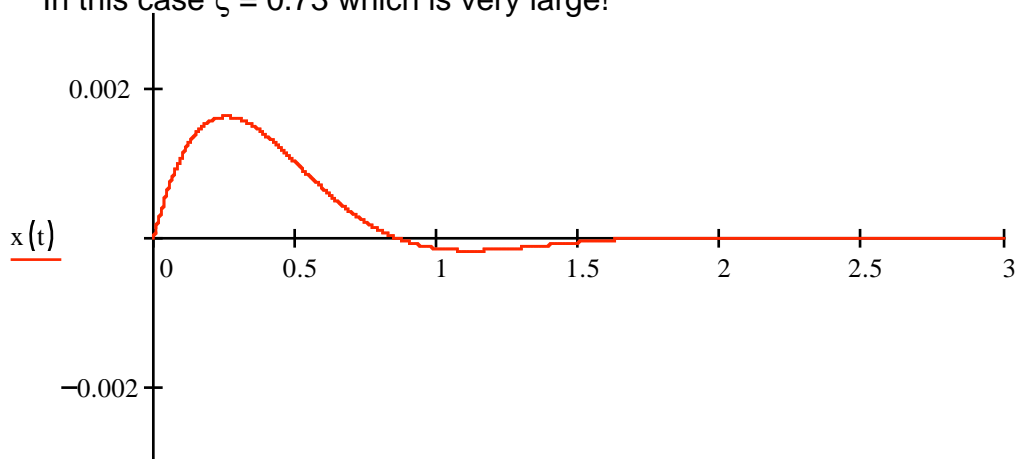
1.50 Derive the equation of motion of the system in Figure P1.50 and discuss the effect of gravity on the natural frequency and the damping ratio.
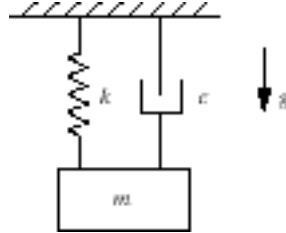
Solution: This requires two free body diagrams. One for the dynamic case and one to show static equilibrium.

ky cdy/dt $k\Delta x$

Δx

mg $x(t)$ mg $y(t)$

(a)   (b)
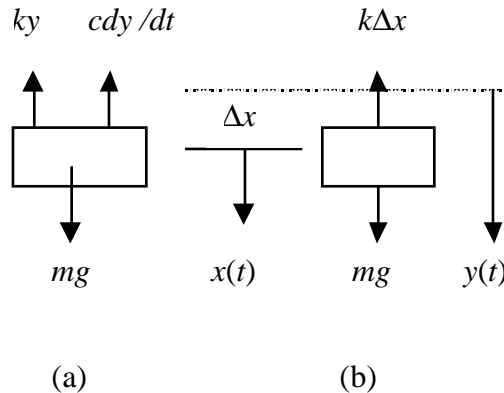
From the free-body diagram of static equilibrium (b) we have that $mg = k\Delta x$, where Δx represents the static deflection. From the free-body diagram of the dynamic case given in (a) the equation of motion is:

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) - mg = 0$$

From the diagram, $y(t) = x(t) + \Delta x$. Since Δx is a constant, differentiating and substitution into the equation of motion yields:

$$\dot{y}(t) = \dot{x}(t) \text{ and } \ddot{y}(t) = \ddot{x}(t) \Rightarrow$$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + \underbrace{(k\Delta x - mg)}_{=0} = 0$$

where the last term is zero from the relation resulting from static equilibrium. Dividing by the mass yields the standard form

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$$

It is clear that gravity has no effect on the damping ratio ζ or the natural frequency ω_n. Not that the damping force is not present in the static case because the velocity is zero.

1.51 Derive the equation of motion of the system in Figure P1.46 and discuss the effect of gravity on the natural frequency and the damping ratio. You may have to make some approximations of the cosine. Assume the bearings provide a viscous damping force only in the vertical direction. (From the A. Diaz-Jimenez, *South African Mechanical Engineer*, Vol. 26, pp. 65-69, 1976)
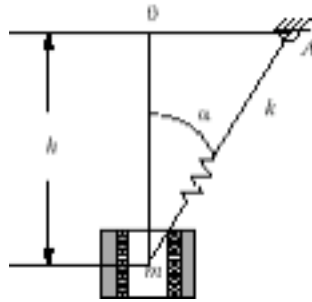
Solution: First consider a free-body diagram of the system:
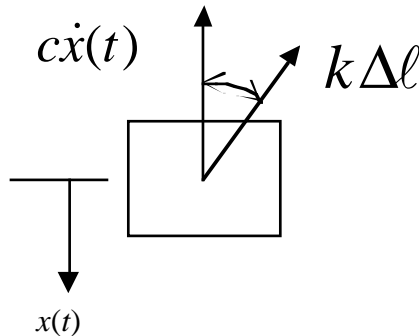
Let α be the angel between the damping and stiffness force. The equation of motion becomes

$$m\ddot{x}(t) = -c\dot{x}(t) - k(\Delta\ell + \delta_s)\cos\alpha$$

From static equilibrium, the free-body diagram (above with $c = 0$ and stiffness force $k\delta_s$) yields: $\sum F_x = 0 = mg - k\delta_s\cos\alpha$. Thus the equation of motion becomes

$$m\ddot{x} + c\dot{x} + k\Delta\ell\cos\alpha = 0 \qquad (1)$$

Next consider the geometry of the dynamic state:
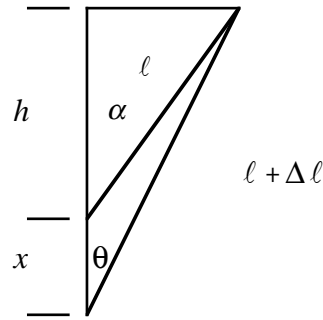
From the definition of cosine applied to the two different triangles:
$$\cos\alpha = \frac{h}{\ell} \quad \text{and} \quad \cos\theta = \frac{h+x}{\ell+\Delta\ell}$$

Next assume small deflections so that the angles are nearly the same $\cos\alpha = \cos\theta$, so that
$$\frac{h}{\ell} \approx \frac{h+x}{\ell+\Delta\ell} \Rightarrow \Delta\ell \approx x\frac{\ell}{h} \Rightarrow \Delta\ell \approx \frac{x}{\cos\alpha}$$

For small motion, then this last expression can be substituted into the equation of motion (1) above to yield:
$$m\ddot{x} + c\dot{x} + kx = 0, \ \alpha \text{ and } x \text{ small}$$

Thus the frequency and damping ratio have the standard values and are not effected by gravity. If the small angle assumption is not made, the frequency can be approximated as

$$\omega_n = \sqrt{\frac{k}{m}\cos^2\alpha + \frac{g}{h}\sin^2\alpha}, \quad \zeta = \frac{c}{2m\omega_n}$$

as detailed in the reference above. For a small angle these reduce to the normal values of

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \text{and} \quad \zeta = \frac{c}{2m\omega_n}$$

as derived here.

Problems and Solutions Section 1.4 (problems 1.52 through 1.65)

1.52 Calculate the frequency of the compound pendulum of Figure 1.20(b) if a mass m_T is added to the tip, by using the energy method.

Solution Using the notation and coordinates of Figure 1.20 and adding a tip mass the diagram becomes:

$$\theta \qquad m_t$$
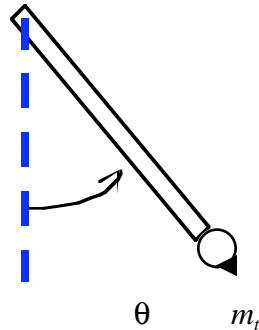
If the mass of the pendulum bar is m, and it is lumped at the center of mass the energies become:

Potential Energy:

$$U = \frac{1}{2}(\ell - \ell\cos\theta)mg + (\ell - \ell\cos\theta)m_t g$$

$$= \frac{\ell}{2}(1 - \cos\theta)(mg + 2m_t g)$$

Kinetic Energy:

$$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_t\dot{\theta}^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2 + \frac{1}{2}m_t\ell^2\dot{\theta}^2$$

$$= (\frac{1}{6}m + \frac{1}{2}m_t)\ell^2\dot{\theta}^2$$

Conservation of energy (Equation 1.52) requires $T + U = $ constant:

$$\frac{\ell}{2}(1 - \cos\theta)(mg + 2m_t g) + (\frac{1}{6}m + \frac{1}{2}m_t)\ell^2\dot{\theta}^2 = C$$

Differentiating with respect to time yields:

$$\frac{\ell}{2}(\sin\theta)(mg + 2m_t g)\dot{\theta} + (\frac{1}{3}m + m_t)\ell^2\dot{\theta}\ddot{\theta} = 0$$

$$\Rightarrow (\frac{1}{3}m + m_t)\ell\ddot{\theta} + \frac{1}{2}(mg + 2m_t g)\sin\theta = 0$$

Rearranging and approximating using the small angle formula $\sin\theta \sim \theta$, yields:

$$\ddot{\theta}(t) + \left(\frac{\dfrac{m}{2} + m_t}{\dfrac{1}{3}m + m_t} \frac{g}{\ell} \right) \theta(t) = 0 \Rightarrow \omega_n = \underline{\sqrt{\frac{3m + 6m_t}{2m + 6m_t}} \sqrt{\frac{g}{\ell}}} \text{ rad/s}$$

Note that this solution makes sense because if $m_t = 0$ it reduces to the frequency of the pendulum equation for a bar, and if $m = 0$ it reduces to the frequency of a massless pendulum with only a tip mass.

1.53 Calculate the total energy in a damped system with frequency 2 rad/s and damping ratio $\zeta = 0.01$ with mass 10 kg for the case $x_0 = 0.1$ and $v_0 = 0$. Plot the total energy versus time.

Solution: Given: $\omega_n = 2$ rad/s, $\zeta = 0.01$, $m = 10$ kg, $x_0 = 0.1$ mm, $v_0 = 0$.

Calculate the stiffness and damped natural frequency:

$$k = m\omega_n^2 = 10(2)^2 = 40 \text{ N/m}$$
$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 2\sqrt{1 - 0.01^2} = 2 \text{ rad/s}$$

The total energy of the damped system is

$$E(t) = \frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}kx(t)$$

where
$$x(t) = Ae^{-0.02t}\sin(2t + \phi)$$
$$\dot{x}(t) = -0.02Ae^{-0.02t}\sin(2t + \phi) + 2Ae^{-0.02t}\cos(2t + \phi)$$

Applying the initial conditions to evaluate the constants of integration yields:

$$x(0) = 0.1 = A\sin\phi$$
$$\dot{x}(0) = 0 = -0.02A\sin\phi + 2A\cos\phi$$
$$\Rightarrow \phi = 1.56 \text{ rad/s}, \quad A = 0.1 \text{ m}$$

Substitution of these values into $E(t)$ yields:

$$E(t) := \left[\frac{e^{-0.04 \cdot t}}{2} \cdot \left[((0.1 \cdot \sin(2 \cdot t + 1.56))) + .2 \cdot \cos(2 \cdot t + 1.56) \right]^2 \right] + 0.5 \cdot e^{-0.04 \cdot t} \cdot (0.1 \cdot \sin(2 \cdot t + 1.56))^2$$
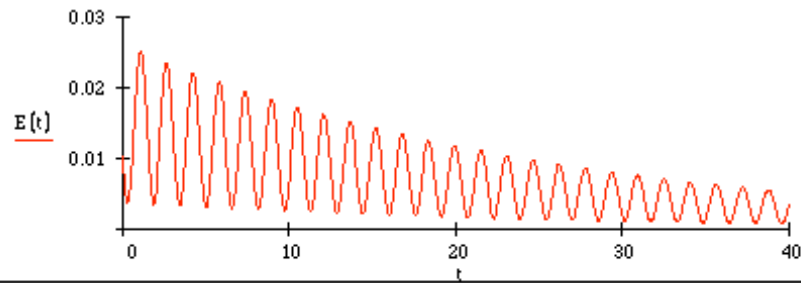
1.54 Use the energy method to calculate the equation of motion and natural frequency of an airplane's steering mechanism for the nose wheel of its landing gear. The mechanism is modeled as the single-degree-of-freedom system illustrated in Figure P1.54.

{Steering wheel}

(Tire-wheel assembly)
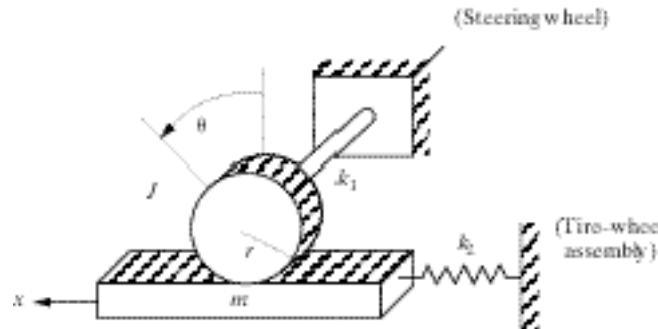
The steering wheel and tire assembly are modeled as being fixed at ground for this calculation. The steering rod gear system is modeled as a linear spring and mass system (m, k_2) oscillating in the x direction. The shaft-gear mechanism is modeled as the disk of inertia J and torsional stiffness k_2. The gear J turns through the angle θ such that the disk does not slip on the mass. Obtain an equation in the linear motion x.

Solution: From kinematics: $x = r\theta, \Rightarrow \dot{x} = r\dot{\theta}$

Kinetic energy:
$$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$$

Potential energy:
$$U = \frac{1}{2}k_2 x^2 + \frac{1}{2}k_1\theta^2$$

Substitute $\theta = \dfrac{x}{r}$: $T + U = \dfrac{1}{2}\dfrac{J}{r^2}\dot{x}^2 + \dfrac{1}{2}m\dot{x}^2 + \dfrac{1}{2}k_2 x^2 + \dfrac{1}{2}\dfrac{k_1}{r^2}x^2$

Derivative:
$$\frac{d(T+U)}{dt} = 0$$

$$\frac{J}{r^2}\ddot{x}\dot{x} + m\ddot{x}\dot{x} + k_2 x\dot{x} + \frac{k_1}{r^2}x\dot{x} = 0$$

$$\left[\left(\frac{J}{r^2}+m\right)\ddot{x} + \left(k_2 + \frac{k_1}{r^2}\right)x\right]\dot{x} = 0$$

Equation of motion: $\left(\dfrac{J}{r^2}+m\right)\ddot{x} + \left(k_2 + \dfrac{k_1}{r^2}\right)x = 0$

Natural frequency: $\omega_n = \sqrt{\dfrac{k_2 + \dfrac{k_1}{r^2}}{\dfrac{J}{r^2}+m}} = \sqrt{\dfrac{k_1 + r^2 k_2}{J + mr^2}}$

1.55 A control pedal of an aircraft can be modeled as the single-degree-of-freedom system of Figure P1.55. Consider the lever as a massless shaft and the pedal as a lumped mass at the end of the shaft. Use the energy method to determine the equation of motion in θ and calculate the natural frequency of the system. Assume the spring to be unstretched at $\theta = 0$.
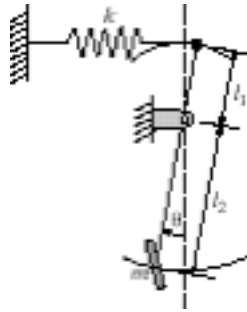
Figure P1.55

Solution: In the figure let the mass at $\theta = 0$ be the lowest point for potential energy. Then, the height of the mass m is $(1-\cos\theta)\ell_2$.

Kinematic relation: $x = \ell_1\theta$

Kinetic Energy: $T = \dfrac{1}{2}m\dot{x}^2 = \dfrac{1}{2}m\ell_2^2\dot{\theta}^2$

Potential Energy: $U = \dfrac{1}{2}k(\ell_1\theta)^2 + mg\ell_2(1-\cos\theta)$

Taking the derivative of the total energy yields:

$$\frac{d}{dt}(T+U) = m\ell_2^2\dot{\theta}\ddot{\theta} + k(\ell_1^2\theta)\dot{\theta} + mg\ell_2(\sin\theta)\dot{\theta} = 0$$

Rearranging, dividing by $d\theta/dt$ and approximating $\sin\theta$ with θ yields:

$$m\ell_2^2\ddot{\theta} + (k\ell_1^2 + mg\ell_2)\theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{k\ell_1^2 + mg\ell_2}{m\ell_2^2}}$$

1.56 To save space, two large pipes are shipped one stacked inside the other as indicated in Figure P1.56. Calculate the natural frequency of vibration of the smaller pipe (of radius R_1) rolling back and forth inside the larger pipe (of radius R). Use the energy method and assume that the inside pipe rolls without slipping and has a mass m.

Large pipe

Small pipe

Truck bed

(a)   (b)
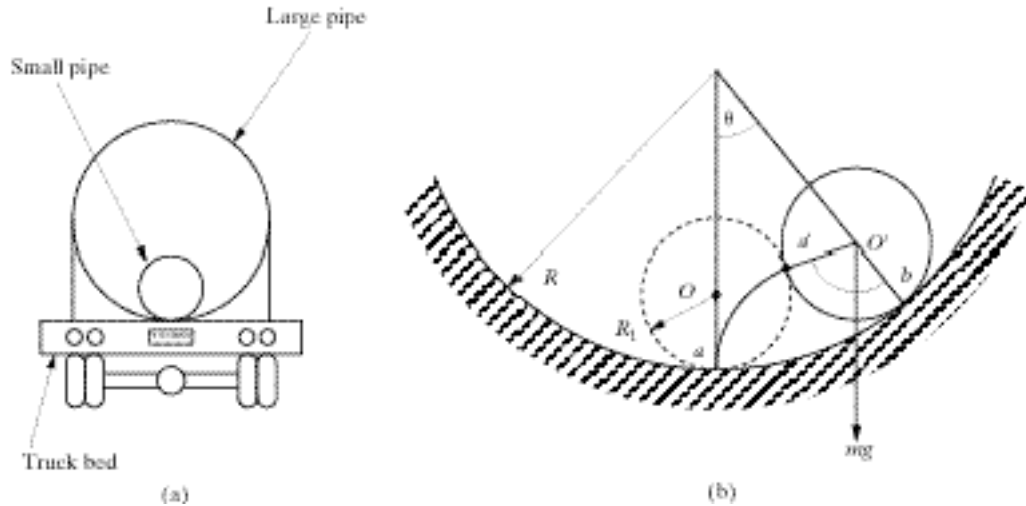
Solution: Let θ be the angle that the line between the centers of the large pipe and the small pipe make with the vertical and let α be the angle that the small pipe rotates through. Let R be the radius of the large pipe and R_1 the radius of the smaller pipe. Then the kinetic energy of the system is the translational plus rotational of the small pipe. The potential energy is that of the rise in height of the center of mass of the small pipe.
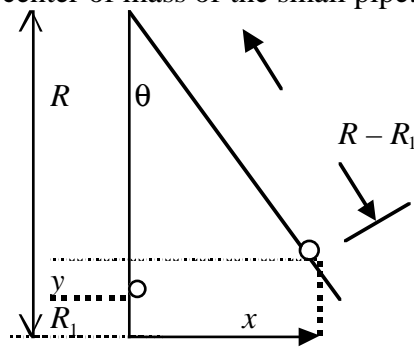
From the drawing:
$$y + (R - R_1)\cos\theta + R_1 = R$$

$$\Rightarrow y = (R - R_1)(1 - \cos\theta)$$

$$\Rightarrow \dot{y} = (R - R_1)\sin(\theta)\dot{\theta}$$

Likewise examination of the value of x yields:
$$x = (R - R_1)\sin\theta$$

$$\Rightarrow \dot{x} = (R - R_1)\cos\theta\dot{\theta}$$

Let β denote the angle of rotation that the small pipe experiences as viewed in the inertial frame of reference (taken to be the truck bed in this case). Then the total

kinetic energy can be written as:

$$T = T_{trans} + T_{rot} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

$$= \frac{1}{2}m(R-R_1)^2(\sin^2\theta + \cos^2\theta)\dot{\theta}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

$$\Rightarrow T = \frac{1}{2}m(R-R_1)^2\dot{\theta}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

The total potential energy becomes just:

$$V = mgy = mg(R-R_1)(1-\cos\theta)$$

Now it remains to evaluate the angel β. Let α be the angle that the small pipe rotates in the frame of the big pipe as it rolls (say) up the inside of the larger pipe. Then

$$\beta = \theta - \alpha$$

were α is the angle "rolled" out as the small pipe rolls from a to b in figure P1.56. The rolling with out slipping condition implies that arc length $a'b$ must equal arc length ab. Using the arc length relation this yields that $R\theta = R_1\alpha$. Substitution of the expression $\beta = \theta - \alpha$ yields:

$$R\theta = R_1(\theta - \beta) = R_1\theta - R_1\beta \Rightarrow (R-R_1)\theta = -R_1\beta$$

$$\Rightarrow \beta = \frac{1}{R_1}(R_1 - R)\theta \text{ and } \dot{\beta} = \frac{1}{R_1}(R_1 - R)\dot{\theta}$$

which is the relationship between angular motion of the small pipe relative to the ground (β) and the position of the pipe (θ). Substitution of this last expression into the kinetic energy term yields:

$$T = \frac{1}{2}m(R-R_1)^2\dot{\theta}^2 + \frac{1}{2}I_0(\frac{1}{R_1}(R_1 - R)\dot{\theta})^2$$

$$\Rightarrow T = m(R-R_1)^2\dot{\theta}^2$$

Taking the derivative of $T+V$ yields

$$\frac{d}{d\theta}(T+V) = 2m(R-R_1)^2\dot{\theta}\ddot{\theta} + mg(R-R_1)\sin\theta\dot{\theta} = 0$$

$$\Rightarrow 2m(R-R_1)^2\ddot{\theta} + mg(R-R_1)\sin\theta = 0$$

Using the small angle approximation for sine this becomes

$$2m(R-R_1)^2\ddot{\theta} + mg(R-R_1)\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{g}{2(R-R_1)}\theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{g}{2(R-R_1)}}$$

1.57 Consider the example of a simple pendulum given in Example 1.4.2. The pendulum motion is observed to decay with a damping ratio of $\zeta = 0.001$. Determine a damping coefficient and add a viscous damping term to the pendulum equation.

Solution: From example 1.4.2, the equation of motion for a simple pendulum is

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

So $\omega_n = \sqrt{\dfrac{g}{\ell}}$. With viscous damping the equation of motion in normalized form becomes:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0 \text{ or with } \zeta \text{ as given :}$$
$$\Rightarrow \ddot{\theta} + 2(.001)\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

The coefficient of the velocity term is

$$\frac{c}{J} = \frac{c}{m\ell^2} = (.002)\sqrt{\frac{g}{\ell}}$$
$$c = (0.002)m\sqrt{g\ell^3}$$

1.58 Determine a damping coefficient for the disk-rod system of Example 1.4.3. Assuming that the damping is due to the material properties of the rod, determine c for the rod if it is observed to have a damping ratio of $\zeta = 0.01$.

Solution: The equation of motion for a disc/rod in torsional vibration is

$$J\ddot{\theta} + k\theta = 0$$

or
$$\ddot{\theta} + \omega_n^2\theta = 0 \quad \text{where } \omega_n = \sqrt{\frac{k}{J}}$$

Add viscous damping:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

$$\ddot{\theta} + 2(.01)\sqrt{\frac{k}{J}}\dot{\theta} + \omega_n^2\theta = 0$$

From the velocity term, the damping coefficient must be

$$\frac{c}{J} = (0.02)\sqrt{\frac{k}{J}}$$

$$\Rightarrow c = 0.02\sqrt{kJ}$$

1.59 The rod and disk of Window 1.1 are in torsional vibration. Calculate the damped natural frequency if $J = 1000 \text{ m}^2 \cdot$ kg, $c = 20$ N$\cdot$ m$\cdot$ s/rad, and $k = 400$ N$\cdot$m/rad.

Solution: From Problem 1.57, the equation of motion is

$$J\ddot{\theta} + c\dot{\theta} + k\theta = 0$$

The damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

where
$$\omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{400}{1000}} = 0.632 \text{ rad/s}$$

and
$$\zeta = \frac{c}{2\sqrt{kJ}} = \frac{20}{2\sqrt{400 \times 1000}} = 0.0158$$

Thus the damped natural frequency is $\omega_d = 0.632$ rad/s

1.60 Consider the system of P1.60, which represents a simple model of an aircraft landing system. Assume, $x = r\theta$. What is the damped natural frequency?
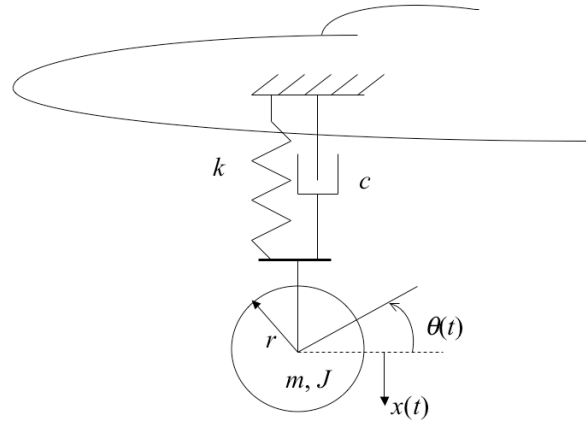
Solution: From Example 1.4.1, the undamped equation of motion is

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + kx = 0$$

From examining the equation of motion the natural frequency is:

$$\omega_n = \sqrt{\frac{k}{m_{eq}}} = \sqrt{\frac{k}{m + \frac{J}{r^2}}}$$

An add hoc way do to this is to add the damping force to get the damped equation of motion:

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + c\dot{x} + kx = 0$$

The value of ζ is determined by examining the velocity term:

$$\frac{c}{m + \frac{J}{r^2}} = 2\zeta\omega_n \Rightarrow \zeta = \frac{c}{m + \frac{J}{r^2}} \frac{1}{2\sqrt{\frac{k}{m + \frac{J}{r^2}}}}$$

$$\Rightarrow \zeta = \frac{c}{2\sqrt{k\left(m + \frac{J}{r^2}\right)}}$$

Thus the damped natural frequency is

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{\dfrac{k}{m + \dfrac{J}{r^2}}\left[1 - \left(\dfrac{c}{2\sqrt{k\left(m + \dfrac{J}{r^2}\right)}}\right)^2\right]}$$

$$\Rightarrow \omega_d = \sqrt{\dfrac{k}{m + \dfrac{J}{r^2}} - \dfrac{c^2}{4\left(m + \dfrac{J}{r^2}\right)^2}} = \dfrac{r}{2(mr^2 + J)}\sqrt{4(kmr^2 + kJ) - c^2 r^2}$$

1.61   Consider Problem 1.60 with $k = 400{,}000$ N·m, $m = 1500$ kg, $J = 100$ m²·kg, $r = 25$ cm, and $c = 8000$ N·m·s. Calculate the damping ratio and the damped natural frequency. How much effect does the rotational inertia have on the undamped natural frequency?

Solution: From problem 1.60:

$$\zeta = \dfrac{c}{2\sqrt{k\left(m + \dfrac{J}{r^2}\right)}} \quad \text{and} \quad \omega_d = \sqrt{\dfrac{k}{m + \dfrac{J}{r^2}} - \dfrac{c^2}{4\left(m + \dfrac{J}{r^2}\right)^2}}$$

Given:

$$k = 4 \times 10^5 \text{ Nm/rad}$$
$$m = 1.5 \times 10^3 \text{ kg}$$
$$J = 100 \text{ m}^2\text{kg}$$
$$r = 0.25 \text{ m and}$$
$$c = 8 \times 10^3 \text{ N} \cdot \text{m} \cdot \text{s/rad}$$

Inserting the given values yields

$$\underline{\zeta = 0.114} \text{ and } \underline{\omega_d = 11.16 \text{ rad/s}}$$

For the undamped natural frequency, $\omega_n = \sqrt{\dfrac{k}{m + J/r^2}}$

With the rotational inertia, $\omega_n = 36.886$ rad/s

Without rotational inertia, $\omega_n = 51.64$ rad/s

The effect of the rotational inertia is that it lowers the natural frequency by almost 33%.

1.62 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.62. Model each of the brackets as a spring of stiffness k, and assume the inertia of the pulleys is negligible.
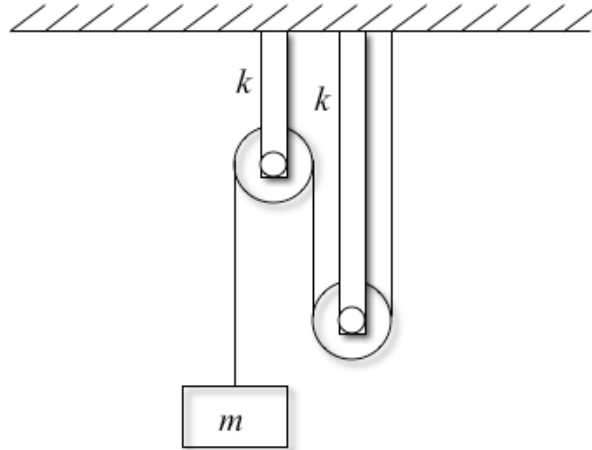
Figure P1.62

Solution: Let x denote the distance mass m moves, then each spring will deflects a distance $x/4$. Thus the potential energy of the springs is

$$U = 2 \times \frac{1}{2} k \left(\frac{x}{4} \right)^2 = \frac{k}{16} x^2$$

The kinetic energy of the mass is

$$T = \frac{1}{2} m \dot{x}^2$$

Using the Lagrange formulation in the form of Equation (1.64):

$$\frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}} \left(\frac{1}{2} m \dot{x}^2 \right) \right) + \frac{\partial}{\partial x} \left(\frac{kx^2}{16} \right) = 0 \Rightarrow \frac{d}{dt} \left(m \dot{x} \right) + \frac{k}{8} x = 0$$

$$\Rightarrow m \ddot{x} + \frac{k}{8} x = 0 \Rightarrow \omega_n = \frac{1}{2} \sqrt{\frac{k}{2m}} \text{ rad/s}$$

1.63 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.63. This figure represents a simplified model of a jet engine mounted to a wing through a mechanism which acts as a spring of stiffness k and mass m_s. Assume the engine has inertial J and mass m and that the rotation of the engine is related to the vertical displacement of the engine, $x(t)$ by the "radius" r_0 (i.e. $x = r_0 \theta$).
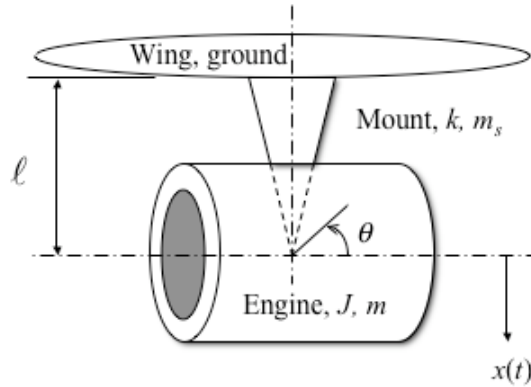
Figure P1.63

Solution: This combines Examples 1.4.1 and 1.4.4. The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 + T_{spring} = \frac{1}{2}\left(m + \frac{J}{r_0^2}\right)\dot{x}^2 + T_{spring}$$

The kinetic energy in the spring (see example 1.4.4) is

$$T_{spring} = \frac{1}{2}\frac{m_s}{3}\dot{x}^2$$

Thus the total kinetic energy is

$$T = \frac{1}{2}\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\dot{x}^2$$

The potential energy is just

$$U = \frac{1}{2}kx^2$$

Using the Lagrange formulation of Equation (1.64) the equation of motion results from:

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}}\left(\frac{1}{2}\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) = 0$$

$$\Rightarrow \underline{\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\ddot{x} + kx = 0}$$

$$\Rightarrow \omega_n = \underline{\sqrt{\frac{k}{\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)}}} \text{ rad/s}$$

1.64 Lagrange's formulation can also be used for non-conservative systems by adding the applied non-conservative term to the right side of equation (1.64) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial R_i}{\partial\dot{q}_i} = 0$$

Here R_i is the *Rayleigh dissipation function* defined in the case of a viscous damper attached to ground by

$$R_i = \frac{1}{2}c\dot{q}_i^2$$

Use this extended Lagrange formulation to derive the equation of motion of the damped automobile suspension of Figure P1.64
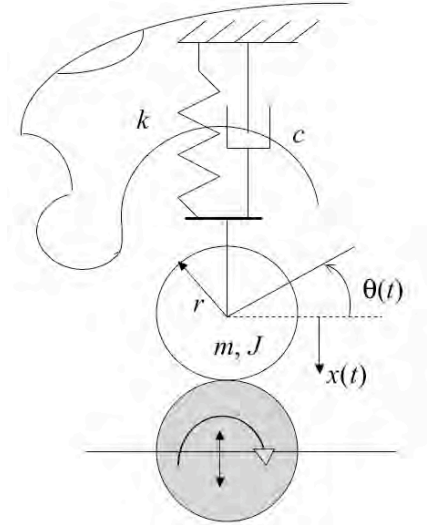
Figure P1.64

Solution: The kinetic energy is (see Example 1.4.1):

$$T = \frac{1}{2}(m + \frac{J}{r^2})\dot{x}^2$$

The potential energy is:

$$U = \frac{1}{2}kx^2$$

The Rayleigh dissipation function is

$$R = \frac{1}{2}c\dot{x}^2$$

The Lagrange formulation with damping becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial R_i}{\partial \dot{q}_i} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}(m + \frac{J}{r^2})\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) + \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}c\dot{x}^2\right) = 0$$

$$\Rightarrow (m + \frac{J}{r^2})\ddot{x} + c\dot{x} + kx = 0$$

1.65 Consider the disk of Figure P1.65 connected to two springs. Use the energy method to calculate the system's natural frequency of oscillation for small angles $\theta(t)$.

$m = $ mass
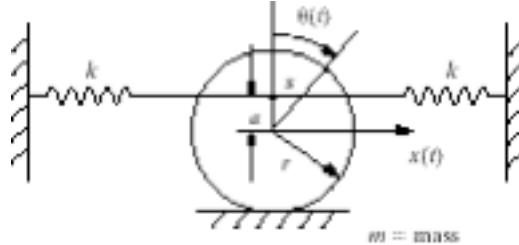
Solution:

Known: $x = r\theta, \dot{x} = r\dot{\theta}$ and $J_o = \dfrac{1}{2}mr^2$

Kinetic energy:

$$T_{rot} = \frac{1}{2}J_o \ \dot{\theta}^2 = \frac{1}{2}\left(\frac{mr^2}{2}\right) \dot{\theta}^2 = \frac{1}{4}mr^2\dot{\theta}^2$$

$$T_{trans} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mr^2\dot{\theta}^2$$

$$T = T_{rot} + T_{trans} = \frac{1}{4}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\theta}^2 = \frac{3}{4}mr^2\dot{\theta}^2$$

Potential energy: $\quad U = 2\left(\dfrac{1}{2}k[(a+r)\theta]^2\right) = k(a+r)^2\theta^2$

Conservation of energy:

$$T + U = \text{ Constant}$$

$$\frac{d}{dt}(T + U) = 0$$

$$\frac{d}{dt}\left(\frac{3}{4}mr^2\dot{\theta}^2 + k(a+r)^2\theta^2\right) = 0$$

$$\frac{3}{4}mr^2\left(2\dot{\theta}\ddot{\theta}\right) + k(a+r)^2\left(2\dot{\theta}\theta\right) = 0$$

$$\frac{3}{2}mr^2\ddot{\theta} + 2k(a+r)^2\theta = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{2k(a+r)^2}{\frac{3}{2}mr^2}}$$

$$\omega_n = 2\frac{a+r}{r}\sqrt{\frac{k}{3m}} \text{ rad/s}$$

Problems and Solutions Section 1.5 (1.66 through 1.74)

1.66 A helicopter landing gear consists of a metal framework rather than the coil spring based suspension system used in a fixed-wing aircraft. The vibration of the frame in the vertical direction can be modeled by a spring made of a slender bar as illustrated in Figure 1.21, where the helicopter is modeled as ground. Here $l = 0.4$ m, $E = 20 \times 10^{10}$ N/m^2, and $m = 100$ kg. Calculate the cross-sectional area that should be used if the natural frequency is to be $f_n = 500$ Hz.

Solution: From Figure 1.21

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{EA}{lm}} \tag{1}$$

and

$$\omega_n = 500 \text{ Hz} \left(\frac{2\pi \, \text{rad}}{1 \text{ cycle}} \right) = 3142 \text{ rad/s}$$

Solving (1) for A yields:

$$A = \frac{\omega_n^2 lm}{E} = \frac{(3142)^2 (.4)(100)}{20 \times 10^{10}}$$

$$A = 0.0019 \text{ m}^2 = 19 \text{cm}^2$$

1.67 The frequency of oscillation of a person on a diving board can be modeled as the transverse vibration of a beam as indicated in Figure 1.24. Let m be the mass of the diver ($m = 100$ kg) and $l = 1$ m. If the diver wishes to oscillate at 3 Hz, what value of EI should the diving board material have?

Solution: From Figure 1.24,

$$\omega_n^2 = \frac{3EI}{ml^3}$$

and

$$\omega_n = 3Hz\left(\frac{2\pi \, \text{rad}}{1 \, \text{cycle}}\right) = 6\pi \, \text{rad/s}$$

Solving for EI

$$EI = \frac{\omega_n^2 ml^3}{3} = \frac{(6\pi)^2 (100)(1)^3}{3} = \underline{11843.5 \, \text{Nm}^2}$$

1.68 Consider the spring system of Figure 1.29. Let $k_1 = k_5 = k_2 = 100$ N/m, $k_3 = 50$ N/m, and $k_4 = 1$ N/m. What is the equivalent stiffness?

Solution: Given: $k_1 = k_2 = k_5 = 100$ N/m, $k_3 = 50$ N/m, $k_4 = 1$ N/m

From Example 1.5.4

$$k_{eq} = k_1 + k_2 + k_5 + \frac{k_3 k_4}{k_3 + k_4}$$

$$\Rightarrow \underline{k_{eq} = 300.98 \, \text{N/m}}$$

1.69 Springs are available in stiffness values of 10, 100, and 1000 N/m. Design a spring system using these values only, so that a 100-kg mass is connected to ground with frequency of about 1.5 rad/s.

Solution: Using the definition of natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

With $m = 100$ kg and $\omega_n = 1.5$ rad/s the equivalent stiffness must be:

$$k_{eq} = m\omega_n^2 = (100)(1.5)^2 = 225 \text{ N/m}$$

There are many configurations of the springs given and no clear way to determine one configuration over another. Here is one possible solution. Choose two 100 N/m springs in parallel to get 200 N/m, then use four 100 N/m springs in series to get an equivalent spring of 25 N/m to put in parallel with the other 3 springs since

$$k_{eq} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2} + \dfrac{1}{k_3} + \dfrac{1}{k_4}} = \frac{1}{4/100} = 25$$

Thus using six 100 N/m springs in the following arrangement will produce an equivalent stiffness of 225 N/m
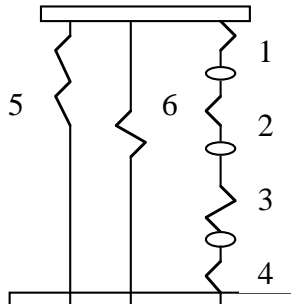
1.70   Calculate the natural frequency of the system in Figure 1.29(a) if $k_1 = k_2 = 0$. Choose m and nonzero values of k_3, k_4, and k_5 so that the natural frequency is 100 Hz.

Solution: Given: $k_1 = k_2 = 0$ and $\omega_n = 2\pi(100) = 628.3$ rad/s

From Figure 1.29, the natural frequency is

$$\omega_n = \sqrt{\frac{k_5 k_3 + k_5 k_4 + k_3 k_4}{m(k_3 + k_4)}} \quad \text{and} \quad k_{eq} = \left(k_5 + \frac{k_3 k_4}{k_3 + k_4}\right)$$

Equating the given value of frequency to the analytical value yields:

$$\omega_n^2 = (628.3)^2 = \frac{k_5 k_3 + k_5 k_4 + k_3 k_4}{m(k_3 + k_4)}$$

Any values of k_3, k_4, k_5, and m that satisfy the above equation will do. Again, the answer is *not unique*. One solution is

$$k_3 = 1\,\text{N/m}, k_4 = 1\,\text{N/m}, k_5 = 50{,}000\,\text{N/m, and } m = 0.127\,\text{kg}$$

1.71* Example 1.4.4 examines the effect of the mass of a spring on the natural frequency of a simple spring-mass system. Use the relationship derived there and plot the natural frequency versus the percent that the spring mass is of the oscillating mass. Use your plot to comment on circumstances when it is no longer reasonable to neglect the mass of the spring.

Solution: The solution here depends on the value of the stiffness and mass ratio and hence the frequency. Almost any logical discussion is acceptable as long as the solution indicates that for smaller values of m_s, the approximation produces a reasonable frequency. Here is one possible answer. For

$k := 1000 \qquad m := 100$

$p := 0, 0.01 .. 0.1$

+

$$\omega ns(p) := \sqrt{\dfrac{k}{m + \dfrac{p \cdot m}{3}}} \qquad\qquad \omega n(p) := \sqrt{\dfrac{k}{m}}$$
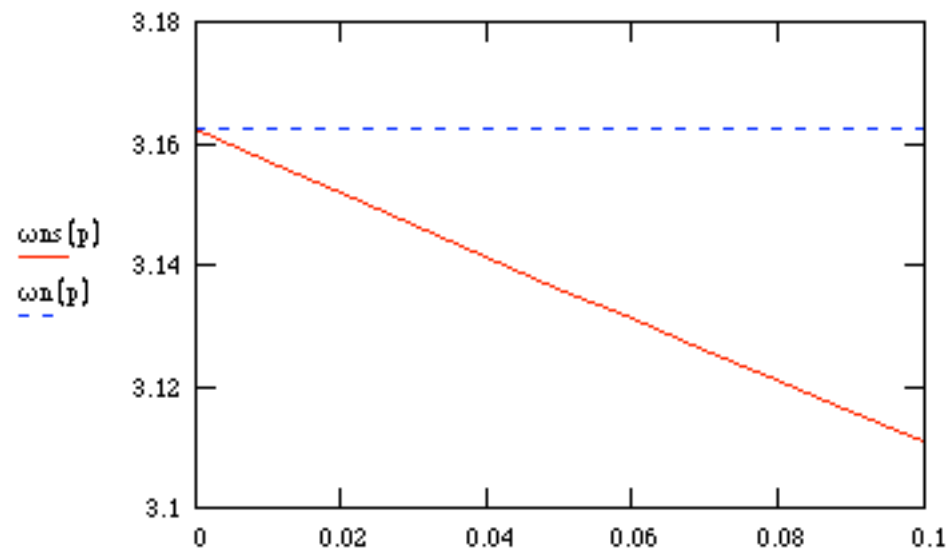
From this plot, for these values of *m* and *k* it **looks like a 10 % spring mass causes less then a 1 % error in the frequency.**

1.72 Calculate the natural frequency and damping ratio for the system in Figure P1.72 given the values $m = 10$ kg, $c = 100$ kg/s, $k_1 = 4000$ N/m, $k_2 = 200$ N/m and $k_3 = 1000$ N/m. Assume that no friction acts on the rollers. Is the system overdamped, critically damped or underdamped?
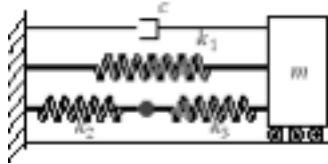
Figure P1.72

Solution: Following the procedure of Example 1.5.4, the equivalent spring constant is:

$$k_{eq} = k_1 + \frac{k_2 k_3}{k_2 + k_3} = 4000 + \frac{(200)(1000)}{1200} = 4167 \text{ N/m}$$

Then using the standard formulas for frequency and damping ratio:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4167}{10}} = 20.412 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{100}{2(10)(20.412)} = 0.245$$

Thus the system is underdamped.

1.73 Repeat Problem 1.72 for the system of Figure P1.73.
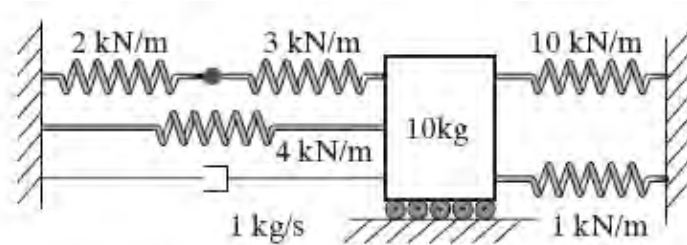
Figure P1.73

Solution: Again using the procedure of Example 1.5.4, the equivalent spring constant is:

$$k_{eq} = k_1 + k_2 + k_3 + \frac{k_4 k_5}{k_4 + k_5} = (10 + 1 + 4 + \frac{2 \times 3}{2 + 3}) \text{kN/m} = 16.2 \text{ kN/m}$$

Then using the standard formulas for frequency and damping ratio:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{16.2 \times 10^3}{10}} = 40.25 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{1}{2(10)(40.25)} = 0.00158$$

Thus the system is underdamped.

1.74   A manufacturer makes a cantilevered leaf spring from steel ($E = 2$ x 10^{11} N/m^2) and sizes the spring so that the device has a specific frequency. Later, to save weight, the spring is made of aluminum ($E = 7.1$ x 10^{10} N/m^2). Assuming that the mass of the spring is much smaller than that of the device the spring is attached to, determine if the frequency increases or decreases and by how much.

Solution: Use equation (1.68) to write the expression for the frequency twice:

$$\omega_{al} = \sqrt{\frac{3E_{al}}{m\ell^3}} \text{ and } \omega_{steel} = \sqrt{\frac{3E_{steel}}{m\ell^3}} \text{ rad/s}$$

Dividing yields:

$$\frac{\omega_{al}}{\omega_{steel}} = \frac{\sqrt{\dfrac{3E_{al}}{m\ell^3}}}{\sqrt{\dfrac{3E_{steel}}{m\ell^3}}} = \sqrt{\frac{7.1 \times 10^{10}}{2 \times 10^{11}}} = 0.596$$

Thus the *frequency is decreased by about 40% by using aluminum*.

Problems and Solutions Section 1.6 (1.75 through 1.81)

1.75 Show that the logarithmic decrement is equal to

$$\delta = \frac{1}{n}\ln\frac{x_0}{x_n}$$

where x_n is the amplitude of vibration after n cycles have elapsed.

Solution:

$$\ln\left[\frac{x(t)}{x(t+nT)}\right] = \ln\left[\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{Ae^{-\zeta\omega_n(t+nt)}\sin(\omega_d t + \omega_d nT + \phi)}\right] \tag{1}$$

Since $n\omega_d T = n(2\pi)$, $\sin(\omega_d t + n\omega_d T + \phi) = \sin(\omega_d t + \phi)$

Hence, Eq. (1) becomes

$$\ln\left[\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{Ae^{-\zeta\omega_n(t+nT)}e^{-\zeta\omega_n nt}\sin(\omega_d t + \omega_d nt + \phi)}\right] = \ln\left(e^{\zeta\omega_n nT}\right) = n\zeta\omega_n T$$

Since

$$\ln\left[\frac{x(t)}{x(t+T)}\right] = \zeta\omega_n T = \delta,$$

Then

$$\ln\left[\frac{x(t)}{x(t+nT)}\right] = n\delta$$

Therefore,

$$\delta = \frac{1}{n}\ln\frac{x_o}{x_n} \qquad \begin{array}{l} \leftarrow \text{ original amplitude} \\ \leftarrow \text{ amplitude } n \text{ cycles later} \end{array}$$

Here $x_0 = x(0)$.

1.76 Derive the equation (1.70) for the trifalar suspension system.

Solution: Using the notation given for Figure 1.29, and the following geometry:
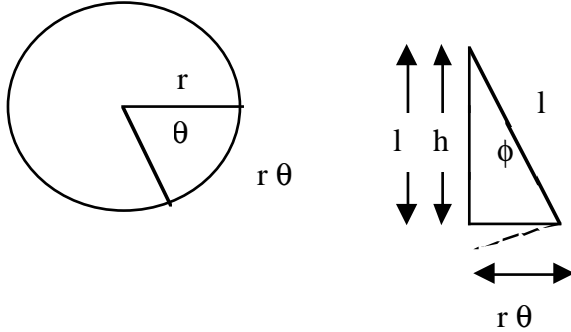
Write the kinetic and potential energy to obtain the frequency:

Kinetic energy:

$$T_{max} = \frac{1}{2}I_o\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2$$

From geometry, $x = r\theta$ and $\dot{x} = r\dot{\theta}$

$$T_{max} = \frac{1}{2}(I_o + I)\frac{\dot{x}^2}{r^2}$$

Potential Energy:

$$U_{max} = (m_o + m)g(l - l\cos\phi)$$

Two term Taylor Series Expansion of $\cos\phi \approx 1 - \dfrac{\phi^2}{2}$:

$$U_{max} = (m_o + m)gl\left(\frac{\phi^2}{2}\right)$$

For geometry, $\sin\phi = \dfrac{r\theta}{l}$, and for small ϕ, $\sin\phi = \phi$ so that $\phi = \dfrac{r\theta}{l}$

$$U_{max} = (m_o + m)gl\left(\frac{r^2\theta^2}{2l^2}\right)$$

$$U_{max} = (m_o + m)g\left(\frac{r^2\theta^2}{2l}\right) \text{ where } r\theta = x$$

$$U_{max} = \frac{(m_o + m)g}{2l}x^2$$

Conservation of energy requires that:

$$T_{max} = U_{max} \quad \Rightarrow$$

$$\frac{1}{2}\frac{(I_o + I)}{r^2}\dot{x}^2 = \frac{(m_o + m)g}{2l}x^2$$

At maximum energy, $x = A$ and $\dot{x} = \omega_n A$

$$\frac{1}{2}\frac{(I_o + I)}{r^2}\omega_n^2 A^2 = \frac{(m_o + m)g}{2l}A^2$$

$$\Rightarrow (I_o + I) = \frac{gr^2(m_o + m)}{\omega_n^2 l}$$

Substitute $\omega_n = 2\pi f_n = \dfrac{2\pi}{T}$

$$(I_o + I) = \frac{gr^2(m_o + m)}{(2\pi/T)^2 l}$$

$$I = \frac{gT^2 r^2(m_o + m)}{4\pi^2 l} - I_o$$

were T is the period of oscillation of the suspension.

1.77 A prototype composite material is formed and hence has unknown modulus. An experiment is performed consisting of forming it into a cantilevered beam of length 1 m and $I = 10^{-9}$ m^4 with a 6-kg mass attached at its end. The system is given an initial displacement and found to oscillate with a period of 0.5 s. Calculate the modulus E.

Solution: Using equation (1.66) for a cantilevered beam,

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{ml^3}{3EI}}$$

Solving for E and substituting the given values yields

$$E = \frac{4\pi^2 ml^3}{3T^2 I} = \frac{4\pi^2 (6)(1)^3}{3(.5)^2(10^{-9})}$$

$$\Rightarrow \underline{E = 3.16 \times 10^{11} \text{ N/m}^2}$$

1.78 The free response of a 1000-kg automobile with stiffness of $k = 400{,}000$ N/m is observed to be of the form given in Figure 1.32. Modeling the automobile as a single-degree-of-freedom oscillation in the vertical direction, determine the damping coefficient if the displacement at t_1 is measured to be 2 cm and 0.22 cm at t_2.

Solution: Given: $x_1 = 2$ cm and $x_2 = 0.22$ cm where $t_2 = T + t_1$

Logarithmic Decrement: $\delta = \ln\dfrac{x_1}{x_2} = \ln\dfrac{2}{0.22} = 2.207$

Damping Ratio: $\zeta = \dfrac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \dfrac{2.207}{\sqrt{4\pi^2 + (2.207)^2}} = 0.331$

Damping Coefficient: $c = 2\zeta\sqrt{km} = 2(0.331)\sqrt{(400{,}000)(1000)} = 13{,}256$ kg/s

1.79 A pendulum decays from 10 cm to 1 cm over one period. Determine its damping ratio.

Solution: Using Figure 1.31: $x_1 = 10$ cm and $x_2 = 1$ cm

Logarithmic Decrement: $\delta = \ln\dfrac{x_1}{x_2} = \ln\dfrac{10}{1} = 2.303$

Damping Ratio: $\zeta = \dfrac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \dfrac{2.303}{\sqrt{4\pi^2 + (2.303)^2}} = 0.344$

1.80 The relationship between the log decrement δ and the damping ratio ζ is often approximated as $\delta = 2\pi\zeta$. For what values of ζ would you consider this a good approximation to equation (1.74)?

Solution: From equation (1.74), $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

For small ζ, $\qquad\qquad\qquad\qquad \delta = 2\pi\zeta$

A plot of these two equations is shown:

$\zeta := 0, 0.001 .. 0.9$

$$\delta1(\zeta) := \frac{2\cdot\pi\cdot\zeta}{\sqrt{1-\zeta^2}} \qquad\qquad \delta2(\zeta) := 2\cdot\pi\cdot\zeta$$
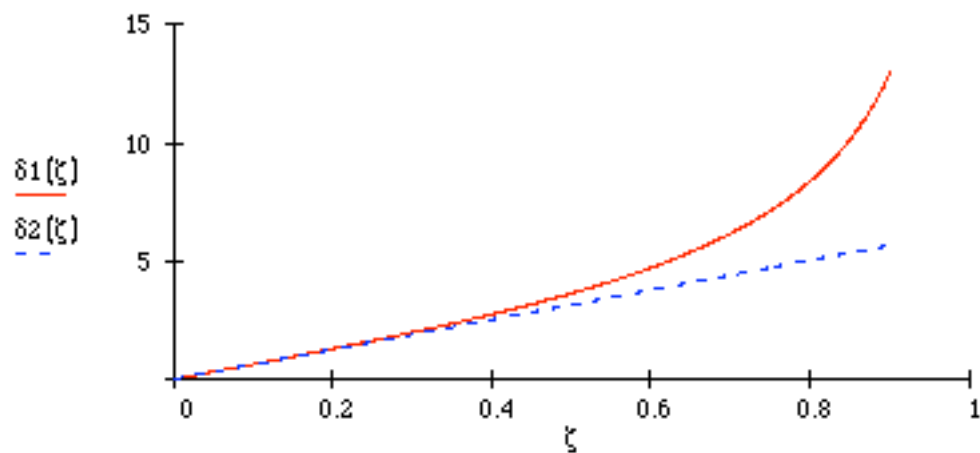
The lower curve represents the approximation for small ζ, while the upper curve is equation (1.74). The approximation appears to be valid to about $\zeta = 0.3$.

1.81   A damped system is modeled as illustrated in Figure 1.10.   The mass of the system is measured to be 5 kg and its spring constant is measured to be 5000 N/m. It is observed that during free vibration the amplitude decays to 0.25 of its initial value after five cycles. Calculate the viscous damping coefficient, c.

Solution:

Note that for any two consecutive peak amplitudes,

$$\frac{x_o}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = e^\delta \text{ by definition}$$

$$\therefore \frac{x_o}{x_5} = \frac{1}{0.25} = \frac{x_0}{x_1} \cdot \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \cdot \frac{x_4}{x_5} = e^{5\delta}$$

So,

$$\delta = \frac{1}{5}\ln(4) = 0.277$$

and

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.044$$

Solving for c,

$$c = 2\zeta \sqrt{km} = 2(0.044)\sqrt{5000(5)}$$
$$c = 13.94 \text{ N - s/m}$$

Problems and Solutions Section 1.7 (1.82 through 1.89)

1.82 Choose a dashpot's viscous damping value such that when placed in parallel with the spring of Example 1.7.2 reduces the frequency of oscillation to 9 rad/s.

Solution:

The frequency of oscillation is $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

From example 1.7.2: $\omega_n = 10$ rad/s, $m = 10$ kg, and $k = 10^3$ N/m

So,
$$9 = 10\sqrt{1 - \zeta^2}$$

$$\Rightarrow 0.9 = \sqrt{1 - \zeta^2} \Rightarrow (0.9)^2 = 1 - \zeta^2$$

$$\zeta = \sqrt{1 - (0.9)^2} = 0.436$$

Then
$$c = 2m\omega_n\zeta = 2(10)(10)(0.436) = \underline{87.2 \text{ kg/s}}$$

1.83 For an underdamped system, $x_0 = 0$ and $v_0 = 10$ mm/s. Determine m, c, and k such that the amplitude is less than 1 mm.

Solution: Note there are multiple correct solutions. The expression for the amplitude is:
$$A^2 = x_0^2 + \frac{(v_o + \zeta\omega_n x_o)^2}{\omega_d^2}$$

$$\text{for } x_o = 0 \Rightarrow A = \frac{v_o}{\omega_d} < 0.001 \text{ m} \Rightarrow \omega_d > \frac{v_o}{0.001} = \frac{0.01}{0.001} = 10$$

So
$$\omega_d = \sqrt{\frac{k}{m}(1 - \zeta^2)} > 10$$

$$\Rightarrow \frac{k}{m}(1 - \zeta^2) > 100, \Rightarrow k = m\frac{100}{1 - \zeta^2}$$

(1) Choose $\zeta = 0.01 \Rightarrow \dfrac{k}{m} > 100.01$

(2) Choose $\underline{m = 1 \text{ kg}} \Rightarrow k > 100.01$

(3) Choose $k = 144$ N/m > 100.01

$$\Rightarrow \omega_n = \sqrt{144}\,\frac{\text{rad}}{\text{s}} = 12\,\frac{\text{rad}}{\text{s}}$$

$$\Rightarrow \omega_d = 11.99\,\frac{\text{rad}}{\text{s}}$$

$$\Rightarrow c = 2m\zeta\omega_n = \underline{\underline{0.24\,\frac{\text{kg}}{\text{s}}}}$$

1.84   Repeat problem 1.83 if the mass is restricted to lie between 10 kg $< m <$ 15 kg.

Solution: Referring to the above problem, the relationship between m and k is

$$k > 1.01 \text{x} 10^{-4} \, m$$

after converting to meters from mm. Choose $m = 10$ kg and repeat the calculation at the end of Problem 1.82 to get ω_n (again taking $\zeta = 0.01$). Then $k = 1000$ N/m and:

$$\Rightarrow \omega_n = \sqrt{\frac{1.0 \times 10^3}{10}} \frac{\text{rad}}{\text{s}} = 10 \, \frac{\text{rad}}{\text{s}}$$

$$\Rightarrow \omega_d = 9.998 \, \frac{\text{rad}}{\text{s}}$$

$$\Rightarrow c = 2m\zeta\omega_n = 2.000 \, \frac{\text{kg}}{\text{s}}$$

1.85 Use the formula for the torsional stiffness of a shaft from Table 1.1 to design a 1-m shaft with torsional stiffness of 10^5 N·m/rad.

Solution: Referring to equation (1.64) the torsional stiffness is
$$k_t = \frac{GJ_p}{\ell}$$

Assuming a solid shaft, the value of the shaft polar moment is given by
$$J_p = \frac{\pi d^4}{32}$$

Substituting this last expression into the stiffness yields:
$$k_t = \frac{G\pi d^4}{32\ell}$$

Solving for the diameter d yields
$$d = \left(\frac{k_t(32)\ell}{G\pi}\right)^{\frac{1}{4}}$$

Thus we are left with the design variable of the material modulus (G). Choose steel, then solve for d. For steel $G = 8 \times 10^{10}$ N/m². From the last expression the numerical answer is

$$d = \left[\frac{10^5 \dfrac{\text{Nm}}{\text{rad}}(32)(1\text{m})}{\left(8 \times 10^{10} \dfrac{\text{N}}{\text{m}^2}\right)(\pi)}\right]^{\frac{1}{4}} = 0.0597 \text{ m}$$

1.86 Repeat Example 1.7.2 using aluminum. What difference do you note?

Solution:

For aluminum $G = 25 \times 10^9$ N/m²

From example 1.7.2, the stiffness is $k = 10^3 = \dfrac{Gd^4}{64nR^3}$ and $d = .01$ m

So, $10^3 = \dfrac{(25 \times 10^9)(.01)^4}{64nR^3}$

Solving for nR^3 yields: $nR^3 = 3.906 \times 10^{-3}\text{m}^3$

Choose $R = 10$ cm $= 0.1$ m, so that

$$n = \frac{3.906 \times 10^{-3}}{(0.1)^3} = 4 \,\text{turns}$$

Thus, aluminum requires 1/3 fewer turns than steel.

1.87 Try to design a bar (see Figure 1.21) that has the same stiffness as the spring of Example 1.7.2. Note that the bar must remain at least 10 times as long as it is wide in order to be modeled by the formula of Figure 1.21.

Solution:

From Figure 1.21, $k = \dfrac{EA}{l}$

For steel, $E = 210 \times 10^9 \,\text{N/m}^2$

From Example 1.7.2, $k = 10^3 \,\text{N/m}$

So, $10^3 = \dfrac{(210 \times 10^9)A}{l}$

$l = (2.1 \times 10^8)A$

If $A = 0.0001 \,\text{m}^2$ (1 cm^2), then

$$l = (2.1 \times 10^8)(10^{-4}) = 21,000 \,\text{m} \;\left(21 \text{km or 13 miles}\right)$$

Not very practical at all.

1.88   Repeat Problem 1.87 using plastic ($E = 1.40 \times 10^9$ N/m^2) and rubber ($E = 7 \times 10^6$ N/m^2). Are any of these feasible?

Solution:

From problem 1.53, $k = 10^3$ N/m $= \dfrac{EA}{l}$

For plastic, $E = 1.40 \times 10^9$ N/m^2

So, $l = 140$ m

For rubber, $E = 7 \times 10^6$ N/m^2

So, $l = 0.7$ m

Rubber may be feasible, plastic would not.

1.89   Consider the diving board of Figure P1.89. For divers, a certain level of static deflection is desirable, denoted by Δ. Compute a design formula for the dimensions of the board (b, h and ℓ) in terms of the static deflection, the average diver's mass, m, and the modulus of the board.
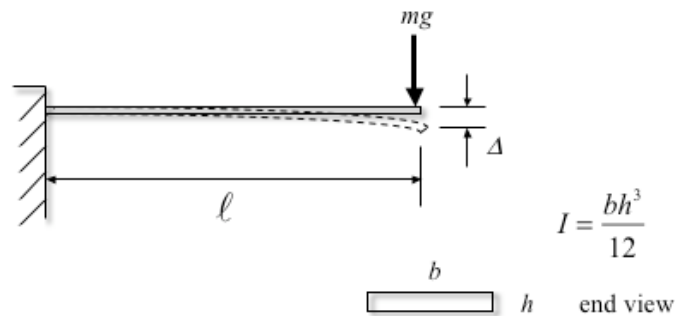
Figure P1.89

Solution: From Figure 1.15 (b), $\Delta k = mg$ holds for the static deflection. The period is:

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/\Delta}} = 2\pi\sqrt{\frac{\Delta}{g}} \qquad (1)$$

From Figure 1.24, we also have that

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m\ell^3}{3EI}} \qquad (2)$$

Equating (1) and (2) and replacing I with the value from the figure yields:

$$2\pi\sqrt{\frac{m\ell^3}{3EI}} = 2\pi\sqrt{\frac{12m\ell^3}{3Ebh^3}} = 2\pi\sqrt{\frac{\Delta}{g}} \Rightarrow \underline{\frac{\ell^3}{bh^3} = \frac{\Delta E}{4mg}}$$

Alternately just use the static deflection expression and the expression for the stiffness of the beam from Figure 1.24 to get

$$\Delta k = mg \Rightarrow \Delta\frac{3EI}{\ell^3} = mg \Rightarrow \underline{\frac{\ell^3}{bh^3} = \frac{\Delta E}{4mg}}$$

Problems and Solutions Section 1.8 (1.90 through 1.93)

1.90 Consider the system of Figure 1.90 and (a) write the equations of motion in terms of the angle, θ, the bar makes with the vertical. Assume linear deflections of the springs and linearize the equations of motion. Then (b) discuss the stability of the linear system's solutions in terms of the physical constants, m, k, and ℓ. Assume the mass of the rod acts at the center as indicated in the figure.
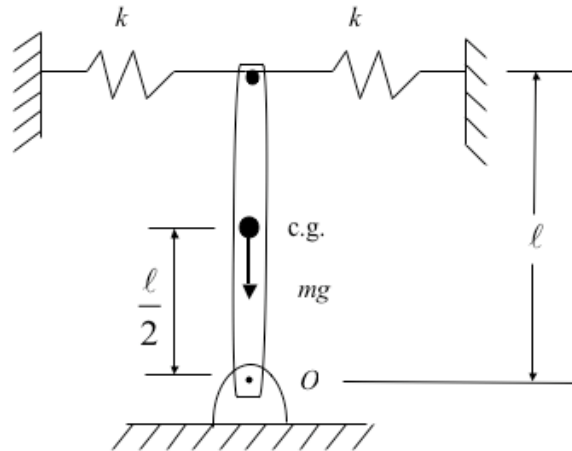
Figure P1.90

Solution: Note that from the geometry, the springs deflect a distance $kx = k(\ell \sin\theta)$ and the cg moves a distance $\frac{\ell}{2}\cos\theta$. Thus the total potential energy is

$$U = 2 \times \frac{1}{2}k(\ell\sin\theta)^2 - \frac{mg\ell}{2}\cos\theta$$

and the total kinetic energy is

$$T = \frac{1}{2}J_o\dot{\theta}^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2$$

The Lagrange equation (1.64) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) + \frac{\partial U}{\partial \theta} = \frac{d}{dt}\left(\frac{m\ell^2}{3}\dot{\theta}\right) + 2k\ell\sin\theta\cos\theta - \frac{1}{2}mg\ell\sin\theta = 0$$

Using the linear, small angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ yields

$$\text{a)} \quad \underline{\frac{m\ell^2}{3}\ddot{\theta} + \left(2k\ell^2 - \frac{mg\ell}{2}\right)\theta = 0}$$

Since the leading coefficient is positive the sign of the coefficient of θ determines the stability.

$$\text{if} \quad 2k\ell - \frac{mg}{2} > 0 \Rightarrow 4k > \frac{mg}{\ell} \Rightarrow \text{ the system is stable}$$

b)   if $4k = mg \Rightarrow \theta(t) = at + b \Rightarrow$ the system is unstable

$$\text{if} \quad 2k\ell - \frac{mg}{2} < 0 \Rightarrow 4k < \frac{mg}{\ell} \Rightarrow \text{ the system is unstable}$$

Note that physically this results states that the system's response is stable as long as the spring stiffness is large enough to over come the force of gravity.

1.91 Consider the inverted pendulum of Figure 1.37 as discussed in Example 1.8.1. Assume that a dashpot (of damping rate c) also acts on the pendulum parallel to the two springs. How does this affect the stability properties of the pendulum?
Solution: The equation of motion is found from the following FBD:
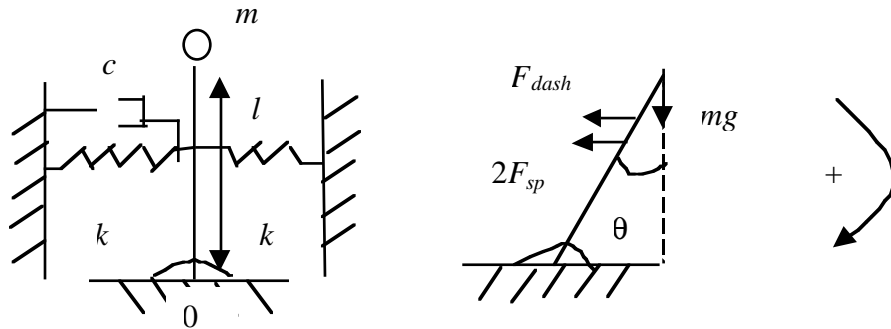
Moment about O: $\Sigma M_o = I\ddot{\theta}$

$$ml^2\ddot{\theta} = mgl\sin\theta - 2\frac{kl}{2}\sin\theta\left(\frac{l}{2}\cos\theta\right) - c\left(\frac{l}{2}\dot{\theta}\right)\left(\frac{l}{2}\cos\theta\right)$$

When θ is small, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$

$$ml^2\ddot{\theta} + \frac{cl^2}{4}\dot{\theta} + \left(\frac{kl^2}{2} - mgl\right)\theta = 0$$

$$ml\ddot{\theta} + \frac{cl}{4}\dot{\theta} + \left(\frac{kl}{2} - mg\right)\theta = 0$$

For stability, $\dfrac{kl}{2} > mg$ and $c \geq 0$.

The result of *adding a dashpot is to make the system asymptotically stable*.

1.92 Replace the massless rod of the inverted pendulum of Figure 1.37 with a solid object compound pendulum of Figure 1.20(b). Calculate the equations of vibration and discuss values of the parameter relations for which the system is stable.

Solution:
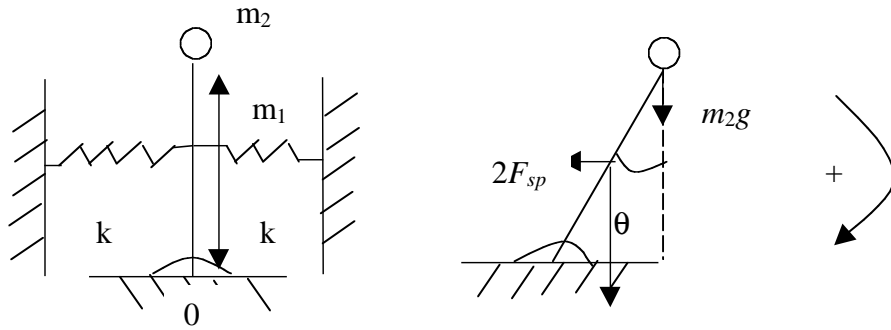
Moment about O: $\Sigma M_o = I\ddot{\theta}$

$$m_1 g \frac{l}{2}\sin\theta + m_2 gl\sin\theta - 2\frac{kl}{2}\sin\theta\left(\frac{l}{2}\cos\theta\right) = \left(\frac{1}{3}m_1 l^2 + m_2 l^2\right)\ddot{\theta}$$

When θ is small, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$.

$$\left(\frac{m_1}{3} + m_2\right)l^2\ddot{\theta} + \left(\frac{kl^2}{2} - \frac{m_1}{2}gl - m_2 gl\right)\theta = 0$$

$$\left(\frac{m_1}{3} + m_2\right)l\ddot{\theta} + \left[\frac{kl}{2} - \left(\frac{m_1}{2} + m_2\right)g\right]\theta = 0$$

For stability, $\dfrac{kl}{2} > \left(\dfrac{m_1}{2} + m_2\right)g$.

1.93 A simple model of a control tab for an airplane is sketched in Figure P1.93. The equation of motion for the tab about the hinge point is written in terms of the angle θ from the centerline to be

$$J\ddot{\theta} + (c - f_d)\dot{\theta} + k\theta = 0.$$

Here J is the moment of inertia of the tab, k is the rotational stiffness of the hinge, c is the rotational damping in the hinge and $f_d\dot{\theta}$ is the negative damping provided

by the aerodynamic forces (indicated by arrows in the figure). Discuss the stability of the solution in terms of the parameters c and f_d .
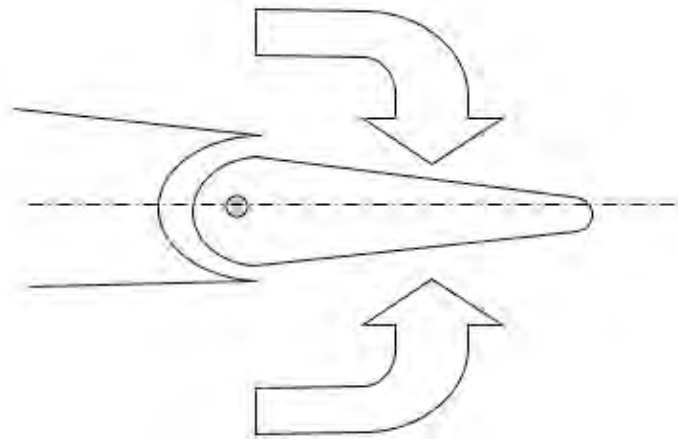
Figure P1.93 A simple model of an airplane control tab

Solution: The stability of the system is determined by the coefficient of $\dot{\theta}$ since the inertia and stiffness terms are both positive. There are three cases

Case 1 $c - f_d > 0$ and the system's solution is of the form $\theta(t) = e^{-at} \sin(\omega_n t + \phi)$ and the solution is asymptotically stable.

Case 2 $c - f_d < 0$ and the system's solution is of the form $\theta(t) = e^{at} \sin(\omega_n t + \phi)$ and the solution is oscillates and grows without bound, and exhibits flutter instability as illustrated in Figure 1.36.

Case 3 $c = f_d$ and the system's solution is of the form $\theta(t) = A \sin(\omega_n t + \phi)$ and the solution is stable as illustrated in Figure 1.34.

Problems and Solutions Section 1.9 (1.94 through 1.101)

1.94* Reproduce Figure 1.38 for the various time steps indicated.

Solution: The code is given here in Mathcad, which can be run repeatedly with different Δt to see the importance of step size. Matlab and Mathematica can also be used to show this.
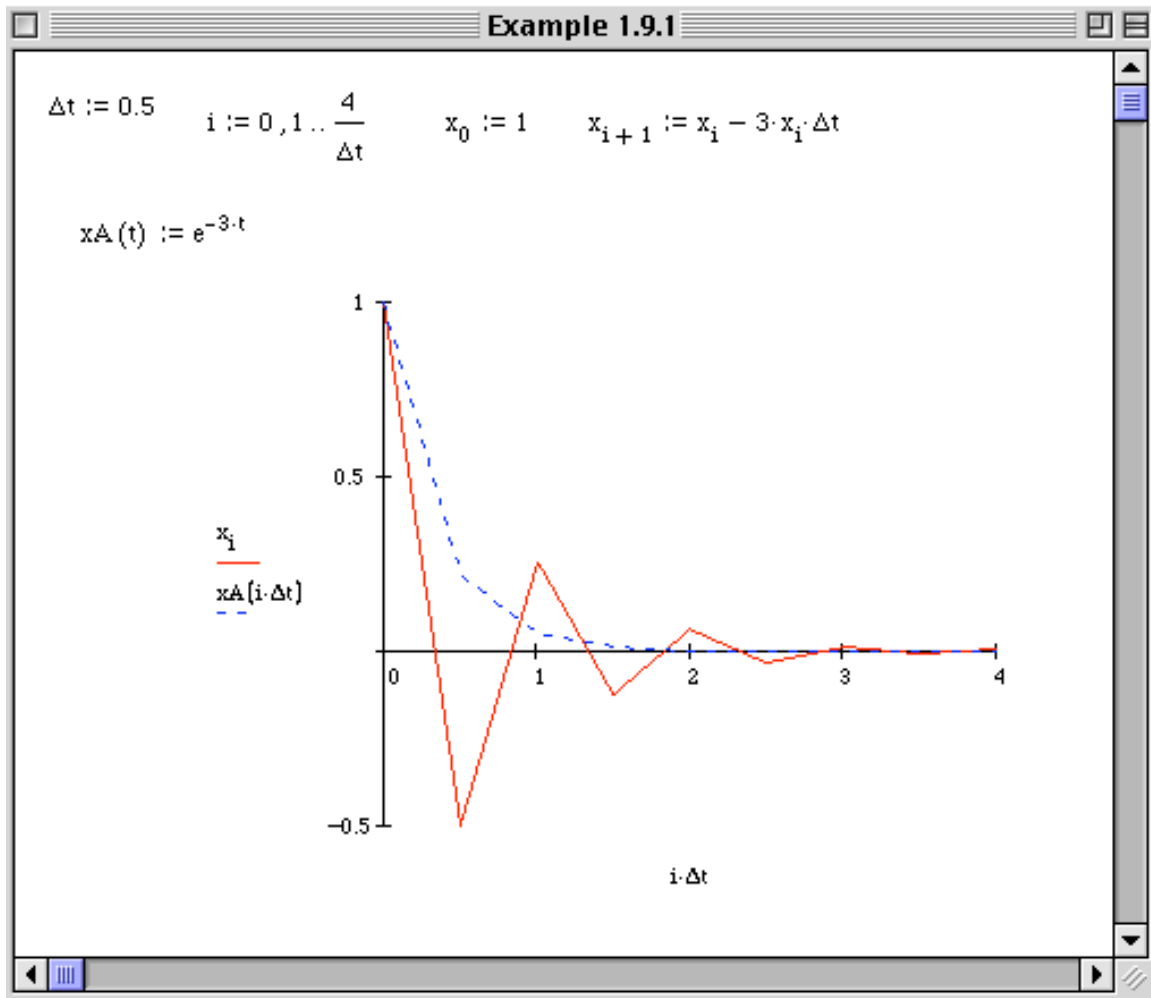
1.95* Use numerical integration to solve the system of Example 1.7.3 with $m = 1361$ kg, $k = 2.688 \times 10^5$ N/m, $c = 3.81 \times 10^3$ kg/s subject to the initial conditions $x(0) = 0$ and $v(0) = 0.01$ mm/s. Compare your result using numerical integration to just plotting the analytical solution (using the appropriate formula from Section 1.3) by plotting both on the same graph.

Solution: The solution is shown here in Mathcad using an Euler integration. This can also been done in the other codes or the Toolbox:

$$k := 2.688 \cdot 10^5 \quad m := 1361 \quad x_0 := 0.0 \quad v_0 := 0.01 \quad c := 3.81 \cdot 10^3 \quad \text{all units in m, kg, s}$$

$$\omega n := \sqrt{\frac{k}{m}} \qquad fn := \frac{\omega n}{2 \cdot \pi} \quad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}} \quad \omega d := \omega n \cdot \sqrt{1 - \zeta^2}$$

$$A := \frac{1}{\omega d} \cdot \sqrt{\left(v_0 + \zeta \cdot x_0 \cdot \omega n\right)^2 + \left(x_0 \cdot \omega d\right)^2}$$

$$T := \frac{2 \cdot \pi}{\omega d}$$

$$\phi := \operatorname{atan}\left[\frac{\omega d \cdot x_0}{v_0 + \zeta \cdot \omega n \cdot x_0}\right] \qquad \zeta = 0.1$$

$$X(t) := A \cdot \sin\left(\omega d \cdot t + \phi\right) \cdot e^{-\zeta \cdot \omega n \cdot t}$$

$$N := 10000 \qquad \Delta := \frac{6 \cdot \pi}{\omega n \cdot N} \qquad y := \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$$

$$i := 0 .. N$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} v_i \cdot \Delta + x_i \\ -\omega n^2 \cdot (\Delta) \cdot x_i - 2 \cdot \zeta \cdot \omega n \cdot v_i \cdot \Delta + v_i \end{bmatrix}$$
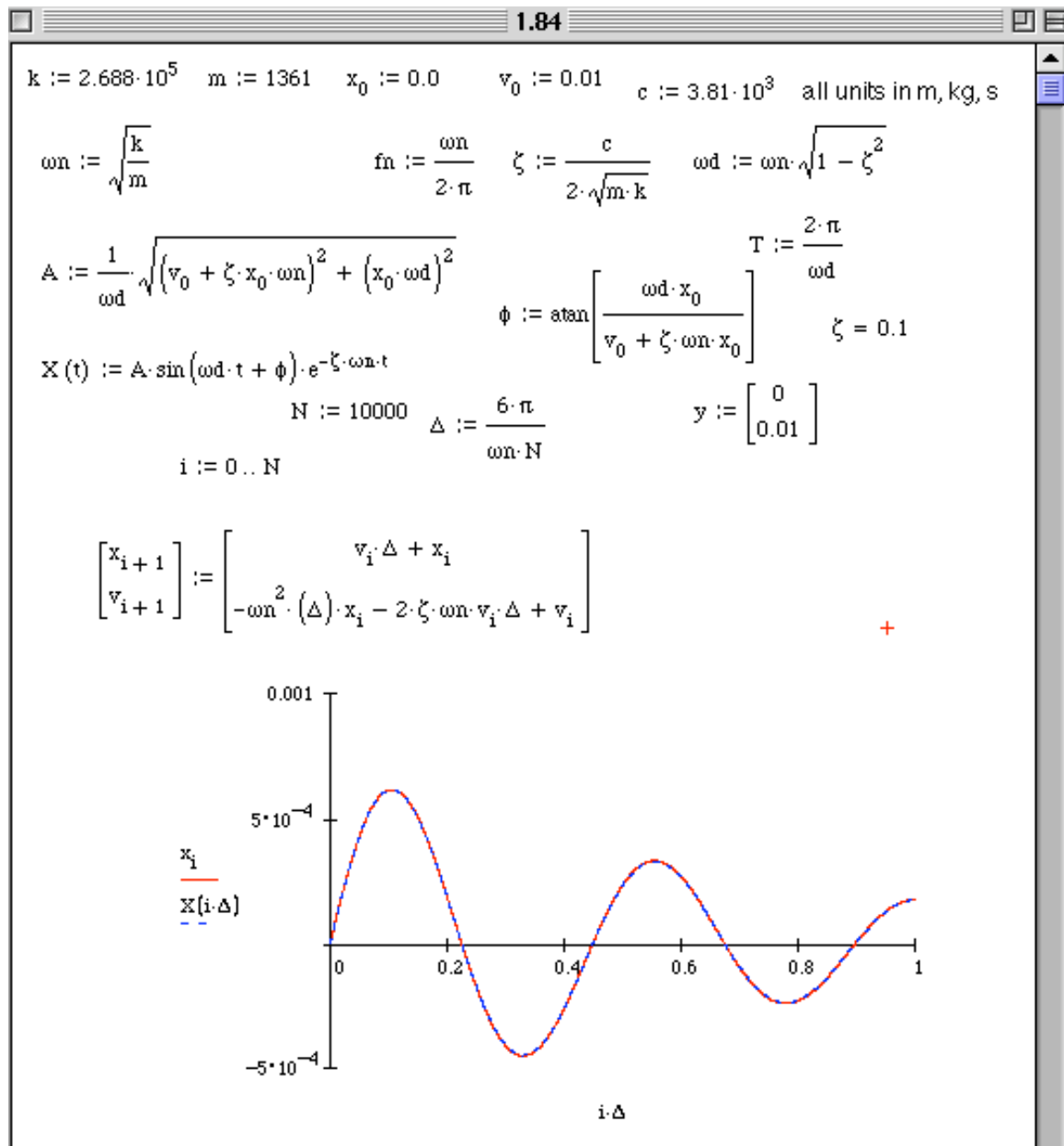
1.96* Consider again the damped system of Problem 1.95 and design a damper such that the oscillation dies out after 2 seconds. There are at least two ways to do this. Here it is intended to solve for the response numerically, following Examples 1.9.2, 1.9.3 or 1.9.4, using different values of the damping parameter *c* until the desired response is achieved.

Solution: Working directly in Mathcad (or use one of the other codes). Changing *c* until the response dies out within about 2 sec yields *c* =6500 kg/s or $\zeta = 0.17$.

1.85

$k := 2.688 \cdot 10^5 \quad m := 1361 \quad x_0 := 0.0 \quad v_0 := 0.01 \quad c := 6.5 \cdot 10^3 \quad$ all units in m, kg, s

$$\omega n := \sqrt{\frac{k}{m}} \qquad fn := \frac{\omega n}{2 \cdot \pi} \qquad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}} \qquad \omega d := \omega n \cdot \sqrt{1 - \zeta^2} \qquad T := \frac{2 \cdot \pi}{\omega d}$$

$$A := \frac{1}{\omega d} \cdot \sqrt{\left(v_0 + \zeta \cdot x_0 \cdot \omega n\right)^2 + \left(x_0 \cdot \omega d\right)^2} \qquad \phi := atan\left[\frac{\omega d \cdot x_0}{v_0 + \zeta \cdot \omega n \cdot x_0}\right] \qquad y := \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

$$D(t,y) := \begin{bmatrix} y_1 \\ -\left(2 \cdot \zeta \cdot \omega n \cdot y_1\right) - \left(\omega n^2\right) \cdot y_0 \end{bmatrix}$$

$Z := rkfixed(y, 0, 20, 1000, D)$

$A = 7.221 \cdot 10^{-4} \quad$ in meters

$\dfrac{\phi}{deg} = 0 \quad$ in degrees $\qquad t := Z^{<0>}$

$\omega n = 14.054 \qquad fn = 2.237 \qquad T = 0.454$

in rad/s $\qquad$ in Hz $\qquad$ in sec $\qquad \zeta = 0.17 \qquad y := Z^{<1>} \qquad +$

$\omega d = 13.849$
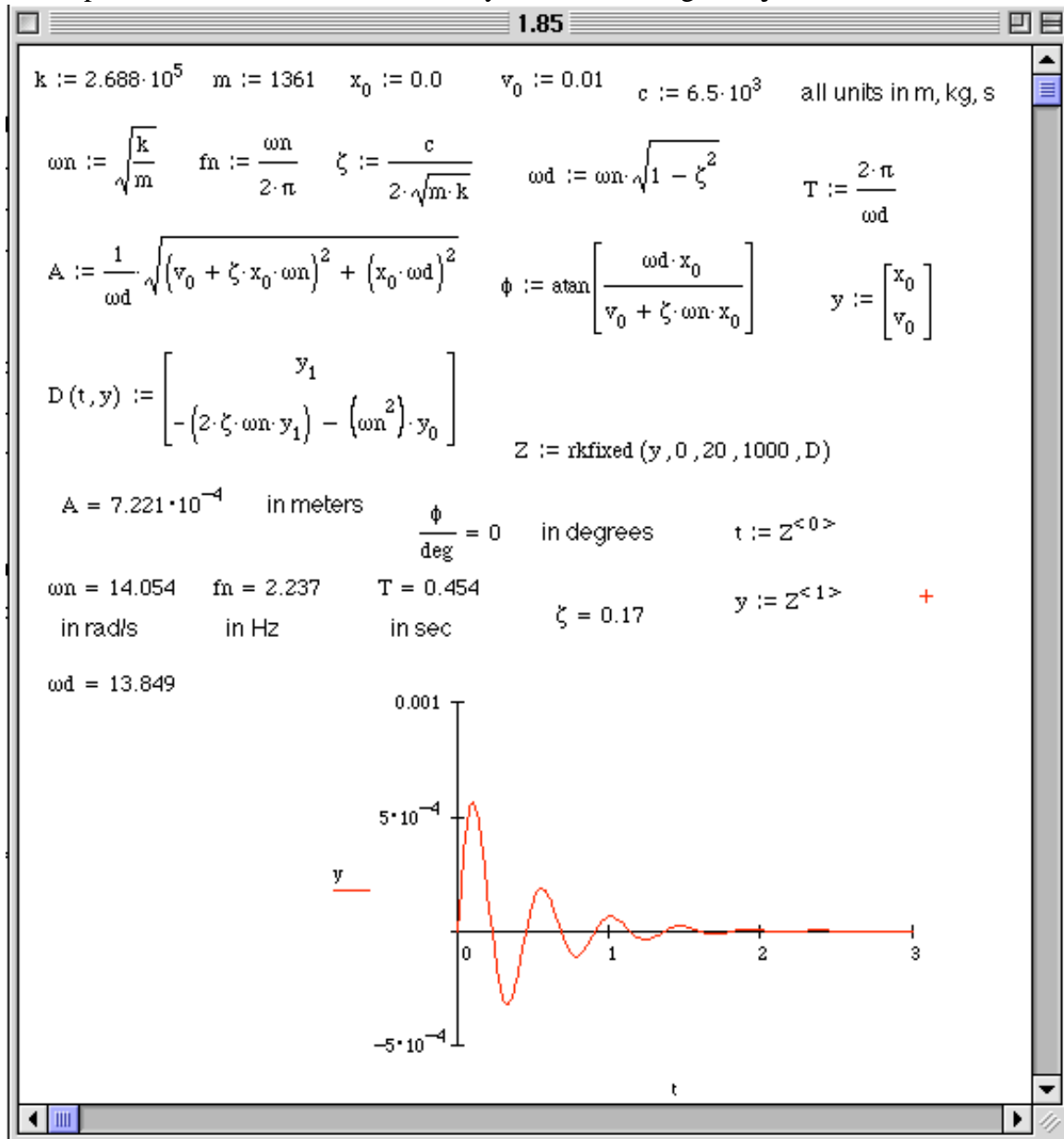
1.97* Consider again the damped system of Example 1.9.2 and design a damper such that the oscillation dies out after 25 seconds. There are at least two ways to do this. Here it is intended to solve for the response numerically, following Examples 1.9.2, 1.9.3 or 1.9.4, using different values of the damping parameter *c* until the desired response is achieved. Is your result overdamped, underdamped or critically damped?

Solution: The following Mathcad program is used to change *c* until the desired response results. This yields a value of $c = 1.1$ kg/s or $\zeta = 0.225$, an underdamped solution.
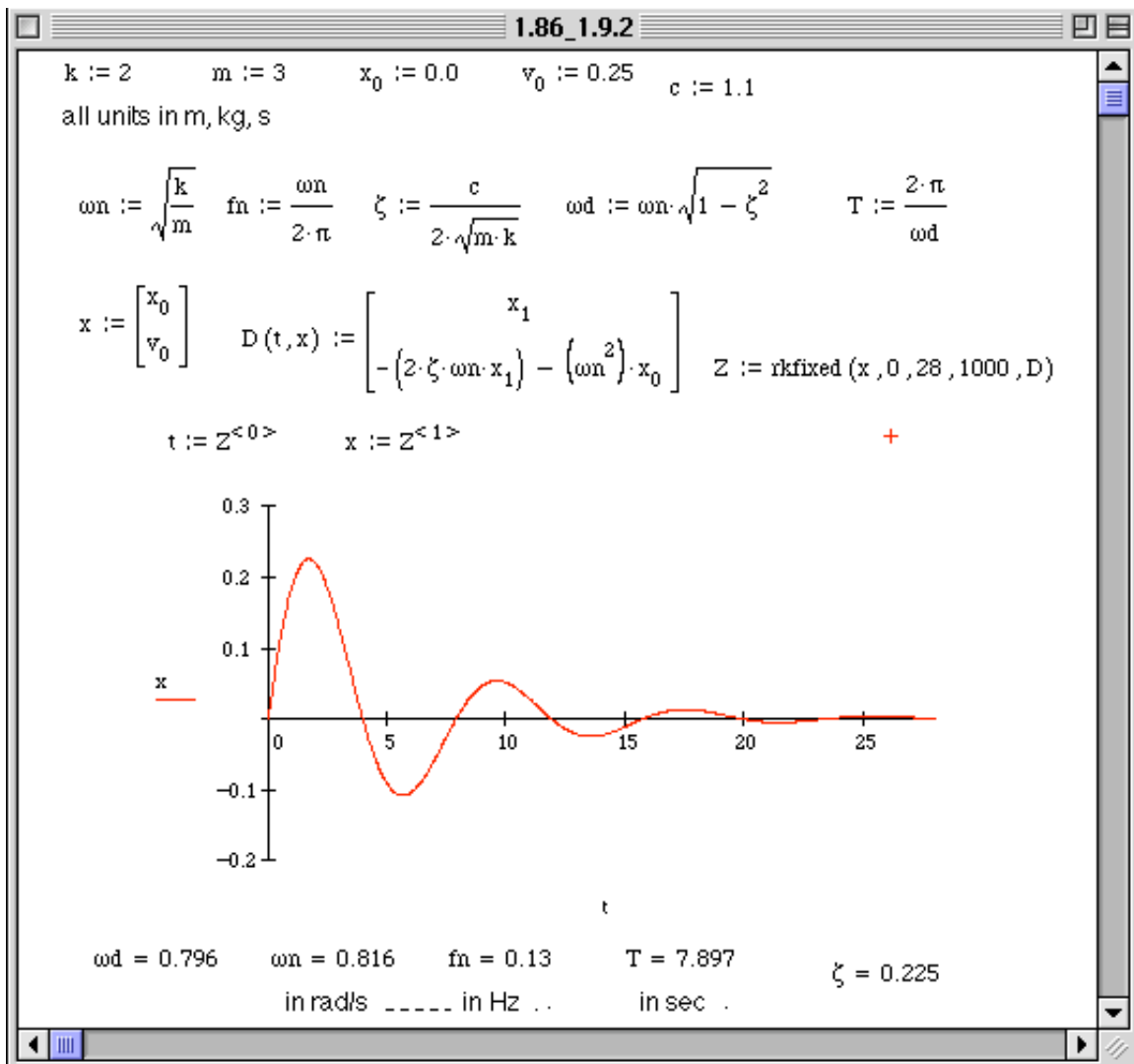
1.98* Repeat Problem 1.96 for the initial conditions $x(0) = 0.1$ m and $v(0) = 0.01$ mm/s.

Solution: Using the code in 1.96 and changing the initial conditions does not change the settling time, which is just a function of ζ and ω_n. Hence the value of c = 6.5×10^3 kg/s (ζ = 0.17) as determined in problem 1.96 will still reduce the response within 2 seconds.

1.99* A spring and damper are attached to a mass of 100 kg in the arrangement given in Figure 1.9. The system is given the initial conditions $x(0) = 0.1$ m and $v(0) = 1$ mm/s. Design the spring and damper (i.e. choose k and c) such that the system will come to rest in 2 s and not oscillate more than two complete cycles. Try to keep c as small as possible. Also compute ζ.

Solution: In performing this numerical search on two parameters, several underdamped solutions are possible. Students will note that increasing k will decrease ζ. But increasing k also increases the number of cycles which is limited to two. A solution with $c = 350$ kg/s and $k = 2000$ N/m is illustrated.

```
                                    1.88
```

$$k := 2000 \qquad m := 100 \qquad x_0 := 0.1 \qquad v_0 := 0.01 \qquad c := 350 \qquad \text{all units in m, kg, s}$$

$$\omega n := \sqrt{\frac{k}{m}} \qquad fn := \frac{\omega n}{2 \cdot \pi} \qquad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}} \qquad \omega d := \omega n \cdot \sqrt{1 - \zeta^2} \qquad T := \frac{2 \cdot \pi}{\omega d}$$

$$x := \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \qquad D(t,x) := \begin{bmatrix} x_1 \\ -\left(2 \cdot \zeta \cdot \omega n \cdot x_1\right) - \left(\omega n^2\right) \cdot x_0 \end{bmatrix} \qquad Z := \text{rkfixed}(x, 0, 4, 1000, D)$$

$$t := Z^{<0>} \qquad x := Z^{<1>}$$

$$\omega d = 4.116 \qquad \omega n = 4.472 \qquad fn = 0.712 \qquad T = 1.527 \qquad \zeta = 0.391$$

$$\qquad\qquad\quad \text{in rad/s} \qquad\quad \text{in Hz} \qquad\quad \text{in sec}$$
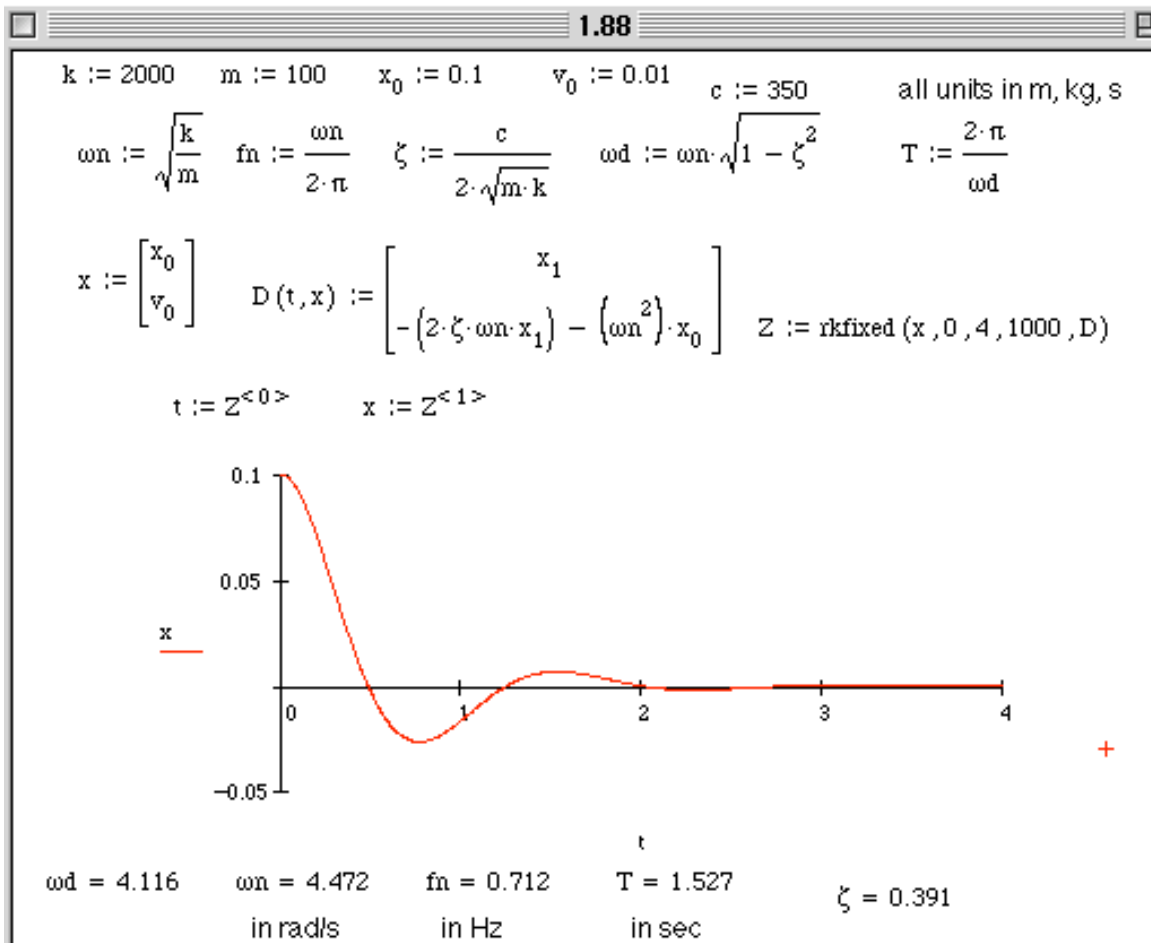
1.100* Repeat Example 1.7.1 by using the numerical approach of the previous 5 problems.

Solution: The following Mathcad session can be used to solve this problem by varying the damping for the fixed parameters given in Example 1.7.1.

1.89

$k := 200$ $m := 2$ $x_0 := 0.$ $v_0 := 0.3$ $c := 8$ all units in m, kg, s

$$\omega n := \sqrt{\frac{k}{m}} \quad fn := \frac{\omega n}{2 \cdot \pi} \quad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}} \quad \omega d := \omega n \cdot \sqrt{1 - \zeta^2} \quad T := \frac{2 \cdot \pi}{\omega d}$$

$$x := \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \quad D(t,x) := \begin{bmatrix} x_1 \\ -\left(2 \cdot \zeta \cdot \omega n \cdot x_1\right) - \left(\omega n^2\right) \cdot x_0 \end{bmatrix} \quad Z := \text{rkfixed}(x, 0, 4, 1000, D)$$

$$t := Z^{<0>} \qquad x := Z^{<1>} \qquad +$$

$\omega d = 9.798$ $\omega n = 10$ $fn = 1.592$ $T = 0.641$ $\zeta = 0.2$

in rad/s   in Hz   in sec
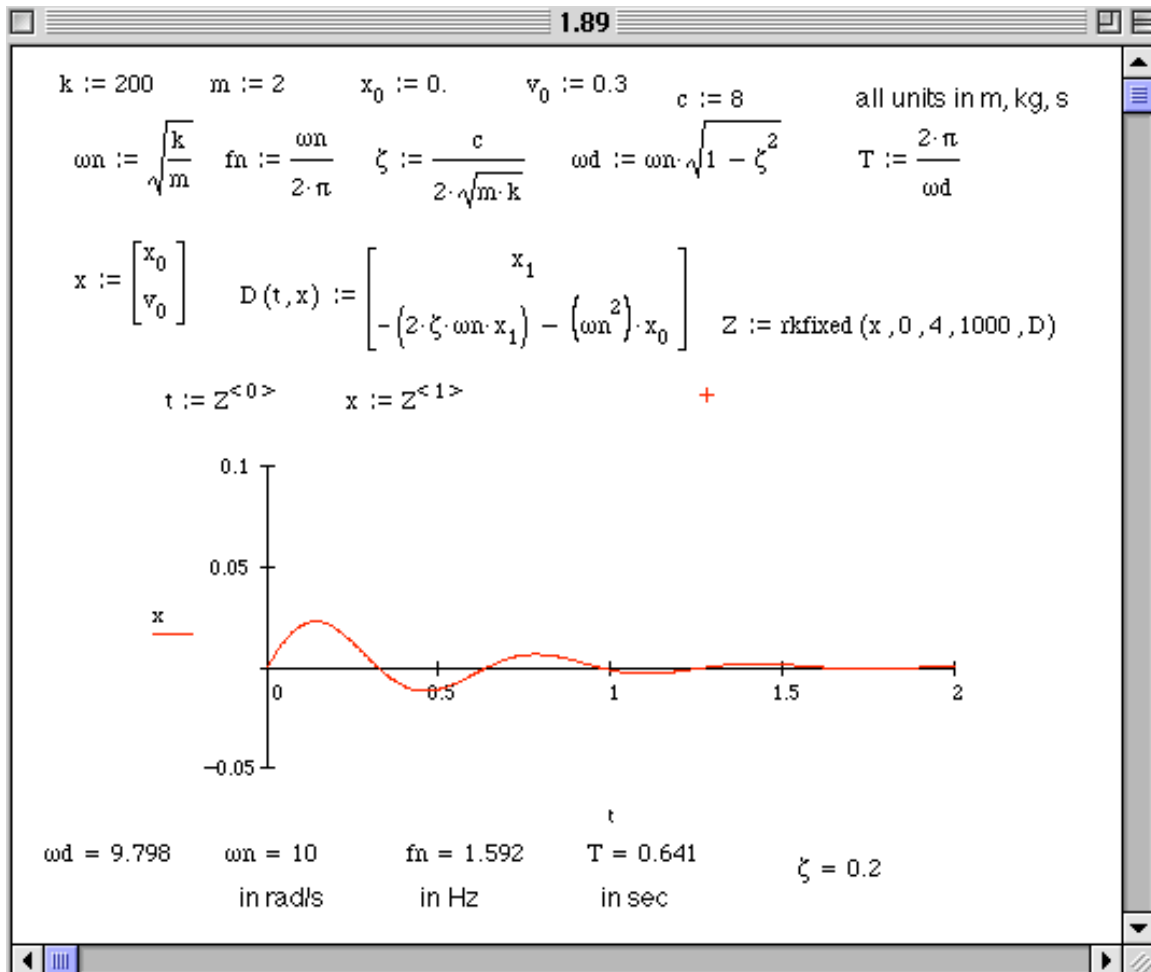
The other codes or the toolbox may also be used to do this.

1.101* Repeat Example 1.7.1 for the initial conditions $x(0) = 0.01$ m and $v(0) = 1$ mm/s.

Solution: The above Mathcad session can be used to solve this problem by varying the damping for the fixed parameters given in Example 1.7.1. For the given values of initial conditions, the solution to Problem 1.100 also works in this case. Note that if $x(0)$ gets too large, this problem will not have a solution.

Problems and Solutions Section 1.10 (1.102 through 1.114)

1.102 A 2-kg mass connected to a spring of stiffness 10^3 N/m has a dry sliding friction force (F_c) of 3 N. As the mass oscillates, its amplitude decreases 20 cm. How long does this take?

Solution: With $m = 2$kg, and $k = 1000$ N/m the natural frequency is just

$$\omega_n = \sqrt{\frac{1000}{2}} = 22.36 \text{ rad/s}$$

From equation (1.101): slope $= \dfrac{-2\mu mg\omega_n}{\pi k} = \dfrac{-2F_c\omega_n}{\pi k} = \dfrac{\Delta x}{\Delta t}$

Solving the last equality for Δt yields:

$$\Delta t = \frac{-\Delta x\pi k}{2f_c\omega_n} = \frac{-(0.20)(\pi)(10^3)}{2(3)(22.36)} = \underline{4.68 \text{ s}}$$

1.103 Consider the system of Figure 1.41 with $m = 5$ kg and $k = 9 \times 10^3$ N/m with a friction force of magnitude 6 N. If the initial amplitude is 4 cm, determine the amplitude one cycle later as well as the damped frequency.

Solution: Given $m = 5$ kg, $k = 9\times10^3$ N/m, $f_c = 6$ N, $x_0 = 0.04$ m, the amplitude after one cycle is $x_1 = x_0 - \dfrac{4f_c}{k} = 0.04 - \dfrac{(4)(6)}{9\times10^3} = \underline{0.0373 \text{ m}}$

Note that the damped natural frequency is the same as the natural frequency in the case of Coulomb damping, hence $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{9\times10^3}{5}} = \underline{42.43 \text{ rad/s}}$

1.104* Compute and plot the response of the system of Figure P1.104 for the case where $x_0 = 0.1$ m, $v_0 = 0.1$ m/s, $\mu_\kappa = 0.05$, $m = 250$ kg, $\theta = 20°$ and $k = 3000$ N/m. How long does it take for the vibration to die out?
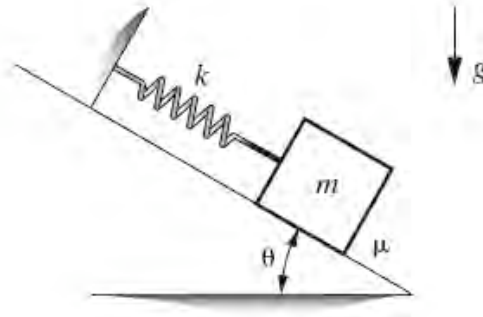
Figure P1.104

Solution: Choose the x y coordinate system to be along the incline and perpendicular to it. Let μ_s denote the static friction coefficient, μ_k the coefficient of kinetic friction and Δ the static deflection of the spring. A drawing indicating the angles and a free-body diagram is given in the figure:
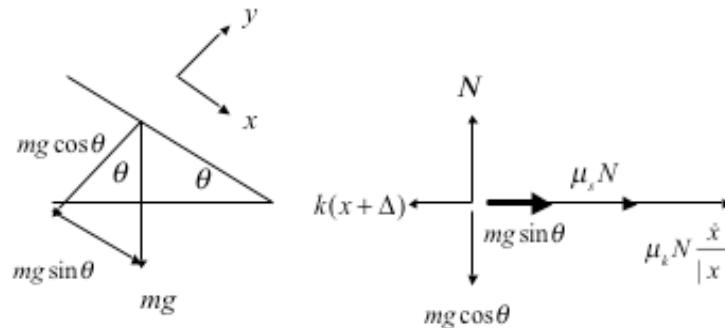
For the static case

$$\sum F_x = 0 \Rightarrow k\Delta = \mu_s N + mg\sin\theta, \text{ and } \sum F_y = 0 \Rightarrow N = mg\cos\theta$$

For the dynamic case

$$\sum F_x = m\ddot{x} = -k(x + \Delta) + \mu_s N + mg\sin\theta - \mu_k N\frac{\dot{x}}{|\dot{x}|}$$

Combining these three equations yields

$$m\ddot{x} + \mu_k mg\cos\theta\frac{\dot{x}}{|\dot{x}|} + kx = 0$$

Note that as the angle θ goes to zero the equation of motion becomes that of a spring mass system with Coulomb friction on a flat surface as it should.

Answer: The oscillation dies out after about 0.9 s. This is illustrated in the following Mathcad code and plot.

$$X := \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \qquad k := 3000 \qquad m := 250 \qquad \mu := 0.05$$

$$D(t, X) := \begin{bmatrix} X_1 \\ -\dfrac{k}{m} \cdot X_0 - \cos(20 \deg) \cdot \mu \cdot g \cdot \dfrac{X_1}{|X_1|} \end{bmatrix}$$

$$Z := \text{rkfixed}(X, 0, 10, 5000, D)$$

$$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle}$$
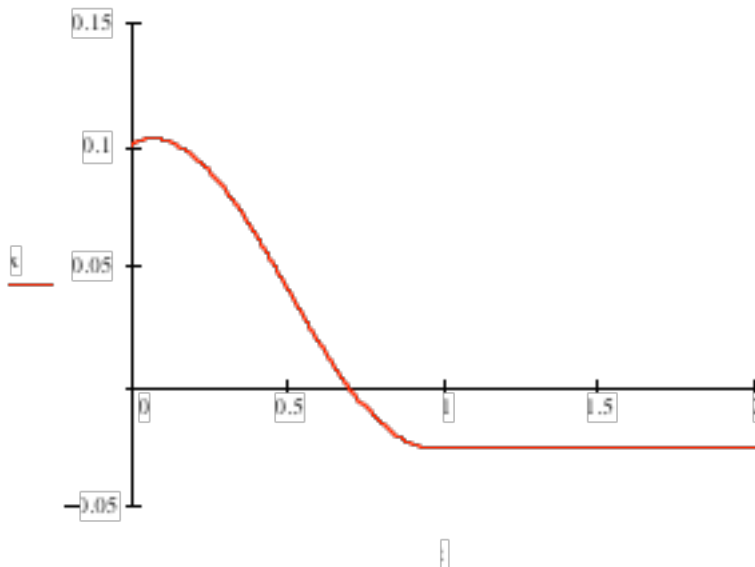
Alternate Solution (Courtesy of Prof. Chin An Tan of Wayne State University):

Static Analysis:

In this problem, $x(t)$ is defined as the displacement of the mass from the equilibrium position of the spring-mass system under friction. Thus, the first issue to address is how to determine this equilibrium position, or what is this equilibrium position. In reality, the mass is attached onto an initially unstretched spring on the incline. The free body diagram of the system is as shown. The governing equation of motion is:

$$m\ddot{X} = -k\overbrace{X}^{\text{zero initially}} - F_f + mg \sin\theta$$

where $X(t)$ is defined as the displacement measured from the unstretched position of the spring. Note that since the spring is initially unstretched, the spring force $F_s = kX$ is zero
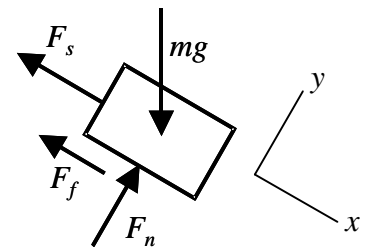
initially. If the coefficient of static friction μ_s is sufficiently large, i.e., $\mu_s > \tan(\theta)$, then the mass remains stationary and the spring is unstretched with the mass-spring-friction in equilibrium. Also, in that case, the friction force $F_f \leq \mu_s \underbrace{mg \cos\theta}_{F_N}$, not necessarily equal to the maximum static friction. In other words, these situations may hold at equilibrium: (1) the maximum static friction may not be achieved; and (2) there may be no displacement in the spring at all. In this example, $\tan(20^\circ) = 0.364$ and one would expect that μ_s (not given) should be smaller than 0.364 since $\mu_k = 0.05$ (very small). Thus, one would expect the mass to move downward initially (due to weight overcoming the maximum static friction). The mass will then likely oscillate and eventually settle into an equilibrium position with the spring stretched.

Dynamic Analysis:

The equation of motion for this system is:

$$m\ddot{x} = -kx - \mu mg \cos\theta \frac{\dot{x}}{|\dot{x}|}$$

where $x(t)$ is the displacement measured from the equilibrium position. Define $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$. Employing the state-space formulation, we transform the original second-order ODE into a set of two first-order ODEs. The state-space equations (for MATLAB code) are:

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} x_2(t) \\ -\mu g \cos\theta \dfrac{x_2}{|x_2|} - \dfrac{kx_1}{m} \end{Bmatrix}$$

MATLAB Code:

```
x0=[0.1, 0.1];
ts=[0, 5];
[t,x]=ode45('f1_93',ts,x0);
plot(t,x(:,1), t,x(:,2))
title('problem 1.93'); grid on;
xlabel('time (s)');ylabel('displacement (m), velocity (m/s)');


%----------------------------------------------
function xdot = f1_93(t,x)
% computes derivatives for the state-space ODEs
m=250; k=3000; mu=0.05; g=9.81;
angle = 20*pi/180;
xdot(1) = x(2);
xdot(2) = -k/m*x(1) - mu*g*cos(angle)*sign(x(2));
% use the sign function to improve computation time
xdot = [xdot(1); xdot(2)];
```

Plots for $\mu = 0.05$ and $\mu = 0.02$ cases are shown. From the $\mu = 0.05$ simulation results, the oscillation dies out after about 0.96 seconds (using `ginput(1)` command to estimate). Note that the acceleration may be discontinuous at $v = 0$ due to the nature of the friction force.

Effects of μ:

Comparing the figures, we see that reducing μ leads to more oscillations (takes longer time to dissipate the energy). Note that since there is a positive initial velocity, the mass is bounded to move down the incline initially. However, if μ is sufficiently large, there may be no oscillation at all and the mass will just come to a stop (as in the case of

$\mu = 0.05$). This is analogous to an overdamped mass-damper-spring system. On the other hand, when μ is very small (say, close to zero), the mass will oscillate for a long time before it comes to a stop.

problem 1.93

$x_0 = 0.1$ m, $v_0 = 0.1$ m/s
$\mu = 0.05$, m = 250 kg
k = 3,000 N/m, $\theta = 20^\circ$

x(t)

$x_{ss} = -0.0261$

The mass has no oscillation due to sufficiently large friction.

v(t)

problem 1.93

$x_0 = 0.1$ m, $v_0 = 0.1$ m/s
$\mu = 0.02$, m = 250 kg
k = 3,000 N/m, $\theta = 20^\circ$

v(t)

x(t)

$x_{ss} = -0.0114$

a(t) is discontinuous due to friction force changes direction as the mass changes its direction of motion.
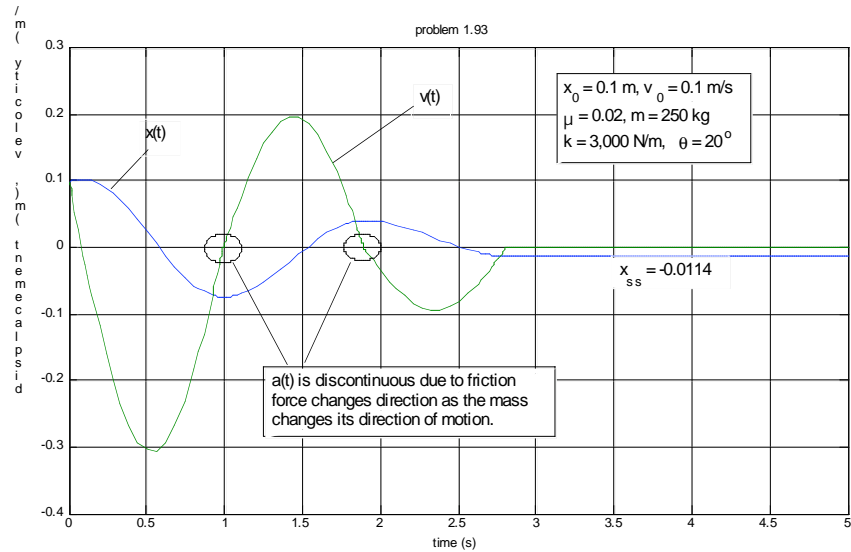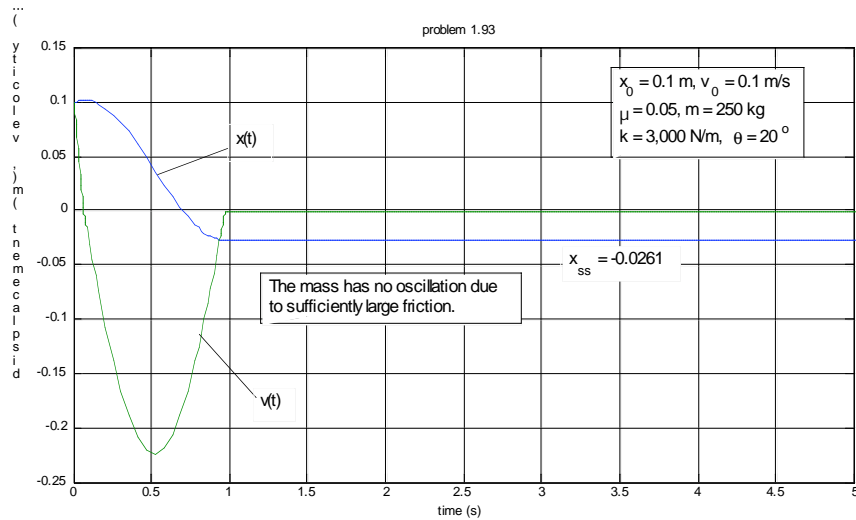
Discussion on the ceasing of motion:

Note that when motion ceases, the mass reaches another state of equilibrium. In both simulation cases, this occurs while the mass is moving upward (negative velocity). Note that the steady-state value of $x(t)$ is very small, suggesting that this is indeed the true equilibrium position, which represents a balance of the spring force, weight component along the incline, and the static friction.

1.105* Compute and plot the response of a system with Coulomb damping of equation (1.90) for the case where $x_0 = 0.5$ m, $v_0 = 0$, $\mu = 0.1$, $m = 100$ kg and $k = 1500$ N/m. How long does it take for the vibration to die out?

Solution: Here the solution is computed in Mathcad using the following code. Any of the codes may be used. The system dies out in about 3.2 sec.
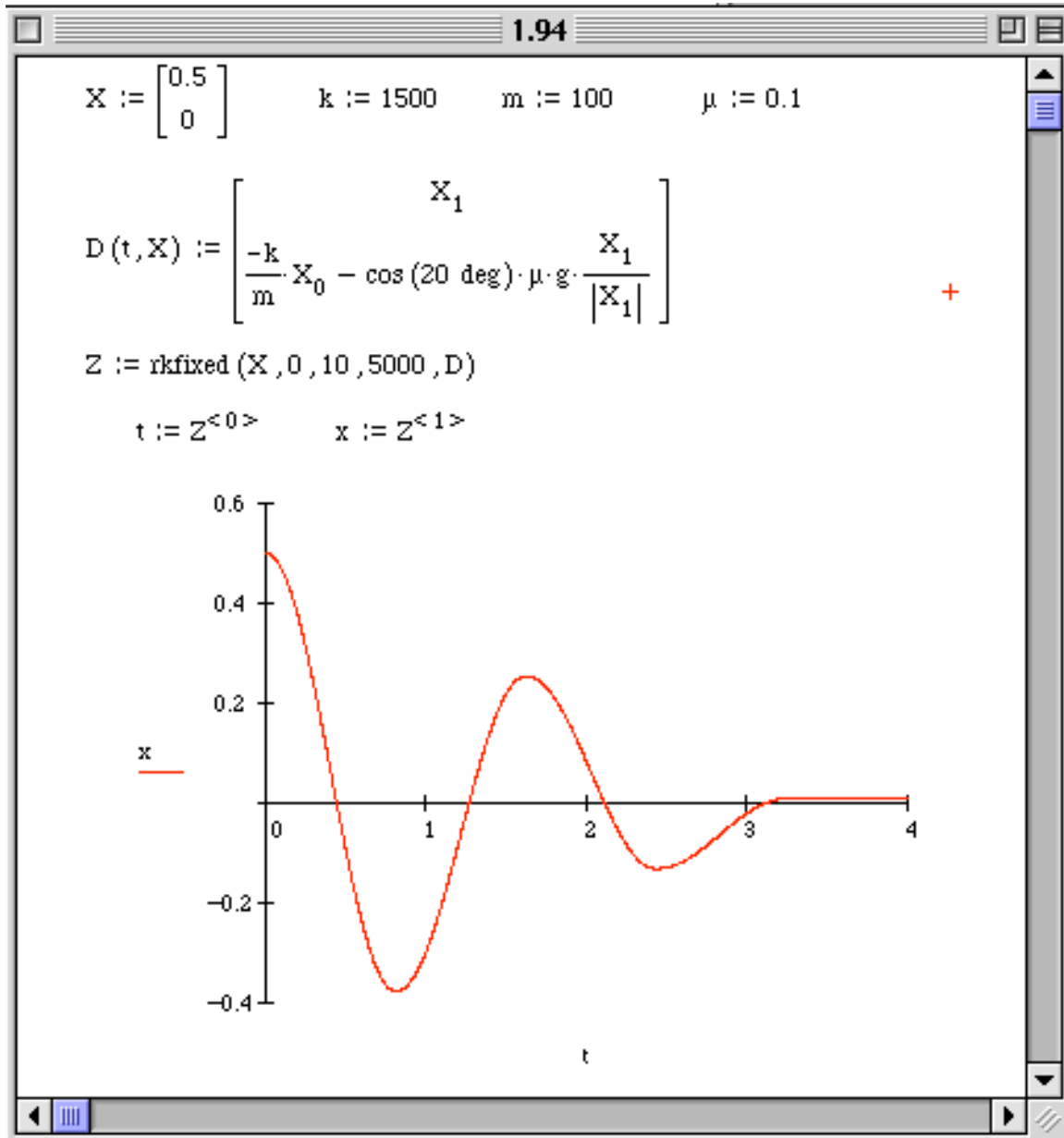
1.106* A mass moves in a fluid against sliding friction as illustrated in Figure P1.106. Model the damping force as a slow fluid (i.e., linear viscous damping) plus Coulomb friction because of the sliding, with the following parameters: $m = 250$ kg, $\mu = 0.01$, $c = 25$ kg/s and $k = 3000$ N/m . a) Compute and plot the response to the initial conditions: $x_0 = 0.1$ m, $v_0 = 0.1$ m/s. b) Compute and plot the response to the initial conditions: $x_0 = 0.1$ m, $v_0 = 1$ m/s. How long does it take for the vibration to die out in each case?
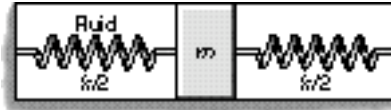
Figure P1.106

Solution: A free-body diagram yields the equation of motion.
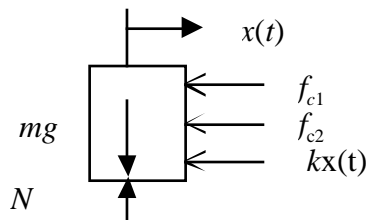
$$m\ddot{x}(t) + \mu mg\,\mathrm{sgn}(\dot{x}) + c\dot{x}(t) + kx(t) = 0$$

where the vertical sum of forces gives

the magnitude $\mu N = \mu mg$ for the

Coulomb force as in figure 1.41.

The equation of motion can be solved by using any of the codes mentioned or by using the toolbox. Here a Mathcad session is presented using a fixed order Runge Kutta integration. Note that the oscillations die out after 4.8 seconds for v_0=0.1 m/s for the larger initial velocity of v_0=1 m/s the oscillations go on quite a bit longer ending only after about 13 seconds. While the next problem shows that the viscous damping can be changed to reduce the settling time, this example shows how dependent the response is on the value of the initial conditions. In a linear system the settling time, or time it takes to die out is only dependent on the system parameters, not the initial conditions. This makes design much more difficult for nonlinear systems.

c := 25   k := 3000   m := 250 μ := 0.01 $X := \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$ $X2 := \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$

$$D(t,X) := \begin{bmatrix} X_1 \\ \dfrac{-k}{m} \cdot X_0 - \mu \cdot g \cdot \dfrac{X_1}{|X_1|} - \dfrac{c}{m} \cdot X_1 \end{bmatrix} \qquad D2(t,X2) := \begin{bmatrix} X2_1 \\ \dfrac{-k}{m} \cdot X2_0 - \mu \cdot g \cdot \dfrac{X2_1}{|X2_1|} - \dfrac{c}{m} \cdot X2_1 \end{bmatrix}$$

Z := rkfixed(X, 0, 15, 3000, D)   Z2 := rkfixed(X2, 0, 15, 3000, D2)

$t := Z^{<0>}$ $x := Z^{<1>}$ $x2 := Z2^{<1>}$
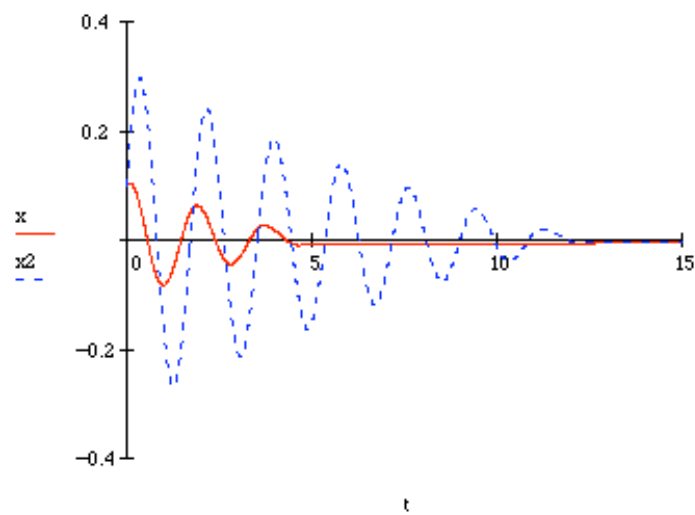
1.107* Consider the system of Problem 1.106 part (a), and compute a new damping coefficient, c, that will cause the vibration to die out after one oscillation.

Solution: Working in any of the codes, use the simulation from the last problem and change the damping coefficient c until the desired response is obtained. A Mathcad solution is given which requires an order of magnitude higher damping coefficient,

$$c = 275 \text{ kg/s}$$
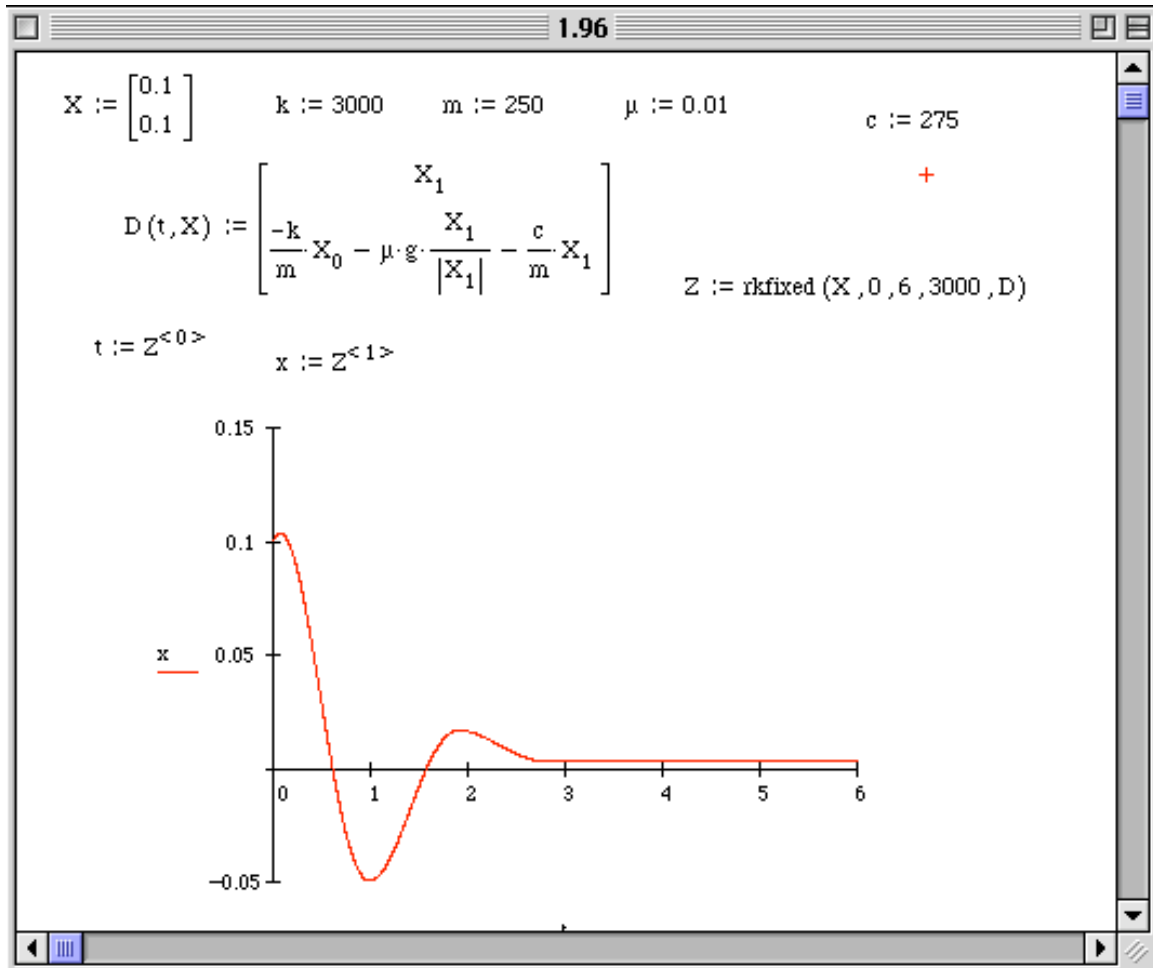
1.108 Compute the equilibrium positions of $\ddot{x} + \omega_n^2 x + \beta x^2 = 0$. How many are there?

Solution: The equation of motion in state space form is
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_n^2 x_1 - \beta x_1^2$$
The equilibrium points are computed from:
$$x_2 = 0$$
$$-\omega_n^2 x_1 - \beta x_1^2 = 0$$
Solving yields the two equilibrium points:
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\omega_n^2 \big/ \beta \\ 0 \end{bmatrix}$$

1.109 Compute the equilibrium positions of $\ddot{x} + \omega_n^2 x - \beta^2 x^3 + \gamma x^5 = 0$. How many are there?

Solution: The equation of motion in state space form is
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_n^2 x_1 + \beta^2 x_1^3 - \gamma x_1^5$$
The equilibrium points are computed from:
$$x_2 = 0$$
$$-\omega_n^2 x_1 + \beta^2 x_1^3 - \gamma x_1^5 = 0$$
Solving yields the five equilibrium points (one for each root of the previous equation). The first equilibrium (the linear case) is:
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
Next divide $-\omega_n^2 x_1 + \beta^2 x_1^3 - \gamma x_1^5 = 0$ by x_1 to obtain:
$$-\omega_n^2 + \beta^2 x_1^2 - \gamma x_1^4 = 0$$
which is quadratic in x_1^2 and has the following roots which define the remaining four equilibrium points: $x_2 = 0$ and
$$x_1 = \pm\sqrt{\frac{-\beta^2 + \sqrt{\beta^4 - 4\gamma\omega_n^2}}{-2\gamma}}$$
$$x_1 = \pm\sqrt{\frac{-\beta^2 - \sqrt{\beta^4 - 4\gamma\omega_n^2}}{-2\gamma}}$$

1.110* Consider the pendulum example 1.10.3 with length of 1 m an initial conditions of $\theta_0 = \pi/10$ rad and $\dot{\theta}_0 = 0$. Compare the difference between the response of the linear version of the pendulum equation (i.e. with $\sin(\theta) = \theta$) and the response of the nonlinear version of the pendulum equation by plotting the response of both for four periods.

> **Solution:** First consider the linear solution. Using the formula's given in the text the solution of the linear system is just: $\theta(t) = 0.314\sin(3.132t + \pi/2)$. The following Mathcad code, plots the linear solution on the same plot as a numerical solution of the nonlinear system.

$i := 0.. \ 800$

$\Delta t := 0.01$

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} := \begin{bmatrix} \dfrac{\pi}{10} \\ 0 \end{bmatrix}$$

$$\theta_i := 0.314 \cdot \sin\left(3.132 \cdot \Delta t \cdot i + \frac{\pi}{2}\right)$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} x_i + v_i \cdot \Delta t \\ v_i - \Delta t \cdot \left(\sin\left(x_i\right)\right) \ 9.81 \end{bmatrix}$$
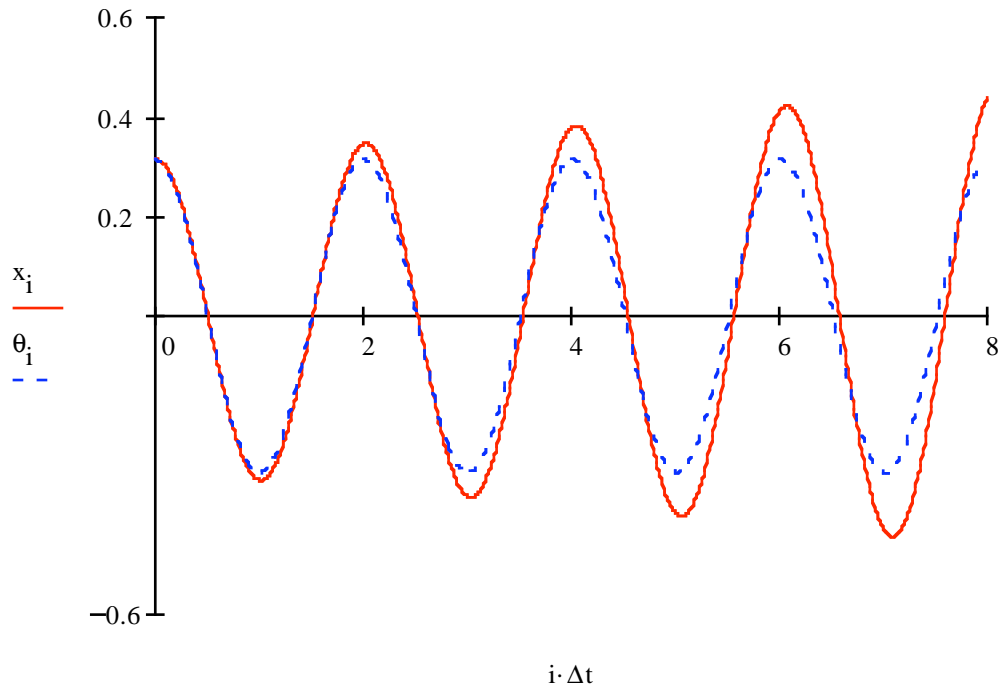
Note how the amplitude of the nonlinear system is growing. The difference between the linear and the nonlinear plots are a function of the ration of the linear spring stiffness and the nonlinear coefficient, and of course the size of the initial condition. It is work it to investigate the various possibilities, to learn just when the linear approximation completely fails.

1.111* Repeat Problem 1.110 if the initial displacement is $\theta_0 = \pi/2$ rad.

Solution: The solution in Mathcad is:

$i := 0..80000$

$\omega := \sqrt{9.81}$

$\Delta t := 0.001$

$$\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} := \begin{pmatrix} \dfrac{\pi}{2} \\ 0 \end{pmatrix}$$

$$\theta_i := \frac{\pi}{2} \cdot \sin\left(\omega \cdot \Delta t \cdot i + \frac{\pi}{2} \right)$$

$$\begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} := \begin{bmatrix} x_i + v_i \cdot \Delta t \\ v_i - \Delta t \cdot \left(\sin(x_i) \right) 9.81 \end{bmatrix}$$

Here both solutions oscillate around the "stable" equilibrium, but the nonlinear solution is not oscillating at the natural frequency and is increasing in amplitude.
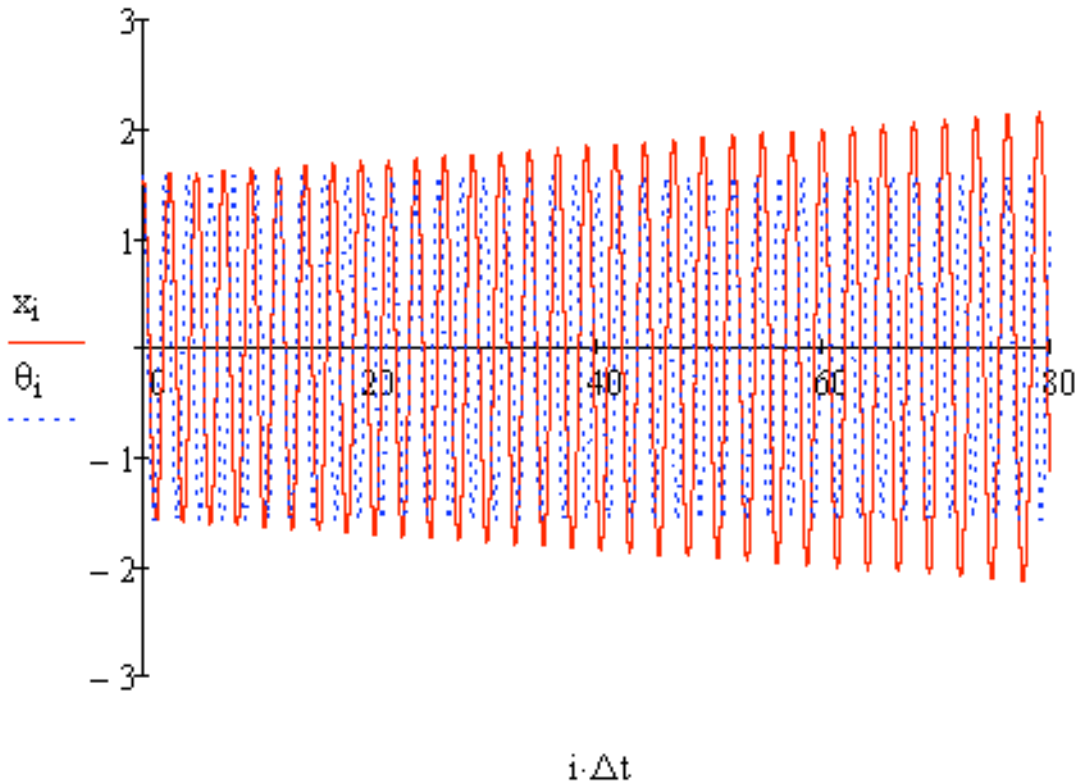
1.112 If the pendulum of Example 1.10.3 is given an initial condition near the equilibrium position of $\theta_0 = \pi$ rad and $\dot{\theta}_0 = 0$, does it oscillate around this equilibrium?

Solution   The pendulum will not oscillate around this equilibrium as it is unstable. Rather it will "wind" around the equilibrium as indicated in the solution to Example 1.10.4.

1.113* Calculate the response of the system of Problem 1.109 for the initial conditions of $x_0 = 0.01$ m, $v_0 = 0$, and a natural frequency of 3 rad/s and for $\beta = 100$, $\gamma = 0$.

Solution: In Mathcad the solution is given using a simple Euler integration as follows:

$\Delta t := 0.01$

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} := \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \qquad \omega := 3 \qquad A := \frac{1}{\omega} \cdot \sqrt{\omega^2 \cdot (x_0)^2}$$

$\beta := 100$

$i := 0 .. \ 1000$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} x_i + v_i \cdot \Delta t \\ v_i - \Delta t \cdot \left[\omega^2 \cdot x_i - \beta^2 \cdot (x_i)^3 \right] \end{bmatrix}$$

$$\theta_i := A \cdot \sin\left(3 \cdot \Delta t \cdot i + \frac{\pi}{2} \right) \qquad \boxed{\text{This is the linear solution } \theta(t)}$$
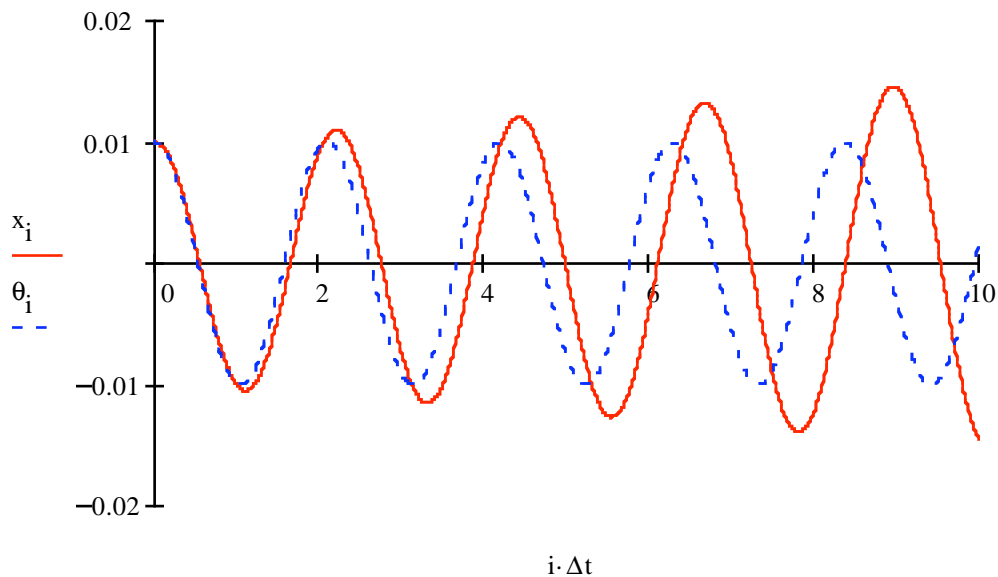
The other codes may be used to compute this solution as well.

1.114* Repeat problem 1.113 and plot the response of the linear version of the system (β =0) on the same plot to compare the difference between the linear and nonlinear versions of this equation of motion.

Solution: The solution is computed and plotted in the solution of Problem 1.113. Note that the linear solution starts out very close to the nonlinear solution. The two solutions however diverge. They look similar, but the nonlinear solution is growing in amplitude and period.

Problems and Solutions Section 2.1 (2.1 through 2.15)

2.1 To familiarize yourself with the nature of the forced response, plot the solution of a forced response of equation (2.2) with $\omega = 2$ rad/s, given by equation (2.11) for a variety of values of the initial conditions and ω_n as given in the following chart:

Case	x_0	v_0	f_0	ω_n
1	0.1	0.1	0.1	1
2	-0.1	0.1	0.1	1
3	0.1	0.1	1.0	1
4	0.1	0.1	0.1	2.1
5	1	0.1	0.1	1

Solution: Given: $\omega = 2$ rad/sec.
From equation (2.11):

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + (x_0 - \frac{f_0}{\omega_n^2 - \omega^2}) \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

Insert the values of x_0, v_0, f_0, and ω_n for each of the five cases.

$x1\,(t) := 0.1 \cdot \sin(t) + 0.133 \cdot \cos(t) - 0.0333 \cdot \cos(2 \cdot t)$

$x2\,(t) := 0.1 \cdot \sin(t) - 0.0667 \cdot \cos(t) - 0.0333 \cdot \cos(2 \cdot t)$

$x3\,(t) := 0.1 \cdot \sin(t) + 0.433 \cdot \cos(t) - 0.0333 \cdot \cos(2 \cdot t)$

$x4\,(t) := 0.0467 \cdot \sin(2.1 \cdot t) + 0.244 \cdot \cos(2 \cdot t) - 0.144 \cdot \cos(2.1 \cdot t)$

$x5\,(t) := 0.1 \cdot \sin(t) + 1.033 \cdot \cos(t) - 0.0333 \cdot \cos(2 \cdot t)$
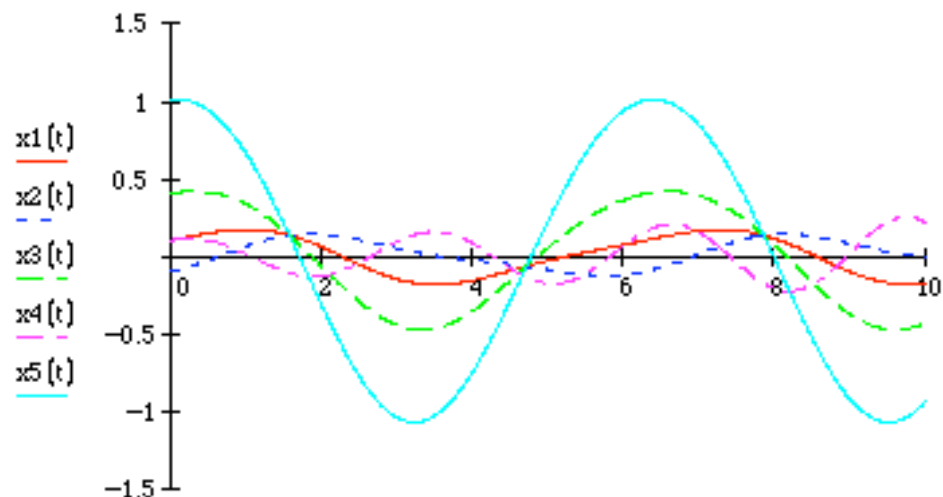
2.2 Repeat the calculation made in Example 2.1.1 for the mass of a simple spring-mass system where the mass of the spring is considered and known to be 1 kg.

Solution: Given: $m_{sp} = 1$ kg, Example 1.4.4 yields that the effective mass is

$$m_e = m + \frac{m_{sp}}{3} = 10 + \frac{1}{3} = 10.333 \text{ kg.}$$

Thus the natural frequency, X and the coefficients in equation (2.11) for the system now become

$$\omega_n = \sqrt{\frac{1000}{10 + \frac{1}{3}}} = 9.837 \text{ rad/s, } \omega = 2\omega_n = 19.675 \text{ rad/s}$$

$$X = \frac{f_0}{\omega_n^2 - \omega^2} = \frac{2.338}{9.837^2 - 19.675^2} = -8.053 \times 10^{-3} \text{ m, } \frac{v_0}{\omega_n} = 0.02033 \text{ m}$$

Thus the response as given by equation (2.11) is

$$x(t) = 0.02033\sin 9.837t + 8.053 \times 10^{-3}(\cos 9.837t - \cos 19.675t) \text{ m}$$

2.3 A spring-mass system is driven from rest harmonically such that the displacement response exhibits a beat of period of 0.2π s. The period of oscillation is measured to be 0.02π s. Calculate the natural frequency and the driving frequency of the system.

Solution: Given: Beat period: $T_b = 0.2\pi$ s, Oscillation period: $T_0 = 0.02\pi$ s

Equation (2.13): $$x(t) = \frac{2f_0}{\omega_n^2 - \omega^2} \sin\left[\frac{\omega_n - \omega}{2}t\right]\sin\left[\frac{\omega_n + \omega}{2}t\right]$$

So, $$T_b = 0.2\pi = \frac{4\pi}{\omega_n - \omega}$$

$$\omega_n - \omega = \frac{4\pi}{0.2\pi} = 20 \text{ rad/s}$$

$$T_0 = 0.02\pi = \frac{4\pi}{\omega_n + \omega}$$

$$\omega_n + \omega = \frac{4\pi}{0.02\pi} = 200 \text{ rad/s}$$

Solving for ω_n and ω gives:
Natural frequency: $\omega_n = 110$ rad/s
Driving frequency: $\omega = 90$ rad/s

2.4 An airplane wing modeled as a spring-mass system with natural frequency 40 Hz is driven harmonically by the rotation of its engines at 39.9 Hz. Calculate the period of the resulting beat.

Solution: Given: $\omega_n = 2\pi(40) = 80\pi$ rad/s, $\omega = 2\pi(39.9) = 79.8\pi$ rad/s

Beat period: $T_b = \dfrac{4\pi}{\omega_n - \omega} = \dfrac{4\pi}{80\pi - 79.8\pi} = 20$ s.

2.5 Derive Equation 2.13 from Equation 2.12 using standard trigonometric identities.

Solution: Equation (2.12): $x(t) = \dfrac{f_0}{\omega_n^2 - \omega^2}[\cos\omega t - \cos\omega_n t]$

Let $A = \dfrac{f_0}{\omega_n^2 - \omega^2}$

$$x(t) = A[\cos\omega t - \cos\omega_n t]$$
$$= A[1 + \cos\omega t - (1 + \cos\omega_n t)]$$
$$= A[2\cos^2\frac{\omega}{2}t - 2\cos^2\frac{\omega_n}{2}t]$$
$$= 2A[(\cos^2\frac{\omega}{2}t - \cos^2\frac{\omega_n}{2} \cos^2\frac{\omega}{2}t) - (\cos^2\frac{\omega_n}{2}t - \cos^2\frac{\omega_n}{2}t \cos^2\frac{\omega}{2}t)]$$
$$= 2A[(1 - \cos^2\frac{\omega_n}{2}t)\cos^2\frac{\omega}{2}t - (1 - \cos^2\frac{\omega}{2}t)\cos^2\frac{\omega_n}{2}t]$$
$$= 2A[\sin^2\frac{\omega}{2}t \cos^2\frac{\omega}{2}t - \cos^2\frac{\omega}{2}t \sin^2\frac{\omega}{2}t]$$
$$= 2A[\sin\frac{\omega_n}{2}t \cos\frac{\omega}{2}t - \cos\frac{\omega_n}{2}t \sin\frac{\omega}{2}t][\sin\frac{\omega_n}{2}t \cos\frac{\omega}{2}t - \cos\frac{\omega_n}{2}t \sin\frac{\omega}{2}t]$$
$$= 2A\sin\left[\frac{\omega_n - \omega}{2}t\right]\sin\left[\frac{\omega_n + \omega}{2}t\right]$$
$$x(t) = \frac{2f_0}{\omega_n^2 - \omega^2}\sin\left[\frac{\omega_n - \omega}{2}t\right]\sin\left[\frac{\omega_n + \omega}{2}t\right] \quad \text{which is Equation (2.13).}$$

2.6 Compute the total response of a spring-mass system with the following values: $k = 1000$ N/m, $m = 10$ kg, subject to a harmonic force of magnitude $F_0 = 100$ N and frequency of 8.162 rad/s, and initial conditions given by $x_0 = 0.01$ m and $v_0 = 0.01$ m/s. Plot the response.

Solution: Given: $k = 1000$ N/m, $m = 10$ kg, $F_0 = 100$ N, $\omega = 8.162$ rad/s
$x_0 = 0.01$ m, $v_0 = 0.01$ m/s
From Eq. (2.11):

$$x(t) = \frac{v_0}{\omega_n}\sin \omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right)\cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos \omega t$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \; rad/s \qquad\qquad f_0 = \frac{F_0}{m} = \frac{100}{10} = 10 \; N/m$$

In Mathcad the solution is

$$x0 := 0.01 \qquad v0 := 0.01 \qquad \omega n := 10 \qquad \omega := 8.162 \qquad f0 := 10$$

$$x(t) := \frac{v0}{\omega n}\cdot\sin(\omega n\cdot t) + \left(x0 - \frac{f0}{\omega n^2 - \omega^2}\right)\cdot\cos(\omega n\cdot t) + \frac{f0}{\left(\omega n^2 - \omega^2\right)}\cdot\cos(\omega\cdot t)$$

$$\frac{v0}{\omega n} = 1 \times 10^{-3} \qquad x0 - \frac{f0}{\omega n^2 - \omega^2} = -0.28956 \qquad \frac{f0}{\left(\omega n^2 - \omega^2\right)} = 0.29956$$

2.7 Consider the system in Figure P2.7, write the equation of motion and calculate the response assuming a) that the system is initially at rest, and b) that the system has an initial displacement of 0.05 m.
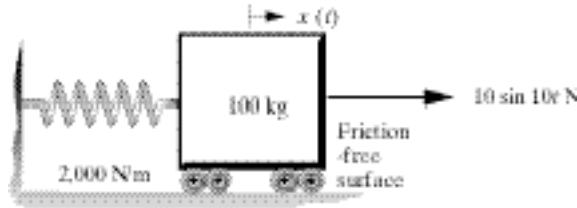
Solution: The equation of motion is

$$m\ddot{x} + kx = 10\sin 10t$$

Let us first determine the general solution for

$$\ddot{x} + \omega_n^2 x = f_0 \sin \omega t$$

Replacing the cosine function with a sine function in Eq. (2.4) and following the same argument, the general solution is:

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\sin \omega t$$

Using the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$, a general expression for the response of a spring-mass system to a harmonic (sine) excitation is:

$$x(t) = (\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n}\cdot\frac{f_0}{\omega_n^2 - \omega^2})\sin \omega_n t + x_0 \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\sin \omega t$$

Given: k=2000 N/m, m=100 kg, ω=10 rad/s,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{100}} = \sqrt{20} \text{ rad/s} = 4.472 \text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{10}{100} = 0.1\,\text{N/kg}$$

a) $x_0 = 0$ m, $v_0 = 0$ m/s

Using the general expression obtained above:

$$x(t) = (0 - \frac{10}{\sqrt{20}}\cdot\frac{0.1}{\sqrt{20}^2 - 10^2})\sin\sqrt{20}t + 0 + \frac{0.1}{\sqrt{20}^2 - 10^2}\sin 10t$$

$$= 2.795\times 10^{-3} \sin 4.472t - 1.25\times 10^{-3}\sin 10t$$

b) $x_0 = 0.05$ m, $v_0 = 0$ m/s

$$x(t) = (0 - \frac{10}{\sqrt{20}}\cdot\frac{0.1}{\sqrt{20}^2 - 10^2})\sin\sqrt{20}t + 0.05\cos\sqrt{20}t + \frac{0.1}{\sqrt{20}^2 - 10^2}\sin 10t$$

$$= 0.002795\sin 4.472t + 0.05\cos 4.472t - 0.00125\sin 10t$$

$$= 5.01\times 10^{-2}\sin(4.472t + 1.515) - 1.25\times 10^{-3}\sin 10t$$

2.8 Consider the system in Figure P2.8, write the equation of motion and calculate the response assuming that the system is initially at rest for the values $k_1 = 100$ N/m, $k_2 = 500$ N/m and $m = 89$ kg.
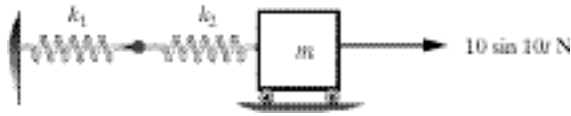
Solution: The equation of motion is

$$m\ddot{x} + kx = 10\sin 10\,t \qquad \text{where} \quad k = \dfrac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

The general expression obtained for the response of an underdamped spring-mass system to a harmonic (sine) input in Problem 2.7 was:

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2}\right)\sin\omega_n t + x_0\cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\sin\omega t$$

Substituting the following values

$$k = 1/(1/100 + 1/500) = 83.333 \text{ N/m}, \quad m = 89 \text{ kg} \qquad \omega = 10 \text{ rad/s}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{83.333}{89}} = 0.968 \text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{10}{89} = 0.112 \text{ N}/kg$$

and initial conditions: $x_0 = 0$, $v_0 = 0$

The response of the system is evaluated as

$$x(t) = 0.0117\sin 0.968\,t - 0.00113\sin 10\,t$$

2.9   Consider the system in Figure P2.9, write the equation of motion and calculate the response assuming that the system is initially at rest for the values $\theta = 30°$, $k = 1000$ N/m and $m = 50$ kg.
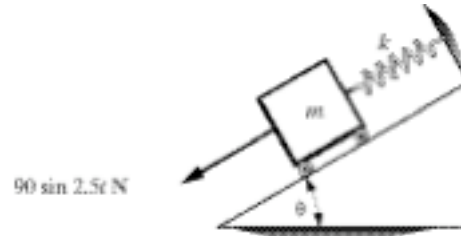
Figure P2.9

Solution: Free body diagram:
Assuming $x = 0$ to be at equilibrium:
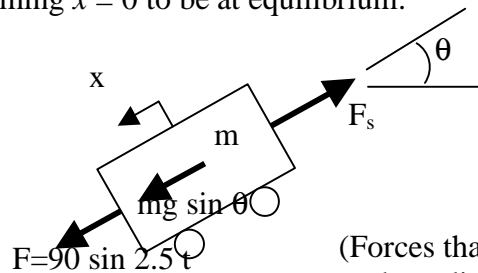
(Forces that are normal to the x direction are neglected)

$$\sum F_x = m\ddot{x} = -k(x + \Delta) + mg\sin\theta + 90\sin 25t \qquad (1)$$

where Δ is the static deflection of the spring. From static equilibrium in the x direction yields

$$-k\Delta + mg\sin\theta \qquad (2)$$

Substitution of (2) onto (1), the equation of motion becomes

$$m\ddot{x} + kx = 90\sin 2.5t$$

The general expression for the response of a mass-spring system to a harmonic (sine) excitation (see Problem 2.7) is:

$$x(t) = (\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2})\sin\omega_n t + x_0 \cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\sin\omega t$$

Given: $v_0 = 0$, $x_0 = 0$, $\omega = 2.5$ rad/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{50}} = \sqrt{20} = 4.472 \text{ rad/s} \quad, \quad f_0 = \frac{F_0}{m} = \frac{90}{50} = \frac{9}{5}\text{N/kg}$$

So the response is:

$$x(t) = -0.0732\sin 4.472t + 0.1309\sin 2.5t$$

2.10   Compute the initial conditions such that the response of :

$$m\ddot{x} + kx = F_0 \cos \omega t$$

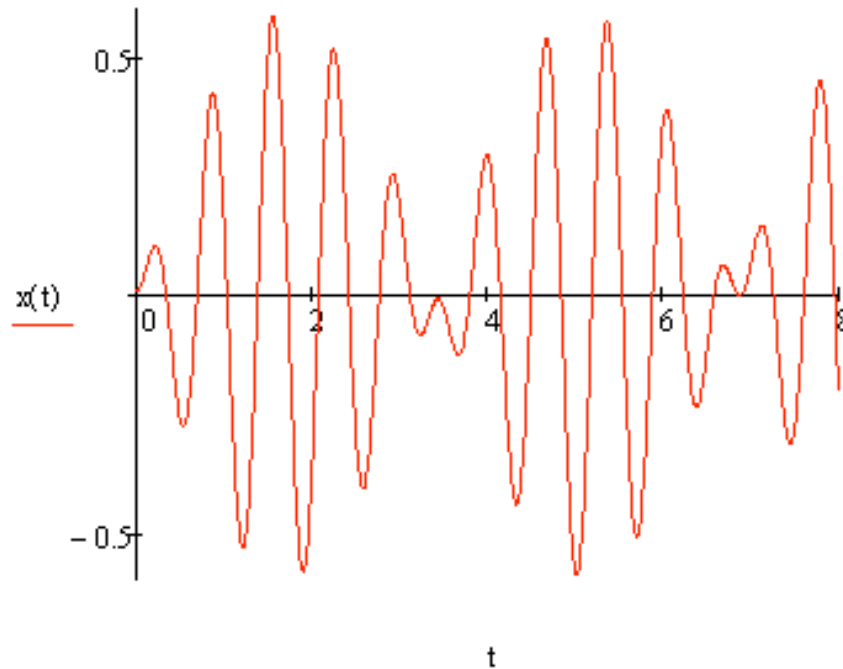oscillates at only one frequency (ω).

Solution: From Eq. (2.11):

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + (x_0 - \frac{f_0}{\omega_n^2 - \omega^2}) \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

For the response of $m\ddot{x} + k x = F_0 \cos \omega t$ to have only one frequency content, namely, of the frequency of the forcing function, ω, the coefficients of the first two terms are set equal to zero. This yields that the initial conditions have to be

$$x_0 = \frac{f_0}{\omega_n^2 - \omega^2} \quad \text{and} \quad v_0 = 0$$

Then the solution becomes

$$x(t) = \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t$$

2.11   The natural frequency of a 65-kg person illustrated in Figure P.11 is measured along vertical, or longitudinal direction to be 4.5 Hz. a) What is the effective stiffness of this person in the longitudinal direction? b) If the person, 1.8 m in length and 0.58 m^2 in cross sectional area, is modeled as a thin bar, what is the modulus of elasticity for this system?

$x(t)$
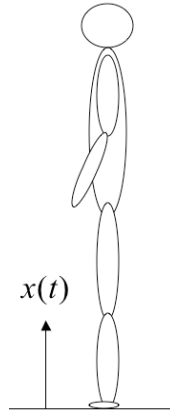
Figure P2.11 Longitudinal vibration of a person

Solution: a) First change the frequency in Hz to rad/s: $\omega_n = 4.5 \dfrac{\text{cycles}}{\text{s}} \dfrac{2\pi \text{ rad}}{\text{cycles}} = 9\pi$ rad/s .

Then from the definition of natural frequency:

$$k = m\omega_n^2 = 65 \cdot (9\pi)^2 = \underline{5.196 \times 10^4 \text{ N/m}}$$

b) From section 1.4, the value of the stiffness for the longitudinal vibration of a beam is

$$k = \frac{EA}{\ell} \Rightarrow E = \frac{k\ell}{A} = \frac{\left(5.196 \times 10^4\right)(1.8)}{0.58} = \underline{1.613 \times 10^5 \text{ N/m}^2} = \underline{1.613 \times 10^5 \text{ Pa}}$$

2.12   If the person in Problem 2.11 is standing on a floor, vibrating at 4.49 Hz with an amplitude of 1 N (very small), what longitudinal displacement would the person "feel"? Assume that the initial conditions are zero.

Solution: Using equation (2.12) for a cosine excitation and zero initial conditions yields (converting the frequency from Hertz to rad/s and using the value of k calculated in 2.11):

$$|X| = \frac{F_0}{m}\left|\frac{1}{\omega_n^2 - \omega^2}\right| = \frac{1}{65}\left|\frac{1}{\dfrac{k}{m} - (4.49 \cdot 2\pi)^2}\right|$$

$$= \frac{1}{65}\left|\frac{1}{\dfrac{5.196 \times 10^4}{65} - (4.49 \cdot 2\pi)^2}\right| = 0.00443347 = \underline{0.0043 \text{ m}}$$

2.13 Vibration of body parts is a significant problem in designing machines and structures. A jackhammer provides a harmonic input to the operator's arm. To model this situation, treat the forearm as a compound pendulum subject to a harmonic excitation (say of mass 6 kg and length 44.2 cm) as illustrated in Figure P2.13. Consider point O as a fixed pivot. Compute the maximum deflection of the hand end of the arm if the jackhammer applies a force of 10 N at 2 Hz.
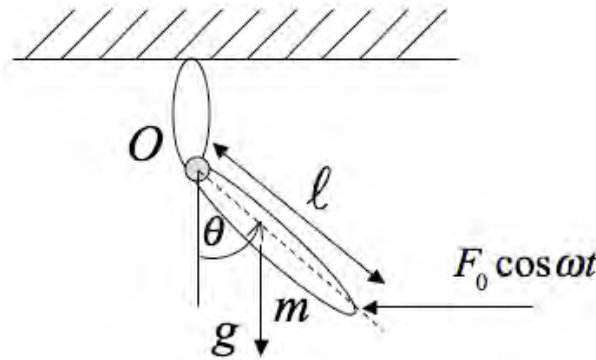
Figure P2.13 Vibration model of a forearm driven by a jackhammer

Solution: Taking moments about point O yields (referring to Example 1.4.6 for the inertial of a compound pendulum):

$$\frac{m\ell^2}{3}\ddot{\theta} + mg\frac{\ell}{2}\sin\theta = F_0 \ell \cos\theta \cos\omega t$$

Using the linear approximation for sine and cosine and dividing through by the inertia yields:

$$\ddot{\theta} + \frac{3g}{2\ell}\theta = \frac{3F_0}{m\ell}\cos\omega t$$

Thus the natural frequency is

$$\omega_n = \sqrt{\frac{3g}{2\ell}} = \sqrt{\frac{3(9.81)}{2(0.442)}} = 5.77 \text{ rad/s} \quad (=0.92 \text{ Hz})$$

and the system is well away from resonance. Referring to equation (2.13), the amplitude for zero initial conditions is (converting the driving frequency from 2 Hertz to $2(2\pi)$ rad/s):

$$|\theta| = \left|\frac{2f_0}{\omega_n^2 - \omega^2}\right| = \left|\frac{2\left(\dfrac{3F_0\ell}{m\ell^2}\right)}{\dfrac{3g}{2\ell} - (2 \cdot 2\pi)^2}\right| = \underline{0.182 \text{ rad}}$$

Note that $\sin(0.182) = 0.181$ so the approximation made above is valid. The maximum linear displacement of the hand end of the arm is just

$$|X| = r|\theta| = 0.442 \cdot 0.182 = \underline{0.08 \text{ m}}$$

2.14 Consider again the camera problem of Example 2.1.3 depicted in Figure P2.14, and determine the torsional natural frequency, the maximum torsional deflection experienced by the camera due to the wind and the linear displacement corresponding to the computed torsional deflection. Model the camera in torsional vibration as suggested in the figure where $J_P = 9.817\text{x}10^{-6}$ m^4 and $L = 0.2$ m. Use the values computed in Example 2.1.3 for the mass ($m = 3$ kg), shaft length ($\ell = 0.55$ m), torque ($M_0 = 15$ x L Nm) and frequency ($\omega = 10$ Hz). Here G is the shear modulus of aluminum and the rotational inertia of the camera is approximated by $J = mL^2$. In the example, torsion was ignored. The purpose of this problem is to determine if ignoring the torsion is a reasonable assumption or not. Please comment on this assumption based on the results of the requested calculation.
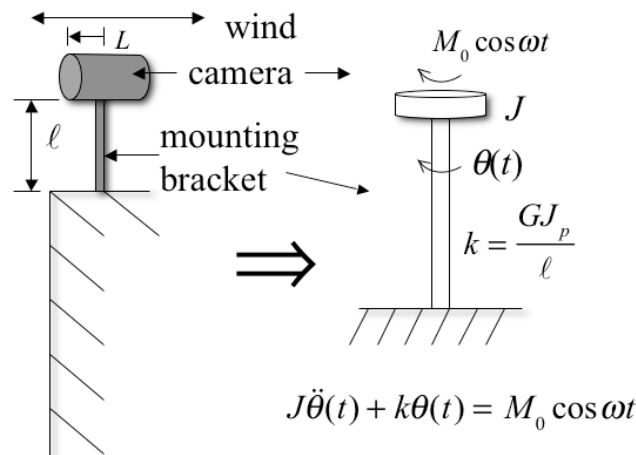
Figure P2.14 Torsional vibration of a camera

Solution: First calculate the rotational stiffness and inertia from the data given:

$$k = \frac{GJ_p}{\ell} = \frac{2.67 \times 10^{10} \times 9.817 \times 10^{-6}}{0.55} = 4.766 \times 10^5 \text{ N} \cdot \text{m}$$

where the modulus is taken from Table 1.2 for aluminum. The inertia is approximated by

$$J = mL^2 = 3(0.2)^2 = 0.12 \text{ kg} \cdot \text{m}^2$$

The torsional natural frequency is thus

$$\omega_n = \sqrt{\frac{k}{J}} = \underline{1.993 \times 10^3} \text{ rad/s}$$

This is well away from the driving frequency. To see the effect, recall equation magnitude of the forced response given in Example 2.1.2:

$$\left| \frac{2f_0}{\omega_n^2 - \omega^2} \right| = \left| \frac{2M_0 / J}{\omega_n^2 - \omega^2} \right| = \underline{1.26 \times 10^{-5}} \text{ rad}$$

Clearly this is very small. To change this to a linear displacement of the camera tip, use

$$X = r\theta = (0.2)(1.26 \times 10^{-5}) = \underline{2.52 \times 10^{-6}} \text{ m}$$

well within the limit imposed on the camera's vibration requirement of 0.01 m. Thus, the assumption to ignore torsional vibration in designing the length of the mounting bracket made in example 2.1.3 is justified.

2.15 An airfoil is mounted in a wind tunnel for the purpose of studying the aerodynamic properties of the airfoil's shape. A simple model of this is illustrated in Figure P2.15 as a rigid inertial body mounted on a rotational spring, fixed to the floor with a rigid support. Find a design relationship for the spring stiffness k in terms of the rotational inertia, J, the magnitude of the applied moment, M_0, and the driving frequency, ω, that will keep the magnitude of the angular deflection less then $5°$. Assume that the initial conditions are zero and that the driving frequency is such that $\omega_n^2 - \omega^2 > 0$.
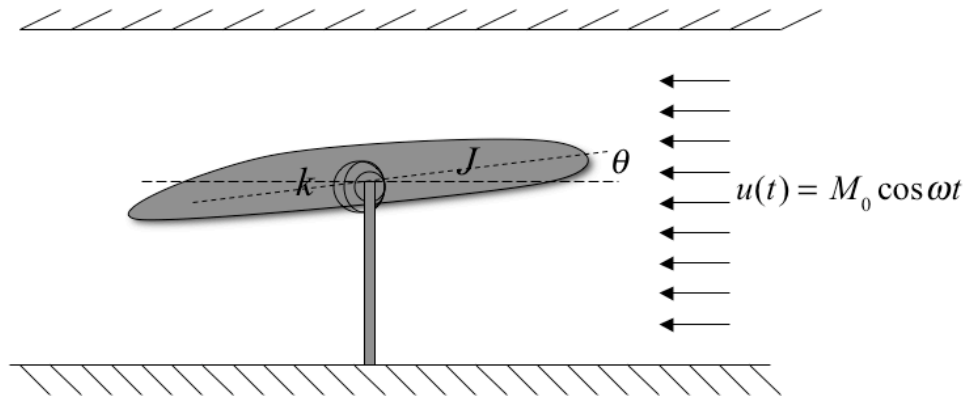
Figure P2.15 Vibration model of a wing in a wind tunnel

Solution: Assuming compatible units, the equation of motion is:

$$J\ddot{\theta}(t) + k\theta(t) = M_0 \cos \omega t \Rightarrow \ddot{\theta}(t) + \frac{k}{J}\theta(t) = \frac{M_0}{J}\cos \omega t$$

From equation (2.12) the maximum deflection for zero initial conditions is

$$\theta_{max} = \left| \frac{2M_0 / J}{\dfrac{k}{J} - \omega^2} \right| < 5° \frac{\pi \text{rad}}{180°} = \frac{\pi}{36} \text{rad}$$

$$\Rightarrow \frac{2M_0}{J} < (\frac{k}{J} - \omega^2) \frac{\pi}{36} \text{rad} \Rightarrow \frac{36J}{\pi} \left(\frac{2M_0}{J} + \frac{\pi \omega^2}{36} \right) < k$$

Problems and Solutions Section 2.2 (2.16 through 2.31)

2.16 Calculate the constants A and ϕ for arbitrary initial conditions, x_0 and v_0, in the case of the forced response given by Equation (2.37). Compare this solution to the transient response obtained in the case of no forcing function (i.e. $F_0 = 0$).

> **Solution:** From equation (2.37)
>
> $$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\cos(\omega t - \theta) \Rightarrow$$
>
> $$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + A\omega_d e^{-\zeta\omega_n t}\cos(\omega_d t + \phi) - X\omega\sin(\omega t - \theta)$$
>
> Next apply the initial conditions to these general expressions for position and velocity to get:
>
> $$x(0) = A\sin\phi + X\cos\theta$$
>
> $$\dot{x}(0) = -\zeta\omega_n A\sin\phi + A\omega_d\cos\phi + X\omega\sin\theta$$
>
> Solving this system of two equations in two unknowns yields:
>
> $$\phi = \tan^{-1}\left(\frac{(x_0 - X\cos\theta)\omega_d}{v_0 + (x_0 - X\cos\theta)\zeta\omega_n - X\omega\sin\theta}\right)$$
>
> $$A = \frac{x_0 - X\cos\theta}{\sin\phi}$$
>
> Recall that X has the form
>
> $$X = \frac{F_0/m}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$
>
> Now if $F_0 = 0$, then $X = 0$ and A and ϕ from above reduce to:
>
> $$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + x_0\zeta\omega_n}\right)$$
>
> $$A = \frac{x_0}{\sin\phi} = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$$
>
> These are identical to the values given in equation (1.38).

2.17   Show that Equations (2.28) and (2.29) are equivalent by verifying Equations (2.29) and (2.30).

Solution: From equation (2.28) and expanding the trig relation yields
$$x_p = X\cos(\omega t - \theta) = X\left[\cos\omega t \cos\theta + \sin\omega t \sin\theta\right]$$
$$= \underbrace{(X\cos\theta)}_{A_s}\cos\omega t + \underbrace{(X\sin\theta)}_{B_s}\sin\omega t$$

Now with A_s and B_s defined as indicated, the magnitude is computed:
$$X = \sqrt{A_s^2 + B_s^2}$$

and

$$\frac{B_s}{A_s} = \frac{X\sin\theta}{X\cos\theta} \Rightarrow \theta = \tan^{-1}\left(\frac{B_s}{A_s}\right)$$

2.18   Plot the solution of Equation (2.27) for the case that $m = 1$ kg, $\zeta = 0.01$, $\omega_n = 2$ rad/s. $F_0 = 3$ N, and $\omega = 10$ rad/s, with initial conditions $x_0 = 1$ m and $v_0 = 1$ m/s.

Solution: The particular solution is given in equations (2.36) and (2.37). Substitution of the values given yields: $x_p = 0.03125\cos(10t + 8.333\times10^{-3})$.
Then the total solution has the form:
$$x(t) = Ae^{-0.02t}\sin(2t + \phi) + 0.03125\cos(10t + 0.008333)$$
$$= e^{-0.02t}\left(A\sin 2t + B\cos 2t\right) + 0.03125\cos(10t + 0.008333)$$
Differentiating then yields
$$\dot{x}(t) = -0.02e^{-0.02t}\left(A\sin 2t + B\cos 2t\right) + \sin(2t + \phi)$$
$$+ 2e^{-0.02t}\left(A\cos 2t - B\sin 2t\right) - 0.3125\sin(10t + 0.008333)$$
Apply the initial conditions to get:
$$x(0) = 1 = B + 0.03125\cos(0.00833) \Rightarrow \underline{B = 0.969}$$
$$\dot{x}(0) = 1 = -0.02B + 2A - 0.3125\sin(0.00833) \Rightarrow A = 0.489$$
So the solution and plot become (using Mathcad):

$F0 := 3 \qquad \omega := 10 \quad \omega n := 2 \quad \zeta := 0.02 \qquad \theta := \operatorname{atan}\!\left(2 \cdot \dfrac{\zeta \cdot \omega n \cdot \omega}{\omega n^2 - \omega^2}\right) = -8.333 \times 10^{-3}$

$$AF := \frac{F0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + (2 \cdot \zeta \cdot \omega \cdot \omega n)^2}} \qquad AF = 0.03125$$

$B := 1 - AF \cdot \cos(\theta) \qquad B = 0.969$

$$A1 := \frac{1 + 10 \cdot AF \cdot \sin(\theta) - 0.02 \cdot B}{2} \qquad A1 = 0.489$$

$$x(t) := e^{-0.02 \cdot t} \cdot (A1 \cdot \sin(2 \cdot t) + B \cdot \cos(2 \cdot t)) + AF \cdot \cos(10 \cdot t - \theta)$$
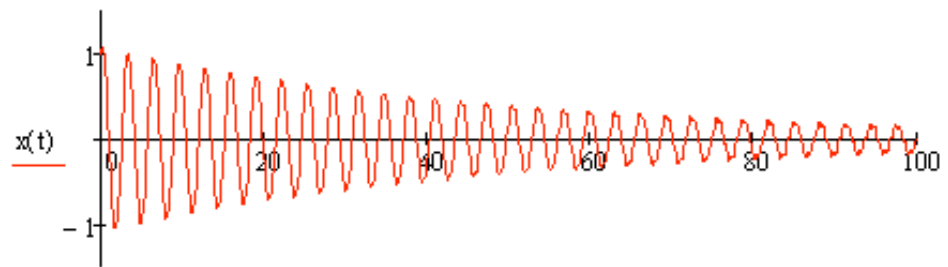
t

2.19 A 100 kg mass is suspended by a spring of stiffness 30×10^3 N/m with a viscous damping constant of 1000 Ns/m. The mass is initially at rest and in equilibrium. Calculate the steady-state displacement amplitude and phase if the mass is excited by a harmonic force of 80 N at 3 Hz.

Solution: Given $m = 100$kg, $k = 30,000$ N/m, $c = 1000$ Ns/m, $F_0 = 80$ N and $\omega = 6\pi$ rad/s:

$$f_0 = \frac{F_0}{m} = \frac{80}{100} = 0.8 \text{ m/s}^2, \quad \omega_n = \sqrt{\frac{k}{m}} = 17.32 \text{ rad/s}$$

$$\zeta = \frac{c}{2\sqrt{km}} = 0.289$$

$$X = \frac{0.8}{\sqrt{\left(17.32^2 + 36\pi^2\right)^2 + \left(2(0.289)(17.32)(6\pi)\right)^2}} = 0.0041 \text{ m}$$

Next compute the angle from

$$\theta = \tan^{-1}\left(\frac{188.702}{-55.323}\right)$$

Since the denominator is negative the angle must be found in the 4th quadrant. To find this use Window 2.3 and then in Matlab type atan2(188.702,-55.323) or use the principle value and add π to it. Either way the phase is $\theta = 1.856$ rad.

2.20 Plot the total solution of the system of Problem 2.19 including the transient.

Solution: The total response is given in the solution to Problem 2.16. For the values given in the previous problem, and with zero initial conditions the response is determined by the formulas:

$$X = 0.0041, \quad \theta = 1.856$$

$$\omega n := 17.32 \quad \zeta := 0.289$$

$$\theta := \operatorname{atan}\left[\frac{2 \cdot 0.289 \cdot 17.32 \cdot 6 \cdot \pi}{17.32^2 - (6 \cdot \pi)^2}\right] + \pi$$

$$X := 0.0041$$

$$\omega d := \omega n \cdot \sqrt{1 - \zeta^2} \qquad\qquad \theta = 1.856$$

$$\phi := \operatorname{atan}\left(\frac{-X \cdot \cos(\theta) \cdot \omega d}{-X \cdot \cos(\theta) \cdot \zeta \cdot \omega n - X \cdot \omega n \cdot \sin(\theta)}\right) + \pi \qquad \phi = 2.844$$

$$A := \frac{-X \cdot \cos(\theta)}{\sin(\phi)} \qquad A = 3.934 \times 10^{-3}$$

$$x(t) := \left(A \cdot e^{-\zeta \cdot \omega n \cdot t} \cdot \sin(\omega d \cdot t + \phi)\right) + X \cdot \cos(6 \cdot \pi \cdot t - \theta)$$

Plotting the result in Mathcad yields
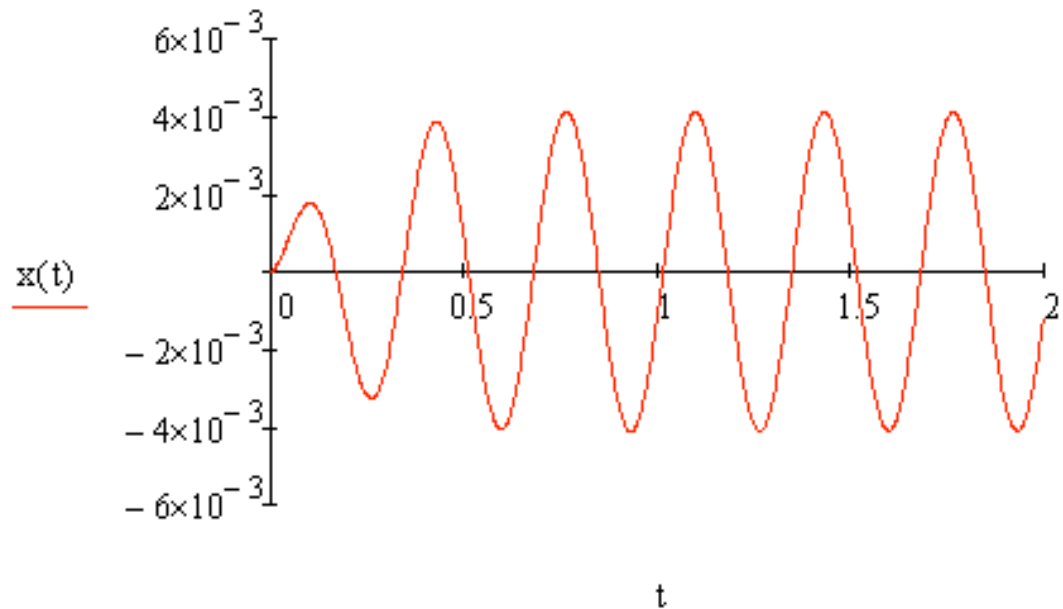
2.21 Consider the pendulum mechanism of Figure P2.21 which is pivoted at point O. Calculate both the damped and undamped natural frequency of the system for small angles. Assume that the mass of the rod, spring, and damper are negligible. What driving frequency will cause resonance?
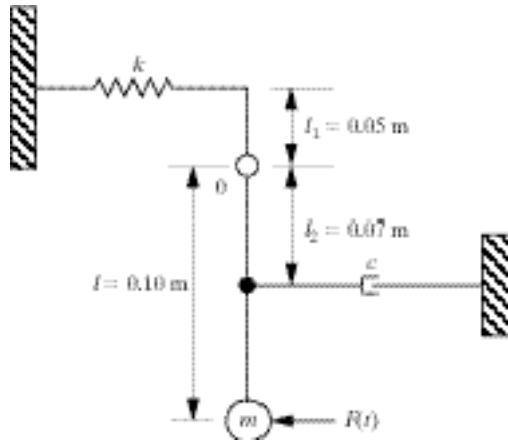
Solution: Assume the driving frequency to be harmonic of the standard form. To get the equation of motion take the moments about point O to get:

$$\sum M_0 = J\ddot{\theta}(t) = m\ell^2\ddot{\theta}(t)$$

$$= -k\ell_1 \sin\theta(\ell_1 \cos\theta) - c\ell_2\dot{\theta}(\ell_2 \cos\theta)$$
$$- mg(\ell\sin\theta) + F_0 \cos\omega t(\ell\cos\theta)$$

Rearranging and approximating $\sin\theta \sim \theta$ and $\cos\theta \sim 1$ yields:

$$m\ell^2\ddot{\theta}(t) + c\ell_2^2\dot{\theta}(t) + (k\ell_1^2 + mg\ell)\theta(t) = F_0\ell\cos\omega t$$

Dividing through by the coefficient of the inertia term and using the standard definitions for ζ and ω yields:

$$\omega_n = \sqrt{\frac{k\ell_1^2 + mg\ell}{m\ell^2}} \text{ which is the resnonant frequency}$$

$$\zeta = \frac{c\ell_2^2}{2\sqrt{(k\ell_1^2 + mg\ell)mg\ell}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k\ell_1^2 + mg\ell}{m\ell^2}\left(1 - \frac{c^2\ell_2^4}{4(k\ell_1^2 + mg\ell)mg\ell}\right)}$$

2.22 Consider the pivoted mechanism of Figure P2.21 with $k = 4 \times 10^3$ N/m. $l_1 = 0.05$ m. $l_2 = 0.07$ m. and $l = 0.10$ m. and $m = 40$ kg. The mass of the beam is 40 kg; it is pivoted at point 0 and assumed to be rigid. Design the dashpot (i.e. calculate c) so that the damping ratio of the system is 0.2. Also determine the amplitude of vibration of the steady-state response if a 10-N force is applied to the mass, as indicated in the figure, at a frequency of 10 rad/s.

Solution: This is similar to the previous problem with the mass of the beam included this time around. The equation of motion becomes:

$$m_{eq}\ddot{\theta} + c_{eq}\dot{\theta} + k_{eq}\theta = F_0 l \cos \omega t$$

Here:

$$m_{eq} = ml^2 + \frac{1}{3}(l^3 + l_1^3)\frac{m_b}{l + l_1} = 0.5 \text{ kg} \cdot \text{m}^2$$

$$c_{eq} = cl_2^2 = 0.25c$$

$$k_{eq} = kl_1^2 + mgl + \frac{1}{2}(l - l_1)m_b g = 4.326 \times 10^3 \text{ Nm}$$

Using the formula the damping ratio and these numbers:

$$\zeta = \frac{l_2^2 c}{2\sqrt{m_{eq}k_{eq}}} = 0.2 \Rightarrow c = 3.797 \cdot 10^3 \text{ kg/s}$$

Next compute the amplitude:

$$X = \frac{10/0.5}{\sqrt{(k_{eq}/m_{eq} - 10^2)^2 + (2 \cdot 0.2 \cdot 10 \cdot \omega_n)^2}} = 2.336 \times 10^3 \text{ rad}$$

2.23 In the design of Problem 2.22, the damping ratio was chosen to be 0.2 because it limits the amplitude of the forced response. If the driving frequency is shifted to 11 rad/s, calculate the change in damping coefficient needed to keep the amplitude less than calculated in Problem 2.22.

Solution: In this case the frequency is far away from resonance so the change in driving frequency does not matter much. This can also be seen numerically by the following Mathcad session.

$L1 := 0.05$ $\qquad$ $k := 4 \cdot 10^3$ $\qquad\qquad$ $L2 := 0.07$ $\qquad\qquad$ $L := 0.1$
$m := 40$ $\qquad$ $mb := 40$

$meq := m \cdot L^2 + \dfrac{1}{3} \cdot \left(L^3 + L1^3\right) \cdot \dfrac{mb}{L + L1}$ $\qquad$ $meq = 0.5$ $\qquad\qquad$ $g := 9.81$

$c := L2^2$ $\qquad$ $c = 4.9 \cdot 10^{-3}$

$keq := k \cdot L1^2 + m \cdot g + \dfrac{1}{2 \cdot (L - L1)} \cdot (mb \cdot g)$ $\qquad$ $keq = 4.326 \cdot 10^3$

$ceq := \dfrac{0.2 \cdot 2 \cdot \sqrt{meq \cdot keq}}{c}$ $\qquad$ $ceq = 3.797 \cdot 10^3$

$\omega n := \sqrt{\dfrac{keq}{meq}}$ $\qquad$ $\omega n = 93.02$ $\qquad$ $X := 10 \cdot \dfrac{L}{meq} \cdot \dfrac{1}{\sqrt{\left(\omega n^2 - 11^2\right)^2 + \left(2 \cdot .2 \cdot \omega n \cdot 11\right)^2}}$

$X = 2.341 \cdot 10^{-4}$

The new amplitude is only slightly larger in this case. The problem would be more meaningful if the driving frequency is near resonance. Then the shift in amplitude will be more substantial and added damping may improve the response.

2.24 Compute the forced response of a spring-mass-damper system with the following values: $c = 200$ kg/s, $k = 2000$ N/m, $m = 100$ kg, subject to a harmonic force of magnitude $F_0 = 15$ N and frequency of 10 rad/s and initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s. Plot the response. How long does it take for the transient part to die off?

Solution:
Calculate the parameters

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{100}} = 4.472 \text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{15}{100} = 0.15 \text{ N/kg}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.472\sqrt{1-0.224^2} = 4.359 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{200}{2\cdot100\cdot4.472} = 0.224$$

Initial conditions: $x_0 = 0.01$ m, $v_0 = 0.1$ m/s

Using equation (2.38) and working in Mathcad yields

$$x(t) = e^{-t}(0.0104\cos 4.359t + 0.025\sin 4.359t) + 1.318\times10^{-6}(0.335\cos 10t + 37.7\sin 10t)$$

$c := 200 \qquad\qquad k := 2000 \qquad\qquad m := 100 \qquad\qquad F0 := 15 \qquad\qquad \omega := 10$

$\omega n := \sqrt{\dfrac{k}{m}} \qquad x0 := 0.01 \qquad\qquad v0 := 0.1 \qquad f0 := \dfrac{F0}{m} \qquad \zeta := \dfrac{c}{2\cdot m\cdot \omega n} \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$

$$A := x0 - \frac{f0\cdot\left(\omega n^2 - \omega^2\right)}{\left(\omega n^2 - \omega^2\right)^2 + (2\cdot\zeta\cdot\omega\cdot\omega n)^2}$$

$$B := \frac{\zeta\cdot\omega n}{\omega d}\cdot\left[\frac{f0\cdot\left(\omega n^2 - \omega^2\right)}{\left(\omega n^2 - \omega^2\right)^2 + (2\cdot\zeta\cdot\omega\cdot\omega n)^2}\right] - \frac{2\cdot\zeta\cdot\omega n\cdot\omega^2}{\omega d\cdot\left[\left(\omega n^2 - \omega^2\right)^2 + (2\cdot\zeta\cdot\omega\cdot\omega n)^2\right]} + \frac{v0}{\omega d}$$

$$C := \frac{f0}{\left(\omega n^2 - \omega^2\right)^2 + (2\cdot\zeta\cdot\omega\cdot\omega n)^2}$$

$$x(t) := e^{-\zeta\cdot\omega n\cdot t}\cdot(A\cdot\cos(\omega d\cdot t) + B\cdot\sin(\omega d\cdot t)) + C\cdot\left[\left(\omega n^2 - \omega^2\right)\cdot\cos(\omega\cdot t) + 2\cdot\zeta\cdot\omega n\cdot\omega\cdot\sin(\omega\cdot t)\right]$$
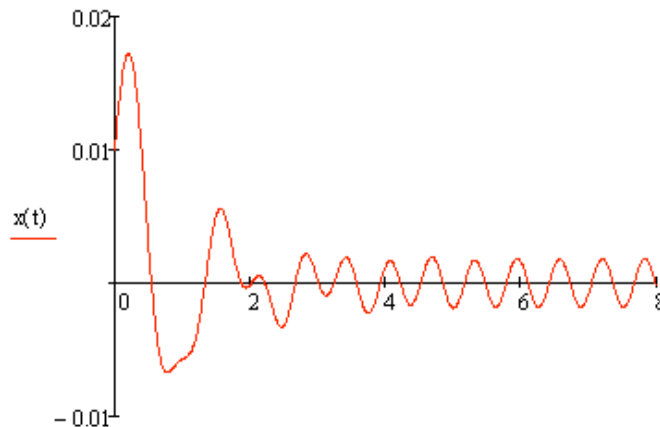
a plot of m vs seconds. The time for the amplitude of the transient response to be reduced, for example, to 0.1 % of the initial (t = 0) amplitude can be determined by:

$$e^{-t} = 0.001, \text{ then } t = -\ln 0.001 = 6.908 \text{ sec}$$

2.25 Show that Equation (2.38) collapses to give Equation (2.11) in the case of zero damping.

Solution:
Eq. (2.38):

$$x(t) = e^{-\zeta\omega_n t} \left\{ \begin{array}{l} \left[(x_0 - \dfrac{f_0(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}) \cos\omega_d t \right. \\[2em] \left. + \left(\begin{array}{c} \dfrac{\zeta\omega_n}{\omega_d}(x_0 - \dfrac{f_0(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}) \\[1.5em] - \dfrac{2\zeta\omega_n\omega^2 f_0}{\omega_d\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right]} + \dfrac{v_0}{\omega_d} \end{array} \right) \sin\omega_d t \right] \end{array} \right\}$$

$$+ \dfrac{f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\left[(\omega_n^2 - \omega^2)\cos\omega t + 2\zeta\omega_n\omega\sin\omega t \right]$$

In case of $\zeta = 0$, this equation becomes:

$$x(t) = 1 \cdot \left\{ \begin{array}{l} \left[(x_0 - \dfrac{f_0}{(\omega_n^2 - \omega^2) + 0}) \cos\omega_d t \right. \\[1.5em] \left. + \left(0 - 0 + \dfrac{v_0}{\omega_d} \right) \sin\omega_d t \right] \end{array} \right\} + \dfrac{f_0}{(\omega_n^2 - \omega^2)}\cos\omega t$$

$$= \dfrac{v_0}{\omega_n}\sin\omega_n t + (x_0 - \dfrac{f_0}{\omega_n^2 - \omega^2})\cos\omega_n t + \dfrac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

(Note: $\omega_d = \omega_n$ for $\zeta = 0$)

2.26 Derive Equation (2.38) for the forced response of an underdamped system.

Solution:
From Sec. 1.3, the homogeneous solution is:
$$x_h(t) = e^{-\zeta\omega_n t}(A_1\sin\omega_d t + A_2\cos\omega_d t)$$
From equations (2.29) and (2.35), the particular solution is:
$$x_p(t) = \dfrac{(\omega_n^2 - \omega^2)f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\cos\omega t + \dfrac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\sin\omega t$$
Then the general solution is:
$$x(t) = x_h(t) + x_p(t) = e^{-\zeta\omega_n t}(A_1\sin\omega_d t + A_2\cos\omega_d t)$$

$$+ \dfrac{(\omega_n^2 - \omega^2)f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\cos\omega t + \dfrac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\sin\omega t$$

Using the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$, the constants, A_1 and A_2, are determined:

$$A_2 = x_0 - \frac{(\omega_n^2 - \omega^2)f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$A_1 = \frac{v_0}{\omega_d} + \frac{\omega}{\omega_d} \cdot \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} + \zeta\frac{\omega_n}{\omega_d}(x_0 - \frac{(\omega_n^2 - \omega^2)f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2})$$

Then, Eq. (2.30) is obtained by substituting the expressions for A_1 and A_2 into the general solution and simplifying the resulting equation.

2.27 Compute a value of the damping coefficient c such that the steady state response amplitude of the system in Figure P2.27 is 0.01 m.
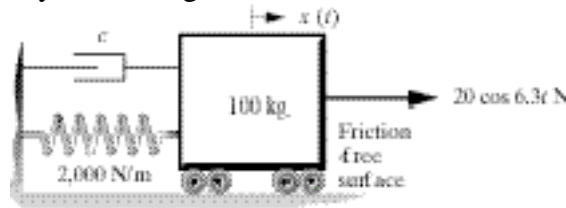
Figure P2.27

Solution:

From Eq. (2.39), the amplitude of the steady state response is given by

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

Then substitute, $2\zeta\omega_n = c/m$, $c = \sqrt{\frac{F_0^2}{\omega^2 \cdot X^2} - m^2\frac{(\omega_n^2 - \omega^2)^2}{\omega^2}}$ into this equation

and solve for c:

Given:

$X = 0.01$m $\qquad\qquad \omega = 6.3\,\mathrm{rad/s} \qquad F_0 = 20$N $\qquad m = 100$kg

$$\omega_n^2 = \frac{k}{m} = \frac{2000}{100} = 20 \;(\mathrm{rad/s})^2 \Rightarrow \underline{c = 55.7 \text{ kg/s}}$$

2.28 Compute the response of the system in Figure P2.28 if the system is initially at rest for the values $k_1 = 100$ N/m, $k_2 = 500$ N/m, $c = 20$ kg/s and $m = 89$ kg.
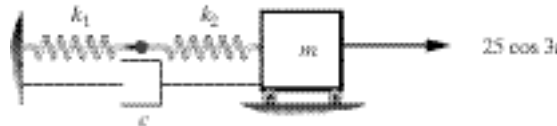
Solution:

The equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = 25\cos 3t \qquad \text{where} \quad k = \frac{1}{1/k_1 + 1/k_2}$$

Using Eq. (2.37) in an alternative form, the general solution is:

$$x(t) = e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t) + X\cos(\omega t - \theta)$$

where

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = \frac{25/89}{\sqrt{(0.966^2 - 3^2)^2 + (2\cdot 0.116 \cdot 0.966 \cdot 3)^2}} = 0.0347\,m$$

$$\theta = \tan^{-1}\cdot\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} = \tan^{-1}\cdot\frac{2\cdot 0.116\cdot 0.966\cdot 3}{0.966^2 - 3^2} = 3.058\,rad \qquad \text{(see Window 2.3)}$$

Using the initial conditions, $x(0) = 0$ and $\dot{x}(0) = 0$, the constants, A_1 and A_2, are determined:

$A_2 = 0.0345$ $\qquad\qquad A_1 = -0.005$

Given: $c = 20$ kg/sec, $m = 89$ kg

$$k = \frac{1}{1/k_1 + 1/k_2} = \frac{1}{1/100 + 1/500} = 83\,\text{N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{83}{89}} = 0.966\,\text{rad/s} \qquad\qquad \zeta = \frac{c}{2m\omega_n} = \frac{20}{2\cdot 89\cdot 0.966} = 0.116$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 0.966\sqrt{1-0.116^2} = 0.9595\,\text{rad/s}$$

Substituting the values into the general solution:

$$x(t) = e^{-0.112t}(-0.005\sin 0.9595t + 0.0345\cos 0.9595t) + 0.0347\cos(3t - 3.058)$$

2.29 Write the equation of motion for the system given in Figure P2.29 for the case that $F(t) = F\cos\omega t$ and the surface is friction free. Does the angle θ effect the magnitude of oscillation?

Solution:

Free body diagram:

(Forces that are normal to the x direction are neglected)
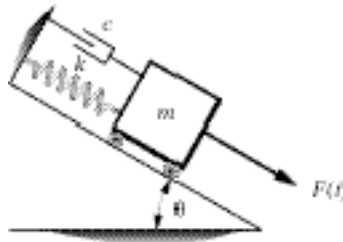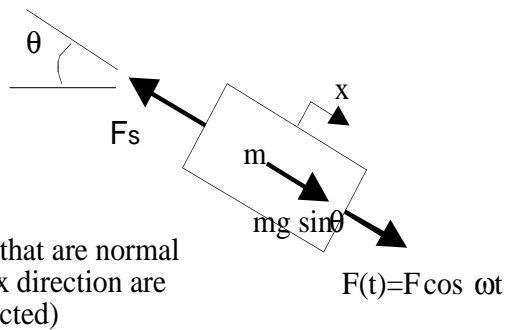
Assuming x = 0 to be at the equilibrium:

$$\sum F_x = F + mg\sin\theta - F_s = m\ddot{x}$$

where $F_s = k(x + \dfrac{mg\sin\theta}{k})$ and $F(t) = F\cos\omega t$

Then the equation of motion is:

$m\ddot{x} + kx = F\cos\omega t$

Note that the equation of motion does not contain θ which means that the magnitude of the response is not affected by the angle of the incline.

2.30 A foot pedal for a musical instrument is modeled by the sketch in Figure P2.30. With $k = 2000$ N/m, $c = 25$ kg/s, $m = 25$ kg and $F(t) = 50\cos 2\pi\,t$ N, compute the steady state response assuming the system starts from rest. Also use the small angle approximation.
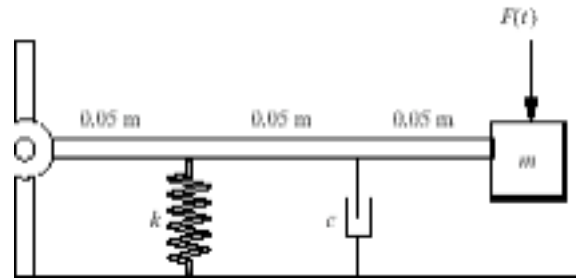
Solution: Free body diagram of pedal follows:
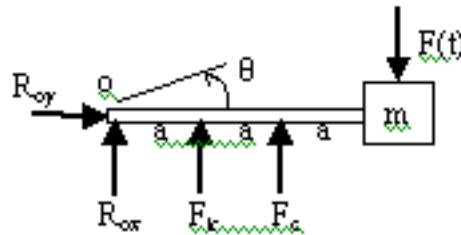
Summing the moments with respect to the point, O:

$$\sum M_0 = F(3\cdot a) - F_c(2\cdot a) - F_s(a) = I_o\ddot{\theta}$$

where $I_o = m(3a)^2 = 9a^2 m$, $F_s = ka\sin\theta$

$$F_c = c(2\cdot a\cdot\sin\theta)' = 2ca\cos\theta\dot{\theta}$$

Substituting these equations and simplifying ($\sin\theta \approx \theta$, $\cos\theta = 1$, for small θ):

$$9a^2 m\ddot{\theta} + 4a^2 c\dot{\theta} + a^2 k\theta = 3a\,F(t)$$

Given: $k = 2000$ N/m, $c = 25$ kg/s , $m = 25$ kg , $F(t) = 50\cos 2\pi t$, $a = 0.05$ m

The equation of motion becomes: $0.5625\ddot{\theta} + 0.25\dot{\theta} + 5\theta = 7.5\cos 2\pi t$

Observing the equation of motion, equivalent mass, damping and stiffness coefficients are:

$$c_{eq} = 0.25, \quad m_{eq} = 0.5625, \quad k_{eq} = 5 \ , \quad f_0 = \dfrac{F_0}{m_{eq}} = \dfrac{7.5}{0.5625} = 13.33 \ , \quad \omega = 2\pi$$

$$\omega_n = \sqrt{\dfrac{k_{eq}}{m_{eq}}} = \sqrt{\dfrac{5}{0.5625}} = 2.981 \qquad\qquad \zeta = \dfrac{c_{eq}}{2m_{eq}\omega_n} = 0.0745$$

From Eq. (2.36), the steady-state response is:

$$\theta(t) = \frac{f_{0eq}}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}\cos(\omega t - \tan^{-1}\frac{2\zeta_{eq}\omega_n\omega}{\omega_n^2 - \omega^2})$$

$$\Rightarrow \theta(t) = 0.434\cos(2\pi t - 3.051) \;\; \text{rad}$$

2.31 Consider the system of Problem 2.15, repeated here as Figure P2.31 with the effects of damping indicated. The physical constants are $J = 25$ kg m^2, $k = 2000$ N/m, and the applied moment is 5 Nm at 1.432 Hz acting through the distance $r = 0.5$ m. Compute the magnitude of the steady state response if the measured damping ratio of the spring system is $\zeta = 0.01$. Compare this to the response for the case where the damping is not modeled ($\zeta = 0$).
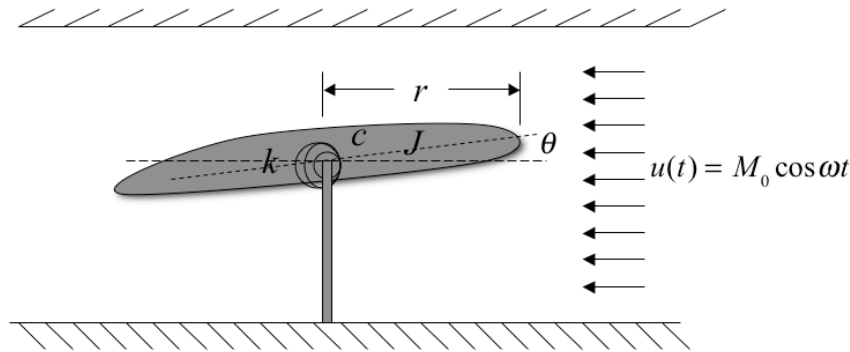
Figure P2.31 Model of an airfoil in at wind tunnel including the effects of damping.

Solution From equation (2.39) the magnitude of the steady state response for an underdamped system is

$$|\theta| = \frac{M_0/J}{\sqrt{\left(\frac{k}{J} - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}}$$

Substitution of the given values yields (here $X = r\theta$)

$$|\theta| = 0.2 \text{ rad and } X = 0.1 \text{ m for } \zeta = 0$$

$$|\theta| = 0.106 \text{ rad and } X = 0.053 \text{ m for } \zeta = 0.01$$

where X is the vertical displacement of the wing tip. Thus a small amount of damping can greatly reduce the amplitude of vibration.

Problems and Solutions Section 2.3 (2.32 through 2.36)

2.32 Referring to Figure 2.10, draw the solution for the magnitude X for the case $m = 100$ kg, $c = 4000$ N s/m, and $k = 10,000$ N/m. Assume that the system is driven at resonance by a 10-N force.

Solution:
Given: $m = 100$ kg, $c = 4000$ N s/m, $k = 10000$ N/m, $F_o = 10$ N,

$$\omega = \omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\phi = \tan^{-1}\left[\frac{cw}{k - m\omega^2}\right] = \tan^{-1}\left[\frac{(40,000)}{(10,000 - 10,000)}\right] = 90° = \frac{\pi}{2} \text{ rad}$$
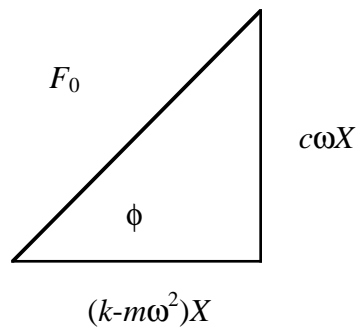
From the figure:
$$X = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{10}{\sqrt{(10,000 - 10,000)^2 + (40,000)^2}}$$
$$X = 0.00025 \text{ m}$$

2.33 Use the graphical method to compute the phase shift for the system of Problem 2.32 if ω = $\omega_n/2$ and again for the case $\omega = 2\omega_n$.

Solution:
From Problem 2.32 ω_n= 10 rad/s

(a) $\omega = \dfrac{\omega_n}{2} = 5$ rad/s

$$X = \frac{10}{\sqrt{(10,000 - 2500)^2 + (20,000)^2}} = .000468 \text{ m}$$

$kX = (10,000)(.000468) = 4.68$ N
$c\omega X = (4000)(5)(.000468) = 9.36$ N
$m\omega^2 X = (100)(5)^2(.000468) = 1.17$ N

From the figure given in problem 2.32:

$$\phi = \tan^{-1}\left[\frac{9.36}{4.68 - 1.17}\right] = 69.4° = 1.21 \text{rad}$$

(b) $\omega = 2\omega_n = 20$ rad/s

$$X = \frac{10}{\sqrt{(10000 - 40000)^2 + (80000)^2}} = .000117 \text{ m}$$

$kX = (10000)(.000117) = 1.17$ N
$c\omega X = (4000)(20)(.000117) = 9.36$ N
$m\omega^2 X = (100)(20)^2(.000117) = 4.68$ N

From the figure:

$$\phi = \tan^{-1}\left[\frac{9.36}{1.17 - 4.68}\right] = -69.4° = -1.21 \text{ rad}$$

2.34 A body of mass 100 kg is suspended by a spring of stiffness of 30 kN/m and dashpot of damping constant 1000 N s/m. Vibration is excited by a harmonic force of amplitude 80 N and a frequency of 3 Hz. Calculate the amplitude of the displacement for the vibration and the phase angle between the displacement and the excitation force using the graphical method.

Solution:
Given: $m = 100$kg, $k = 30$ kN/m, $F_o = 80$ N, $c = 1000$ Ns/m,

$$\omega = 3(2\pi) = 18.85 \text{ rad/s}$$
$$kX = 30000\ X$$
$$c\omega X = 18850\ X$$
$$m\omega^2 X = 35530\ X$$

Following the figure given in problem 2.32:

$$\phi = \tan^{-1}\left[\frac{c\omega X}{\left(k - m\omega^2\right)X}\right]$$

$$\phi = \tan^{-1}\left[\frac{(18850)X}{(30000 - 35530)X}\right] = 106.4° = 1.86 \text{ rad}$$

Also from the figure, $X = \dfrac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$

$$X = \frac{80}{\sqrt{(30000 - 35530)^2 + (18850)^2}} = 0.00407 \text{ m}$$

2.35 Calculate the real part of equation (2.55) to verify that it yields equation (2.36) and hence establish the equivalence of the exponential approach to solving the damped vibration problem.

Solution:

Equation (2.55)

$$x_p(t) = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} e^{j(\omega t - \theta)}$$

where $\theta = \tan^{-1}\left[\dfrac{c\omega}{k - m\omega^2}\right]$

Using Euler's Rule:

$$x_p(t) = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} [\cos(\omega t - \theta) + j\sin(\omega t - \theta)]$$

The real part is:

$$x_p(t) = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \theta)$$

Rearranging:

$$x_p(t) = \frac{F_o/m}{\sqrt{(\omega^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos\left(\omega t - \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right]\right)$$

which is Equation (2.36).

2.36 Referring to equation (2.56) and Appendix B, calculate the solution $x(t)$ by using a table of Laplace transform pairs and show that the solution obtained this way is equivalent to (2.36).

Solution: Taking the Laplace transform of the equation of motion is given in Equation

(2.56):

$$X_p = (ms^2 + cs + k)X(s) = \frac{F_o s}{s^2 + \omega^2}$$

Solving this expression algebraically for X yields

$$X(s) = \frac{F_o s}{(ms^2 + cs + k)(s^2 + \omega^2)} = \frac{f_o s}{(s^2 + 2\zeta\omega_n s + \omega^2)(s^2 + \omega^2)}$$

Using Laplace Transform pairs from the table, this last expression is changed into the time domain to get:

$$x(t) = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos(\omega t - \theta)$$

Problems and Solutions Section 2.4 (2.37 through 2.50)

2.37 A machine weighing 2000 N rests on a support as illustrated in Figure P2.37. The support deflects about 5 cm as a result of the weight of the machine. The floor under the support is somewhat flexible and moves, because of the motion of a nearby machine, harmonically near resonance ($r =1$) with an amplitude of 0.2 cm. Model the floor as base motion, and assume a damping ratio of $\zeta = 0.01$, and calculate the transmitted force and the amplitude of the transmitted displacement.

Rubber mount modeled as a stiffness k and a damper c — Machine of mass m — Δ = static deflection — k — c — Flexible floor — $y(t)$
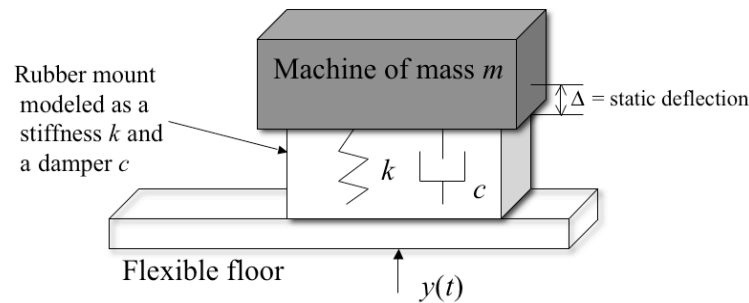
Figure P2.37

Solution:

Given: $Y = 0.2$ cm, $\zeta = 0.01$, $r = 1$, $mg = 2000$N. The stiffness is computed from the static deflection and weight:

Deflection of 5 cm implies: $\quad k = \dfrac{mg}{\Delta} = \dfrac{mg}{5\text{cm}} = \dfrac{2000}{0.05} = 40{,}000$ N/m

Transmitted displacement from equation (2.70): $\quad X = Y\left[\dfrac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2} = 10$ cm

Transmitted force from equation (2.77): $\quad F_T = kYr^2\left[\dfrac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2} = 4001$N

2.38 Derive Equation (2.70) from (2.68) to see if the author has done it correctly.

Solution:

Equation (2.68) states:

$$x_p(t) = \omega_n Y\left[\frac{\omega_n^2+(2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}\right]^{1/2}\cos(\omega_b t-\theta_1-\theta_2)$$

The magnitude is: $\quad X = \omega_n Y\left[\dfrac{\omega_n^2+(2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}\right]^{1/2}$

$$= \omega_n Y \left[\frac{(\omega_n^{-4})(\omega_n^2 + (2\zeta\omega_b)^2)}{(\omega_n^{-4})((\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2)} \right]^{1/2}$$

$$= \omega_n Y \left[\frac{(\omega_n^{-2})(1 + (2\zeta r)^2)}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2} \Rightarrow$$

$$= \omega_n Y \frac{1}{\omega_n} \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2} \Rightarrow$$

$$X = Y \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$

This is equation (2.71).

2.39 From the equation describing Figure 2.13, show that the point ($\sqrt{2}$, 1) corresponds to the value TR > 1 (i.e., for all $r < \sqrt{2}$, TR > 1).

Solution:

Equation (2.71) is $\quad TR = \dfrac{X}{Y} = \left[\dfrac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$

Show $TR > 1$ for $r < \sqrt{2}$

$$TR = \frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2} > 1$$

$$\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} > 1$$

$$1 + (2\zeta r)^2 > (1-r^2)^2 + (2\zeta r)^2$$

$$1 > (1-r^2)^2$$

Take the real solution:

$$1 - r^2 < +1 \text{ or } 1 - r^2 < -1 \Rightarrow$$

$$-r^2 > -2 \Rightarrow r^2 < 2 \Rightarrow r < \sqrt{2}$$

2- 29

2.40 Consider the base excitation problem for the configuration shown in Figure P2.40. In this case the base motion is a displacement transmitted through a dashpot or pure damping element. Derive an expression for the force transmitted to the support in steady state.
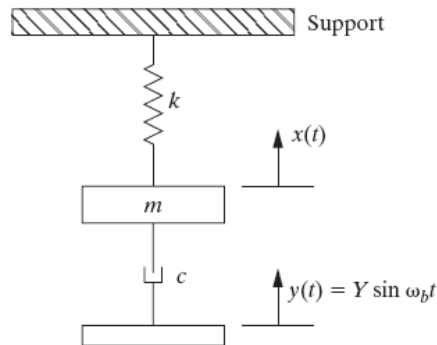
Figure P2.40

Solution: The entire force passes through the spring. Thus the support sees the force $F_T = kX$ where X is the magnitude of the displacement. From equation (2.65)

$$F_T = kX = \frac{2\zeta\omega_n\omega_b kY}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}$$

$$= \frac{2\zeta rkY}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

2.41 A very common example of base motion is the single-degree-of-freedom model of an automobile driving over a rough road. The road is modeled as providing a base motion displacement of $y(t) = (0.01)\sin(5.818t)$ m. The suspension provides an equivalent stiffness of $k = 4 \times 10^5$ N/m, a damping coefficient of $c = 40 \times 10^3$ kg/s and a mass of 1007 kg. Determine the amplitude of the absolute displacement of the automobile mass.

Solution:
From the problem statement we have (working in Mathcad)

$\omega b := 5.818 \qquad k := 4 \cdot 10^5 \quad$ N/m $\qquad c := 40 \cdot 10^3 \quad$ kg/s

$Y := 0.01 \quad$ m $\qquad m := 1007 \quad$ kg

$\omega n := \sqrt{\dfrac{k}{m}} \qquad \zeta := \dfrac{c}{2 \cdot \sqrt{m \cdot k}} \qquad \omega n = 19.93 \qquad \zeta = 0.997 \qquad r := \dfrac{\omega b}{\omega n} \qquad r = 0.292$

still underdamped, but very high damping. From equation (2.70)

$$X := Y \cdot \sqrt{\frac{1 + (2 \cdot \zeta \cdot r)^2}{(1 - r^2)^2 + (2 \cdot \zeta \cdot r)^2}} \qquad X = 0.011 \quad \text{m}$$

2.42   A vibrating mass of 300 kg, mounted on a massless support by a spring of stiffness 40,000 N/m and a damper of unknown damping coefficient, is observed to vibrate with a 10-mm amplitude while the support vibration has a maximum amplitude of only 2.5 mm (at resonance). Calculate the damping constant and the amplitude of the force on the base.

Solution:

Given: $m = 300$ kg, $k = 40,000$ N/m, $\omega_b = \omega_n (r = 1)$, $X = 10$ mm, $Y = 2.5$ mm.
Find damping constant (Equation 2.71)

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2} \Rightarrow \frac{10}{2.5} = \left[\frac{1 + 4\zeta^2}{4\zeta^2} \right]^{1/2} \Rightarrow$$

$$16 = \frac{1 + 4\zeta^2}{4\zeta^2} \Rightarrow \zeta^2 = \frac{1}{60} = \frac{c^2}{4km} \quad \text{or}$$

$$c = \sqrt{\frac{4(40,000)(300)}{60}} = 894.4 \text{ kg/s}$$

Amplitude of force on base: (equation (2.76))

$$F_T = kYr^2 \left[\frac{1 + (2\zeta r)^2}{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2} \right]^{1/2} \Rightarrow$$

$$F_T = (40,000)(0.0025)(1)^2 \left[\frac{1 + 4\left(\dfrac{1}{60}\right)}{4\left(\dfrac{1}{60}\right)} \right]^{1/2} \Rightarrow$$

$$F_T = 400 \text{ N}$$

2.43 Referring to Example 2.4.1, at what speed does car 1 experience resonance? At what speed does car 2 experience resonance? Calculate the maximum deflection of both cars at resonance.

Solution:

Given: $m_1 = 1007$ kg, $m_2 = 1585$ kg, $k = 4\text{x}10^5$ N/m; $c = 2{,}000$ kg/s, $Y = 0.01$m
Velocity for resonance: (from Example 2.4.1)
$$\omega_b = 0.2909v \ (v \text{ in km/h})$$

Car 1: $\omega_1 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4 \times 10^4}{1007}} = \omega_b = 0.2909v_1$

$\qquad\qquad v_1 = 21.7$ km/h

Car 2: $\omega_2 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4 \times 10^4}{1585}} = \omega_b = 0.2909v_2$

$\qquad\qquad v_2 = 17.3$ km/h

Maximum deflection: (Equation 2.71 with $r = 1$)

$$X = Y\left[\frac{1 + 4\zeta^2}{4\zeta^2}\right]^{1/2} \Rightarrow$$

Car 1: $\zeta_1 = \dfrac{c}{2\sqrt{km_1}} = \dfrac{2000}{2\sqrt{(4 \times 10^5)(1007)}} = 0.158$

$\qquad X_1 = (0.01)\left[\dfrac{1 + 4(0.158)^2}{4(0.158)^2}\right]^{1/2} = 0.033$ m

Car 2: $\zeta_2 = \dfrac{c}{2\sqrt{km_2}} = \dfrac{2000}{2\sqrt{(4 \times 10^4)(1585)}} = 0.126$

$\qquad X_2 = (0.01)\left[\dfrac{1 + 4(0.126)^2}{4(0.126)^2}\right]^{1/2} = 0.041$ m

2.44 For cars of Example 2.4.1, calculate the best choice of the damping coefficient so that the transmissibility is as small as possible by comparing the magnitude of $\zeta = 0.01$, $\zeta = 0.1$ and $\zeta = 0.2$ for the case $r = 2$. What happens if the road "frequency" changes?

Solution:

From Equation 2.62, with $r = 2$, the displacement transmissibility is:

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2} = \left[\frac{1 + 16\zeta^2}{9 + 16\zeta^2}\right]^{1/2}$$

For $\zeta = 0.01$, $\dfrac{X}{Y} = 0.334$

For $\zeta = 0.1$, $\dfrac{X}{Y} = 0.356$

For $\zeta = 0.2$, $\dfrac{X}{Y} = 0.412$

The best choice would be $\zeta = 0.01$.

If the road frequency increases, the lower damping ratio would still be the best choice. However, if the frequency decreases, a higher damping ratio would be better because it would approach resonance.

2.45 A system modeled by Figure 2.12, has a mass of 225 kg with a spring stiffness of 3.5 $\times 10^4$ N/m. Calculate the damping coefficient given that the system has a deflection (X) of 0.7 cm when driven at its natural frequency while the base amplitude (Y) is measured to be 0.3 cm.

Solution:

Given: $m = 225$ kg, $k = 3.5 \times 10^4$ N/m, $X = 0.7$ cm, $Y = 0.3$ cm, $\omega = \omega_b$.
Base excitation: (Equation (2.71) with $r = 1$)

$$\frac{X}{Y} = \left[\frac{1 + 4\zeta^2}{4\zeta^2}\right]^{1/2} \Rightarrow \frac{0.7}{0.3} = \left[\frac{1 + 4\zeta^2}{4\zeta^2}\right]^{1/2} \Rightarrow$$

$$\zeta = 0.237 = \frac{c}{2\sqrt{km}}$$

$$c = (0.237)(2)[(3.5 \times 10^4)(225)]^{1/2}$$

$$\underline{c = 1331 \text{ kg/s}}$$

2.46 Consider Example 2.4.1 for car 1 illustrated in Figure P2.46, if three passengers totaling 200 kg are riding in the car. Calculate the effect of the mass of the passengers on the deflection at 20, 80, 100, and 150 km/h. What is the effect of the added passenger mass on car 2?
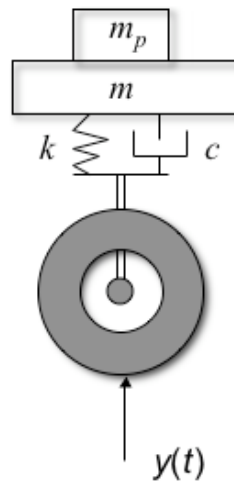
Figure P2.46 Model of a car suspension with the mass of the occupants, m_p, included.

Solution:

Add a mass of 200 kg to each car. From Example 2.4.1, the given values are:
$m_1 = 1207$ kg, $m_2 = 1785$ kg, $k = 4$x10^4 N/m; $c = 2,000$ kg/s, $\omega_b = 0.29v$.

Car 1: $\omega_1 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4 \times 10^4}{1207}} = 5.76$ rad/s

$\zeta_1 = \dfrac{c}{2\sqrt{km_1}} = \dfrac{2000}{2\sqrt{(4 \times 10^5)(1207)}} = 0.144$

Car 2: $\omega_2 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4 \times 10^4}{1785}} = 4.73$ rad/s

$\zeta_2 = \dfrac{c}{2\sqrt{km_2}} = \dfrac{2000}{2\sqrt{(4 \times 10^5)(1785)}} = 0.118$

Using Equation (2.71): $X = Y \left[\dfrac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{1/2}$ produces the following:

Speed (km/h)	ω_b (rad/s)	r_1	r_2	x_1 (cm)	x_2 (cm)
20	5.817	1.01	1.23	3.57	1.77
80	23.271	3.871	4.71	0.107	0.070
100	29.088	5.05	6.15	0.072	0.048
150	2.40	7.58	9.23	0.042	0.028

At lower speeds there is little effect from the passengers weight, but at higher speeds the added weight reduces the amplitude, particularly in the smaller car.

2.47 Consider Example 2.4.1. Choose values of c and k for the suspension system for car 2 (the sedan) such that the amplitude transmitted to the passenger compartment is as small as possible for the 1 cm bump at 50 km/h. Also calculate the deflection at 100 km/h for your values of c and k.

Solution:

For car 2, $m = 1585$ kg.
Also, $\omega_b = 0.2909(50) = 14.545$ rad/s and $Y = 0.01$ m.
From equation (2.70),

$$X = Y\left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

From Figure 2.9, we can choose a value of r away from resonance and a low damping ratio. Choose $r = 2.5$ and $\zeta = 0.05$.

So, $\quad r = 2.5 = \dfrac{\omega_b}{\omega} = \dfrac{14.545}{\sqrt{k/1585}}$

$\quad k = 53{,}650$ N/m

$\quad \zeta = 0.05 = \dfrac{c}{2\sqrt{km}}$

$\quad c = 922.2$ kg/s

So, $\quad X = (0.01)\left[\dfrac{1 + [2(0.05)(2.5)]^2}{\left(1 - \left(2.5\right)^2\right)^2 + [2(0.05)(2.5)]^2}\right]^{1/2} = 0.00196$ m

At 100 km/h, $\omega_b = 29.09$ rad/s and $r = \dfrac{\omega_b}{\sqrt{k/m}} = 5.$

2.48 Consider the base motion problem of Figure 2.12. a) Compute the damping ratio needed to keep the displacement magnitude transmissibility less then 0.55 for a frequency ratio of $r = 1.8$. b) What is the value of the force transmissibility ratio for this system?

Solution: Working with equation (2.71), make a plot of TR versus ζ and use equation (2.77) to compute the value of the force transmissibility. The following Mathcad session illustrates the procedure.

$$r := 1.8$$

$$TR(\zeta) := \sqrt{\frac{1 + (2 \cdot \zeta \cdot r)^2}{(1 - r^2)^2 + (2 \cdot \zeta \cdot r)^2}} \qquad M(\zeta) := 0.55$$
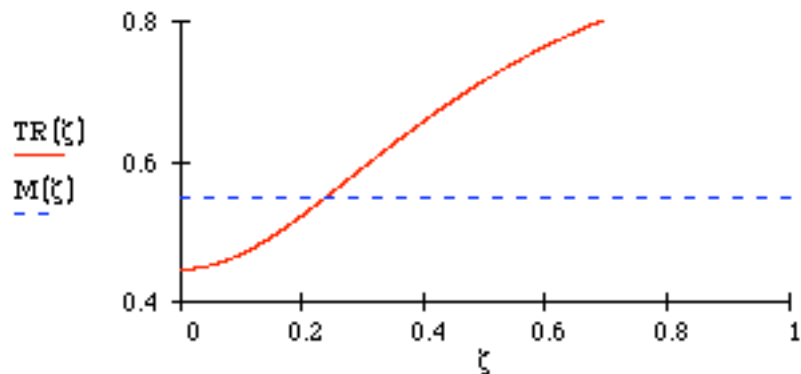
$$F(\zeta) := r^2 \cdot TR(\zeta)$$

$$F(.2) = 1.697$$

From the plot a value of $\zeta = 0.2$ keeps the displacement transmissibility less then 0.55 as desired. The value of the force transmissibility is then 1.697. Precise values can be found by equating the above expression to 0.55.

2.49 Consider the effect of variable mass on an aircraft landing suspension system by modeling the landing gear as a moving base problem similar to that shown in Figure P2.46 for a car suspension. The mass of a regional jet is 13, 236 kg empty and its maximum takeoff mass is 21,523 kg. Compare the maximum deflection for a wheel motion of magnitude 0.50 m and frequency of 35 rad/s, for these two different masses. Take the damping ratio to be $\zeta = 0.1$ and the stiffness to be 4.22 x 10^6 N/m.

Solution: Using a Mathcad worksheet the following calculations result:

$Y := 0.5 \cdot m \quad k = 4.22 \times 10^6 \, kg \cdot s^{-2} \quad mf := 21523 \, kg \qquad me := 13236 \cdot kg$

From equation (2.70):

$re := \dfrac{wb}{\sqrt{\dfrac{k}{me}}} \qquad re = 1.96 \qquad \zeta := 0.1 \qquad\qquad wb := 35 \cdot \dfrac{rad}{sec}$

$Xe := Y \cdot \sqrt{\dfrac{1 + (2 \cdot \zeta \cdot re)^2}{\left(1 - re^2\right)^2 + (2 \cdot \zeta \cdot re)^2}} \qquad\qquad \sqrt{\dfrac{k}{mf}} = 14.002 \, s^{-1}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad Xe = 0.187 \, m$

$rf := \dfrac{wb}{\sqrt{\dfrac{k}{mf}}} \qquad rf = 2.5$

$Xf := Y \cdot \sqrt{\dfrac{1 + (2 \cdot \zeta \cdot rf)^2}{\left(1 - rf^2\right)^2 + (2 \cdot \zeta \cdot rf)^2}} \qquad\qquad Xf = 0.106 \, m$

Note that if the suspension stiffness were defined around the full case, when empty the plane would bounce with a larger amplitude then when full. Note Mathcad does not have a symbol for a Newton so the units on stiffness above are kg/sec^2 in order to allow Mathcad to compute the units.

2.50 Consider the simple model of a building subject to ground motion suggested in Figure P2.50. The building is modeled as a single degree of freedom spring-mass system where the building mass is lumped atop of two beams used to model the walls of the building in bending. Assume the ground motion is modeled as having amplitude of 0.1 m at a frequency of 7.5 rad/s. Approximate the building mass by 10^5 kg and the stiffness of each wall by 3.519 x 10^6 N/m. Compute the magnitude of the deflection of the top of the building.
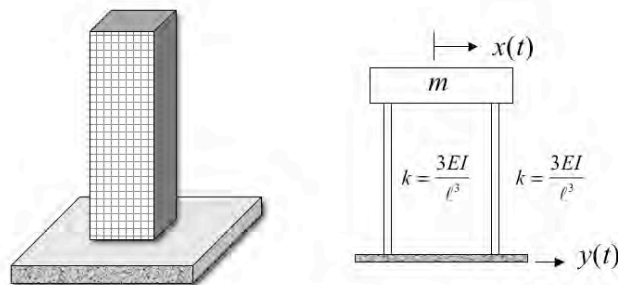
Figure P2.50 A simple model of a building subject to ground motion, such as an earthquake.

Solution: The equation of motion is

$$m\ddot{x}(t) + 2kx(t) = 0.1\cos 7.5t$$

The natural frequency and frequency ratio are

$$\omega_n = \sqrt{\dfrac{2k}{m}} = 8.389 \ \text{rad/s} \quad \text{and} \quad r = \dfrac{\omega}{\omega_n} = \dfrac{7.5}{8.389} = 0.894$$

The amplitude of the steady state response is given by equation (2.70) with $\zeta = 0$ in this case:

$$X = Y \left| \frac{1}{1 - r^2} \right| = 0.498 \text{ m}$$

Thus the earthquake will cause serious motion in the building and likely break.

Problems and Solutions Section 2.5 (2.51 through 2.58)

2.51 A lathe can be modeled as an electric motor mounted on a steel table. The table plus the motor have a mass of 50 kg. The rotating parts of the lathe have a mass of 5 kg at a distance 0.1 m from the center. The damping ratio of the system is measured to be $\zeta = 0.06$ (viscous damping) and its natural frequency is 7.5 Hz. Calculate the amplitude of the steady-state displacement of the motor, assuming $\omega_r = 30$ Hz.

Soltuion:

Given: $m = 50$ kg, $m_o = 5$, $e = 0.1$m, $\zeta = 0.06$, $\omega_n = 7.5$ Hz

Let $\omega_r = 30$ Hz

So, $r = \dfrac{\omega_r}{\omega_n} = 4$

From Equation (2.84),

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{(5)(0.1)}{50} \frac{4^2}{\sqrt{(1-4^2)^2 + [2(0.06)(4)]^{-2}}}$$

$X = 0.011$m

$X = 1.1$ cm

2.52 The system of Figure 2.18 produces a forced oscillation of varying frequency. As the frequency is changed, it is noted that at resonance, the amplitude of the displacement is 10 mm. As the frequency is increased several decades past resonance the amplitude of the displacement remains fixed at 1 mm. Estimate the damping ratio for the system.

Solution: Equation (2.84) is

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

At resonance, $X = 10$ mm $= \dfrac{m_o e}{m} \dfrac{1}{2\zeta}$

$$\frac{10m}{m_o e} = \frac{1}{2\zeta}$$

When r is very large, $\dfrac{Xm}{m_o e} = 1$ and $X = 1$ mm, so

$$\frac{m}{m_o e} = 1$$

Therefore, $10(1) = \dfrac{1}{2\zeta}$

$\zeta = 0.05$

2.53   An electric motor (Figure P2.53) has an eccentric mass of 10 kg (10% of the total mass) and is set on two identical springs ($k = 3200$ /m). The motor runs at 1750 rpm, and the mass eccentricity is 100 mm from the center. The springs are mounted 250 mm apart with the motor shaft in the center. Neglect damping and determine the amplitude of the vertical vibration.

Solution:
Given $m_0 = 10$ kg, $m = 100$ kg, $k = 2$x3.2 N/mm, , $e = 0.1$ m

$$\omega_r = 1750\frac{\text{rev}}{\text{min}}(\frac{\text{min}}{60\sec}\frac{2\pi \text{ rad}}{\text{rev}}) = 183.26\frac{\text{rad}}{s} \text{ rad/s}$$

Vertical vibration:

$$\omega_n = \sqrt{\frac{2(3.2)(1000)}{100}} = 8 \text{ rad/s}$$

$$r = \frac{\omega_r}{\omega_n} = \frac{183.3}{8} = 22.9$$

From equation (2.84)

$$X = e\frac{m_0}{m}\frac{r^2}{|1-r^2|} == 0.01 \text{ m}$$

2.54   Consider a system with rotating unbalance as illustrated in Figure P2.53. Suppose the deflection at 1750 rpm is measured to be 0.05 m and the damping ratio is measured to be $\zeta = 0.1$. The out-of-balance mass is estimated to be 10%. Locate the unbalanced mass by computing e.

Solution: Given: $X = 0.05$ m, $\zeta = 0.1$, $m_e = 0.1m$, and from the solution to problem 2.53 the frequency ratio is calculated to be $r = 22.9$. Solving the rotating unbalance Equation (2.84) for e yields:

$$X = \frac{m_0 e}{m}\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \Rightarrow e = \frac{mX}{m_0}\frac{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}{r^2} = \underline{0.499 \text{ m}}$$

This sort of calculation can be introduced to discuss the application of machinery diagnostics if time permits. Machinery diagnostics deals with determining the location and extend of damage from measurements of the response and input.

2.55   A fan of 45 kg has an unbalance that creates a harmonic force. A spring-damper system is designed to minimize the force transmitted to the base of the fan. A damper is used having a damping ratio of $\zeta = 0.2$. Calculate the required spring stiffness so that only 10% of the force is transmitted to the ground when the fan is running at 10,000 rpm.

Solution: The equation of motion of the fan is
$$m\ddot{x} + c\dot{x} + kx = m_0 e\omega^2 \sin(\omega t + \theta)$$
The steady state solution as given by equation (2.84) is

$$x(t) = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin \omega t$$

where r is the standard frequency ratio. The force transmitted to the ground is

$$F(t) = kx + c\dot{x} = \frac{m_0 e}{m} \frac{kr^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin \omega t + \frac{m_0 e}{m} \frac{c\omega r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cos \omega t$$

Taking the magnitude of this quantity, the magnitude of the force transmitted becomes

$$F_0 = \frac{m_0 e}{m} \frac{r^2 \sqrt{k^2 + c^2 \omega^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = m_0 e \omega \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

From equation (2.81) the magnitude of the force generated by the rotating mass F_r is

$$F_r = m_0 e \omega^2$$

The limitation stated in the problem is that $F_0 = 0.1 F_r$, or

$$m_0 e \omega^2 \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 0.1 m_0 e \omega^2$$

Setting $\zeta = 0.2$ and solving for r yields:

$$r^4 - 17.84 r^2 - 99 = 0$$

which yields only one positive solution for r^2, which is

$$r^2 = 22.28 = \frac{\omega^2}{k/m} \Rightarrow \frac{k}{m} = \left(\frac{10000 \times 2\pi}{60} \right)^2 \frac{1}{22.28}$$

$$\Rightarrow k = 45 \left(\frac{10000 \times 2\pi}{60} \right)^2 \frac{1}{22.28} = 2.21 \times 10^6 \text{ N/m}$$

2.56 Plot the normalized displacement magnitude versus the frequency ratio for the out of balance problem (i.e., repeat Figure 2.20) for the case of $\zeta = 0.05$.

Solution: Working in Mathcad using equation (2.84) yields:

$$Xm(\zeta, r) := \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + \left(2 \cdot \zeta \cdot r\right)^2}}$$
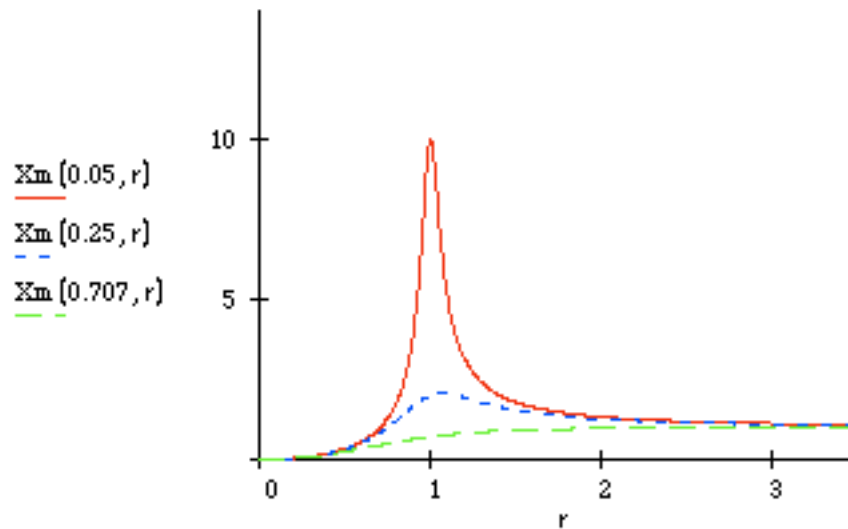
2.57 Consider a typical unbalanced machine problem as given in Figure P2.57 with a machine mass of 120 kg, a mount stiffness of 800 kN/m and a damping value of 500 kg/s. The out of balance force is measured to be 374 N at a running speed of 3000 rev/min. a) Determine the amplitude of motion due to the out of balance. b) If the out of balance mass is estimated to be 1% of the total mass, estimate the value of the e.
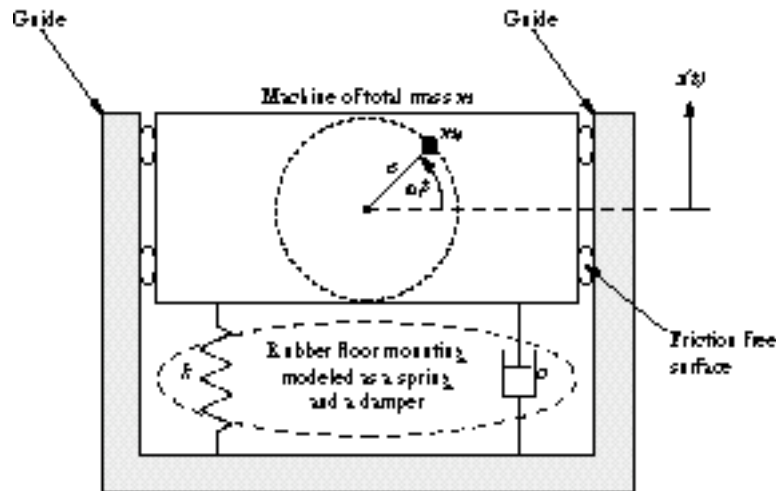
Figure P2.57 Typical unbalance machine problem.

Solution:

a) Using equation (2.84) with $m_0 e = F_0/\omega_r^2$ yields:

$k := 800 \cdot 1000$ $\qquad$ $m := 120$ $\quad$ $c := 500$ $\qquad$ $F0 := 374$

$\omega r := 100 \cdot \pi$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $k = 8 \cdot 10^5$

$$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2 \cdot \sqrt{k \cdot m}}$$

$$r := \frac{\omega r}{\omega n} \qquad \omega n = 81.65 \qquad\qquad r = 3.848 \qquad\qquad \zeta = 0.026$$

$$X := \frac{F0}{\omega r^2 \cdot m} \cdot \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + \left(2 \cdot \zeta \cdot r\right)^2}} \quad X = 3.386 \cdot 10^{-5}$$

b) Use the fact that $F_0 = m_0 e \omega_r^2$ to get

$$e := \frac{F0}{\omega r^2 \cdot (0.01 \cdot m)} \qquad e = 3.158 \cdot 10^{-3}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ in meters.

2.58   Plot the response of the mass in Problem 2.57 assuming zero initial conditions.

Solution: The steady state response is the particular solution given by equation (2.84) and is plotted here in Mathcad:

$m := 120 \qquad k := 120 \qquad c := 500 \qquad F0 := 374$

$$\omega n := \sqrt{\frac{k}{m}} \qquad \omega r := \frac{3000 \cdot 2 \cdot \pi}{60} \qquad r := \frac{\omega r}{\omega n} \qquad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}}$$

$$X := \frac{F0}{\omega r^2 \cdot m} \cdot \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + \left(2 \cdot \zeta \cdot r\right)^2}} \qquad \theta := atan\left(\frac{2 \cdot \zeta \cdot r}{1 - r^2}\right)$$

$$x(t) := X \cdot \sin\left(\omega r \cdot t - \theta\right)$$
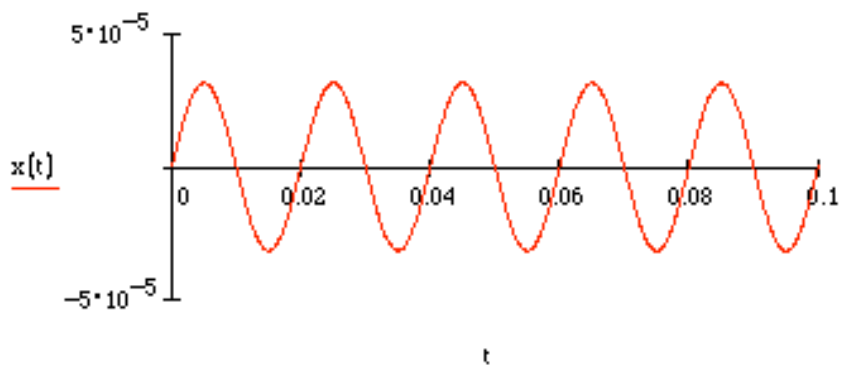
Problems and Solutions Section 2.6 (2.59 through 2.62)

2.59 Calculate damping and stiffness coefficients for the accelerometer of Figure 2.23 with moving mass of 0.04 kg such that the accelerometer is able to measure vibration between 0 and 50 Hz within 5%. (*Hint*: For an accelerometer it is desirable for $Z / \omega_b^2 Y =$ constant.)

Solution: Use equation (2.90):

Given: $m = 0.04$ kg with error $< 5\%$

$$0.2f = 50 \text{ Hz} \rightarrow f = 250 \text{ Hz} \rightarrow \omega = 2\pi f = 1570.8 \text{ rad/s}$$

Thus, $\qquad\qquad\qquad k = m\omega^2 = 98,696$ N/m

When $r = .2$, $\qquad\qquad 0.95 < \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} < 1.05 \qquad (\pm 5\% \text{ error})$

This becomes $\qquad\qquad 0.8317 + 0.1444\zeta^2 < 1 < 1.016 + 0.1764\zeta^2$

Therefore, $\qquad\qquad \zeta = 0.7 = \dfrac{c}{2\sqrt{km}}$

$$c = 2(.7)\sqrt{(98696)(.04)}$$

$$c = 87.956 \text{ Ns/m}$$

2.60 The damping constant for a particular accelerometer of the type illustrated in Figure 2.23 is 50 N s/m. It is desired to design the accelerometer (i.e., choose m and k) for a maximum error of 3% over the frequency range 0 to 75 Hz.

Solution: Given $0.2f = 75 \text{ Hz} \rightarrow f = 375 \text{ Hz} \rightarrow \omega_n = 2\pi f = 2356.2 \text{ rad/s}$. Using equation (2.93) when $r = 0.2$:

$$0.97 < \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} < 1.03 \qquad (\pm 3\% \text{ error})$$

This becomes $\qquad\qquad 0.8671 + 0.1505\zeta^2 < 1 < 0.9777 + 0.1697\zeta^2$

Therefore, $\qquad\qquad 0.3622 < \zeta < 0.9395$

Choose $\qquad\qquad \zeta = 0.7 = \dfrac{c}{2m\omega} = \dfrac{50}{2m(2356.2)}$

$$m = 0.015 \text{ kg}$$

$$k = m\omega_n^2 = 8.326 \times 10^4 \text{ N/m}$$

2.61 The accelerometer of Figure 2.23 has a natural frequency of 120 kHz and a damping ratio of 0.2. Calculate the error in measurement of a sinusoidal vibration at 60 kHz.

Solution:

Given: $\omega = 120$ kHz, $\zeta = .2$, $\omega_b = 60$ kHz

So, $\dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \dfrac{1}{\sqrt{(1-.5^2)^2 + (2(.2)(.5))^2}} = 1.288 > 1$

The error is $\dfrac{1.288 - 1}{1} \times 100\% = 28.8\%$

2.62 Design an accelerometer (i.e., choose m, c and k) configured as in Figure 2.23 with very small mass that will be accurate to 1% over the frequency range 0 to 50 Hz.

Solution:

Given: error $< 1\%$, $0.2f = 50$ Hz $\rightarrow f = 250$ Hz $\rightarrow \omega = 2\pi f = 1570.8$ rad/s

When $r = 0.2$, $\qquad 0.99 < \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} < 1.01 \qquad (\pm 1\%$ error)

This becomes $\qquad 0.9032 + 0.1568\zeta^2 < 1 < 0.9401 + 0.1632\zeta^2$

Therefore, $\qquad 0.6057 < \zeta < 0.7854$

Choose $\qquad m = 0.01$ kg , then $k = m\omega^2 = 24{,}674$ N/m

Thus $\qquad \zeta = 0.7 = \dfrac{c}{2\sqrt{km}}$ implies that: $c = 21.99$ Ns/m

Problems and Solutions Section 2.7 (2.63 through 2.79)

2.63 Consider a spring-mass sliding along a surface providing Coulomb friction, with stiffness 1.2×10^4 N/m and mass 10 kg, driven harmonically by a force of 50 N at 10 Hz. Calculate the approximate amplitude of steady-state motion assuming that both the mass and the surface that it slides on, are made of lubricated steel.

Solution: Given: $m = 10$ kg, $k = 1.2 \times 10^4$ N/m, $F_o = 50$ N, $\omega = 10(2\pi) = 20\pi$ rad/s

$\omega = \sqrt{\dfrac{k}{m}} = 34.64$ rad/s

for lubricated steel, $\mu = 0.07$

From Equation (2.109) $X = \dfrac{F_o}{k} \dfrac{\sqrt{1 - \left[\dfrac{4\mu mg}{\pi(F_o)}\right]^2}}{\left|(1 - r^2)\right|}$

$X = \dfrac{50}{1.2 \times 10^4} \dfrac{\sqrt{1 - \left[\dfrac{4(.07)(10)(9.81)}{\pi(50)}\right]^2}}{\left|\left(1 - \left(\dfrac{20\pi}{34.64}\right)^2\right)\right|}$

$X = 1.79 \times 10^{-3}$ m

2.64 A spring-mass system with Coulomb damping of 10 kg, stiffness of 2000 N/m, and coefficient of friction of 0.1 is driven harmonically at 10 Hz. The amplitude at steady state is 5 cm. Calculate the magnitude of the driving force.

Solution:
Given: $m = 10$ kg, $k = 2000$ N/m, $\mu = 0.1$, $\omega = 10(2\pi) = 10(2\pi) = 20\pi$ rad/s,

$\omega_n = \sqrt{\dfrac{k}{m}} = 14.14$ rad/s, $X = 5$ cm

Equation (2.108) $X = \dfrac{\dfrac{F_0}{k}}{\sqrt{(1 - r^2)^2 + \left[\dfrac{4\mu mg}{\pi kX}\right]^2}} \Rightarrow F_0 = Xk\sqrt{(1 - r^2)^2 + \left[\dfrac{4\mu mg}{\pi kX}\right]^2}$

$F_0 = (0.05)(2000)\sqrt{\left(1 - \left[\dfrac{20\pi}{14.14}\right]^2\right)^2 + \left(\dfrac{4(0.1)(10)(9.81)}{\pi(2000)(.05)}\right)^2} = 1874$ N

2.65 A system of mass 10 kg and stiffness 1.5×10^4 N/m is subject to Coulomb damping. If the mass is driven harmonically by a 90-N force at 25 Hz, determine the equivalent viscous damping coefficient if the coefficient of friction is 0.1.

Solution:

Given: $m = 10$ kg, $k = 1.5\mathrm{x}10^4$ N/m, $F_0 = 90$ N, $\omega = 25(2\pi) = 50\pi$ rad/s,

$$\omega_n = \sqrt{\frac{k}{m}} = 38.73 \text{ rad/s}, \ \mu = 0.1$$

Steady-state Amplitude using Equation (2.109) is

$$X = \frac{F_0}{k} \cdot \frac{\sqrt{1 - \left[\dfrac{4\mu mg}{\pi(F_o)}\right]^2}}{\left|(1-r^2)\right|} = \frac{90}{1.5\times 10^4} \cdot \frac{\sqrt{1 - \left[\dfrac{4(0.1)(10)(9.81)}{\pi(90)}\right]^2}}{\left|1 - \left(\dfrac{50\pi}{38.73}\right)^2\right|} = 3.85\times 10^{-4} \text{ m}$$

From equation (2.105), the equivalent Viscous Damping Coefficient becomes:

$$c_{eq} = \frac{4\mu mg}{\pi \omega X} = \frac{4(0.1)(10)(9.81)}{\pi(50\pi)(3.85\times 10^{-4})} = 206.7 \text{ Ns/m}$$

2.66   a. Plot the free response of the system of Problem 2.65 to initial conditions of $x(0) = 0$ and $\dot{x}(0) = |F_0/m| = 9$ m/s using the solution in Section 1.10.

b. Use the equivalent viscous damping coefficient calculated in Problem 2.65 and plot the free response of the "equivalent" viscously damped system to the same initial conditions.

Solution: See Problem 2.65

(a) $x(0) = 0$ and $\dot{x}(0) = \dfrac{F_o}{m} = 9$ m/s

$$\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{1.5x10^4}{10}} = 38.73 \text{ rad/s}$$

From section 1.10:

$$m\ddot{x} + kx = \mu mg \text{ for } \dot{x} < 0$$
$$m\ddot{x} + kx = -\mu mg \text{ for } \dot{x} > 0$$

Let $F_d = \mu mg = (0.1)(10)(9.81) = 9.81$ N

To start, $\dot{x}(0) = \omega_n B_1 = 9$

Therefore, $A_1 = \dfrac{F_d}{k}$ and $B_1 = \dfrac{9}{\omega_n}$

So, $x(t) = \dfrac{F_d}{k}\cos\omega_n t + \dfrac{9}{\omega}\sin\omega_n t - \dfrac{F_d}{k}$

This will continue until $\dot{x} = 0$, which occurs at time t_1:

$$x(t) = A_2 \cos\omega_n t + B_2 \sin\omega_n t + \dfrac{F_d}{k}$$
$$\dot{x}(t) = -\omega_n A_2 \sin\omega_n t + \omega_n B_2 \cos\omega_n t$$
$$x(t_1) = A_2 \cos\omega_n t_1 + B_2 \sin\omega_n t_1 + \dfrac{F_d}{k}$$
$$\dot{x}(t_1) = 0 = -\omega_n A_2 \sin\omega_n t_1 + \omega_n B_2 \cos\omega_n t_1$$

Therefore, $A_2 = \left(x(t_1) - F_d/k\right)\cos\omega_n t_1$ and $B_2 = \left(x(t_1) - F_d/k\right)\sin\omega_n t_1$

So, $x(t) = \left[\left(x(t_1) - F_d/k\right)\cos\omega_n t_1\right]\cos\omega_n t + \left[\left(x(t_1) - F_d/k\right)\sin\omega_n t_1\right]\sin\omega_n t + \dfrac{F_d}{k}$

Again, when $\dot{x} = 0$ at time t_2, the motion will reverse:

$$x(t) = A_3 \cos\omega_n t + B_3 \sin\omega_n t - \dfrac{F_d}{k}$$
$$\dot{x}(t) = -\omega_n A_3 \sin\omega_n t + \omega_n B_3 \cos\omega_n t$$
$$x(t_2) = A_3 \cos\omega_n t_2 + B_3 \sin\omega_n t_2 - \dfrac{F_d}{k}$$

$$\dot{x}(t_2) = 0 = -\omega_n A_3 \sin\omega_n t_2 + \omega_n B_3 \cos\omega_n t_2$$

Therefore, $\quad A_3 = \left(x(t_2) + F_d / k\right)\cos\omega_n t_2$ and $B_3 = \left(x(t_2) - F_d / k\right)\sin\omega_n t_2$

So, $x(t) = \left[\left(x(t_2) + F_d / k\right)\cos\omega_n t_2\right]\cos\omega_n t + \left[\left(x(t_2) + F_d / k\right)\sin\omega_n t_2\right]\sin\omega_n t - \dfrac{F_d}{k}$

This continues until $\dot{x} = 0$ and $kx < \mu mg = 9.81$ N

t
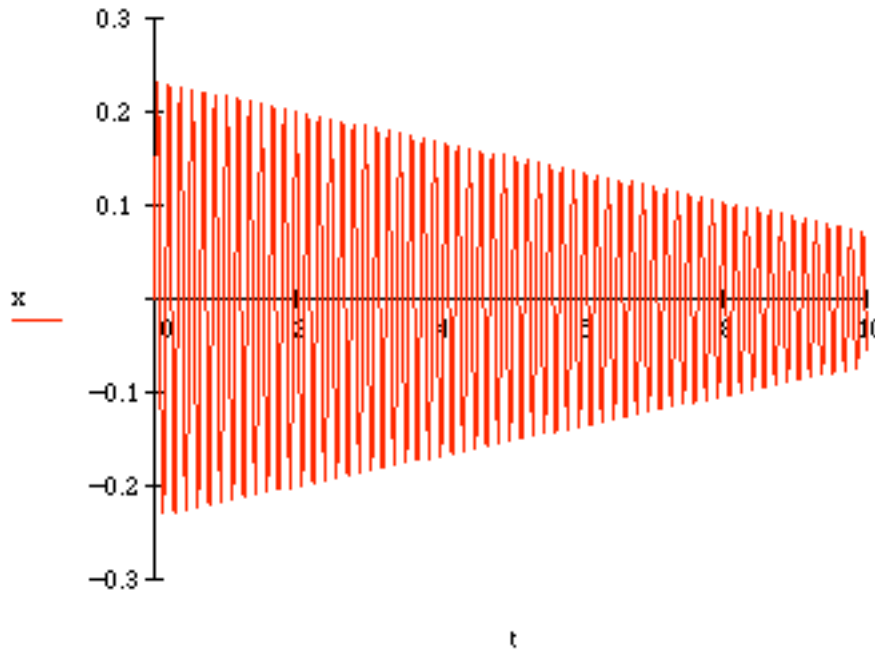
(b)　　From Problem 2.65, $c_{eq} = 206.7$ kg/s

The equivalently damped system would be:

$$m\ddot{x} + c_{eq}\dot{x} + kx = 0$$

Also, $\quad \omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{1.5x10^4}{10}} = 38.73$ rad/s

$$\zeta = \dfrac{c_{eq}}{2\sqrt{km}} = \dfrac{206.7}{2\sqrt{(1.5x10^4)(10)}} \ 0.2668$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 37.33 \text{ rad/s}$$

The solution would be found from Equation 1.36:

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$
$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \omega_d Ae^{-\zeta\omega_n t}\cos(\omega_d t + \phi)$$
$$x(0) = A\sin\phi = 0$$
$$\dot{x}(0) = -\zeta\omega_n A\sin\phi + \omega_d A\cos\phi = 9$$

Therefore, $A = \dfrac{9}{\omega_d} = 0.2411$m and $\phi = 0$ rad

So, $x(t) = 0.2411e^{-10.335t} \sin(37.33t)$

$t := 0, 0.01 .. 1$

$x(t) := 0.2411 \cdot e^{-10.35 \cdot t} \cdot \sin(37.33 \cdot t)$
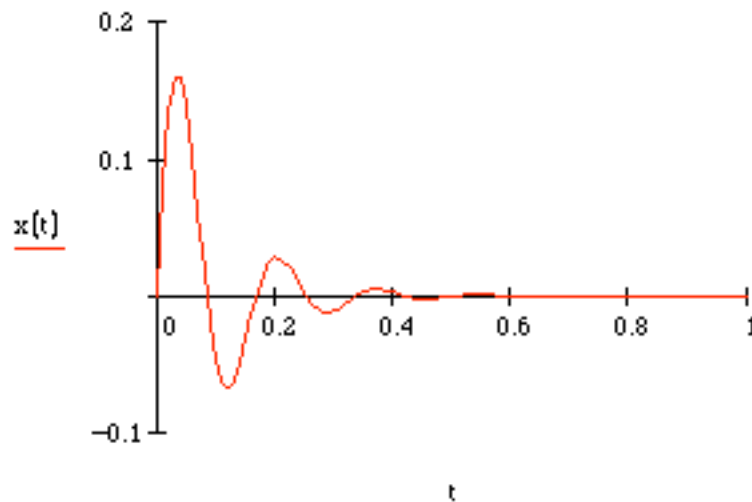
2.67 Referring to the system of Example 2.7.1, calculate how large the magnitude of the driving force must be to sustain motion if the steel is lubricated. How large must this magnitude be if the lubrication is removed?

Solution:

From Example 2.7.1 $m = 10$ kg, $k = 1.5 \times 10^4$ N/m, $F_o = 90$ N,

$\omega = 25(2\pi) = 50\pi$ rad/s

Lubricated Steel $\mu = 0.07$
Unlubricated Steel $\mu = 0.3$

Lubricated: $F_o > \dfrac{4\mu mg}{\pi} = \dfrac{4(0.07)(10)(9.81)}{\pi}$

$F_o = 8.74$ N

Unlubricated: $F_o > \dfrac{4\mu mg}{\pi} = \dfrac{4(0.3)(10)(9.81)}{\pi}$

$F_o = 37.5$ N

2.68   Calculate the phase shift between the driving force and the response for the system of Problem 2.67 using the equivalent viscous damping approximation.

Solution:

From Problem 2.67: $m = 10$ kg, $k = 1.5 \times 10^4$ N/m, $F_o = 90$ N,

$$\omega = 25(2\pi) = 157.1 \, \text{rad/s}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 38.73 \, \text{rad/s}$$

From Equation (2.111), and since $r > 1$

$$\theta = \tan^{-1}\left[\frac{-4\mu mg}{\pi F_0 \sqrt{1 - \left(\dfrac{4\mu mg}{\pi F_o}\right)^2}}\right]$$

Since in Problem 2.67, $\pi F_o = 4\mu mg$, this reduces to

$$\theta = \tan^{-1}\left[\frac{-1}{0}\right] = \frac{-\pi}{2} \, \text{rad} = \text{-90}°$$

2.69 Derive the equation of vibration for the system of Figure P2.69 assuming that a viscous dashpot of damping constant c is connected in parallel to the spring. Calculate the energy loss and determine the magnitude and phase relationships for the forced response of the equivalent viscous system.
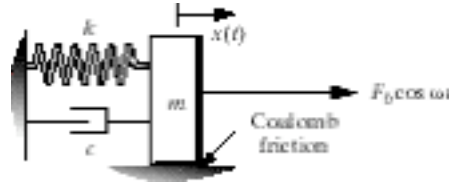
Solution: Sum of the forces in Figure P2.69

$$m\ddot{x} = -kx - c\dot{x} - \mu mg \,\text{sgn}(\dot{x})$$

$$m\ddot{x} + c\dot{x} + \mu mg \,\text{sgn}(\dot{x}) + kx = 0$$

Assume the mass is moving to the left ($\dot{x}(0) = 0, x(0) = x_0$)

$$m\ddot{x} - c\dot{x} + \mu mg + kx = 0$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} - \mu g + \omega_n^2 x = 0$$

The solution of the form:

$$x(t) = ae^{rt} + \frac{\mu g}{\omega_n^2}$$

Substituting:

$$ar^2 e^{rt} + 2\zeta\omega_n are^{rt} - \mu g + \omega_n^2 ae^{rt} + \mu g = 0$$

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0$$

$$r = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

So, $\quad x(t) = a_1 e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + a_1 e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t} + \frac{\mu g}{\omega_n^2}$

$$x(t) = e^{-\zeta\omega_n t}(a_1 e^{-\zeta\omega_d t} + a_2 e^{-\zeta\omega_d t}) + \frac{\mu g}{\omega_n^2}$$

$$x(t) = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \theta) + \frac{\mu g}{\omega_n^2}$$

Initial conditions

$$x(0) = X\sin(\theta) + \frac{\mu g}{\omega_n^2} = x_o$$

$$\dot{x}(0) = X(-\zeta\omega_n)(\sin\theta) + X\omega_d\cos\theta = 0$$

$$-X\zeta\omega_n\sin\theta + X\omega_d\cos\theta = 0$$

$$\tan\theta = \frac{\omega_d}{\zeta\omega_n} \Rightarrow \theta = \tan^{-1}\left[\frac{\omega_d}{\zeta\omega_n}\right]$$

$$X = \frac{\left(x_o - \dfrac{\mu g}{\omega_n^{\,2}}\right)\sqrt{\omega_d^{\,2} + (\zeta\omega_n)^2}}{\omega_d}$$

$$x(t) = \frac{\left(x(0) - \dfrac{\mu g}{\omega_n^{\,2}}\right)\sqrt{\omega_d^{\,2} + (\zeta\omega_n)^2}}{\omega_d}\, e^{-\zeta\omega_n t}\sin\left(\omega_d t + \tan^{-1}\left[\frac{\omega_d}{\zeta\omega_n}\right]\right) + \frac{\mu g}{\omega_n^{\,2}} \quad (1)$$

This will occur until $\dot{x}(t) = 0$:

$$\dot{x}(t) = X(-\zeta\omega_n)e^{-\zeta\omega_n t}\sin(\omega_d t + \theta) + A_0 e^{-\zeta\omega_n t}\omega_d\cos(\omega_d t + \theta) = 0$$
$$-\zeta\omega_n\sin(\omega_d t + \theta) + \omega_d\cos(\omega_d t + \theta) = 0$$

$$\Rightarrow \tan(\omega_d t + \theta) = \frac{\omega_d}{\zeta\omega_n}$$

$$t = \frac{\pi}{\omega_d}$$

So Equation (1) is valid from $0 \le t \le \dfrac{\pi}{\omega_d}$

For motion to the right

Initial conditions (From Equation (1)):

$$x\left(\frac{\pi}{\omega_d}\right) = Xe^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)}\cos\theta + \frac{\mu g}{\omega_n^{\,2}} = \frac{\left(x(0) - \dfrac{\mu g}{\omega_n^{\,2}}\right)\zeta\omega_n}{\omega_d}\, e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)} + \frac{\mu g}{\omega_n^{\,2}}$$

$$\dot{x}\left(\frac{\pi}{\omega_d}\right) = 0$$

$$x(t) = A_1 e^{-\zeta\omega_n t}\sin(\omega_d t + \theta_1) - \frac{\mu g}{\omega_n^{\,2}}$$

$$x(0) = A_1\sin\theta_1 - \frac{\mu g}{\omega_n^{\,2}} = \frac{\left(x(0) - \dfrac{\mu g}{\omega_n^{\,2}}\right)\zeta\omega}{\omega_d}\, e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)} + \frac{\mu g}{\omega_n^{\,2}}$$

$$\dot{x}(0) = A_1(-\zeta\omega_n)\sin\theta_1 + X\omega_d\cos\theta_1 = 0$$

Solution: $\quad x(t) = A_1 e^{-\zeta\omega_n t}\sin(\omega_d t + \theta_1) - \dfrac{\mu g}{\omega_n^{\,2}}$

$$A_1 = \frac{\sqrt{\omega_d^2 + (\zeta\omega_n)^2}}{\omega_d}\left[\frac{\left(x(0) - \dfrac{\mu g}{\omega_n^{\,2}}\right)\zeta\omega_n}{\omega_d}\, e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)} + \frac{\mu g}{\omega_n^{\,2}}\right]$$

$$\theta = \tan^{-1}\left[\frac{\omega_d}{\zeta\omega_n}\right]$$

Forced Case:

$$m\ddot{x} - c\dot{x} + \mu mg\,\text{sgn}\,(\dot{x}) + kx = F_o\cos(\omega t)$$

Approximate Steady-state Response:

$$x_{ss}(t) = X\sin(\omega t - \theta)$$

Energy Dissipated per Cycle:

$$\Delta E = \int F_d dx = \int_{2\pi}^{\frac{2\pi}{\omega}}\left[c\dot{x}\frac{dx}{dt} + \mu mg\,\text{sgn}\,\dot{x}\frac{dx}{dt}\right]dt$$

$$= \int_{2\pi}^{\frac{2\pi}{\omega}}(c\dot{x}^2 dt) + \mu mg\int_{2\pi}^{\frac{2\pi}{\omega}}\text{sgn}(\dot{x})\dot{x}dt$$

$$\Delta E = \pi c\omega X^2 + 4\mu mgX$$

This results in an equivalent viscously damped system:

$$\ddot{x} + 2(\zeta + \zeta_{eq})\omega_n\dot{x} + \omega_n^2 x = F_o\cos\omega t$$

$$\text{where }\zeta_{eq} = \frac{2\mu g}{\pi\omega_n\omega X}$$

The magnitude is:

$$X = \frac{\dfrac{F_0}{k}}{\sqrt{(1-r)^2 + (2(\zeta + \zeta_{eq})r)^2}}$$

Solving for X:

$$X = \frac{\left(\dfrac{8\mu gcr^2}{\pi k\omega}\right) + \sqrt{\left(\dfrac{8\mu gcr^2}{\pi k\omega}\right) - 2\left[(1-r^2)^2 + \dfrac{c^2 r^2}{km}\right]\left[\left(\dfrac{4\mu gr}{\pi\omega_n\omega}\right)^2 - \left(\dfrac{F_0}{k}\right)^2\right]}}{4\left[(1-r^2)^2 + \dfrac{c^2 r^2}{km}\right]}$$

The phase is:

$$\theta = \tan^{-1}\left[\frac{2(\zeta + \zeta_{eq})r}{1-r^2}\right] = \tan^{-1}\left[\frac{2\zeta r + \dfrac{4\mu gr}{\pi\omega_n\omega X}}{1-r^2}\right]$$

2.70 A system of unknown damping mechanism is driven harmonically at 10 Hz with an adjustable magnitude. The magnitude is changed, and the energy lost per cycle and amplitudes are measured for five different magnitudes. The measured quantities are:

$\Delta E(J)$	0.25	0.45	0.8	1.16	3.0
$X (M)$	0.01	0.02	0.04	0.08	0.15

Is the damping viscous or Coulomb?

Solution:

For <u>viscous</u> damping, $\Delta E = \pi c \omega X^2$

For <u>Coulomb</u> damping, $\Delta E = 4\mu mgX$

$$x := \begin{bmatrix} 0.01 \\ 0.02 \\ 0.04 \\ 0.08 \\ 0.15 \end{bmatrix} \quad + \qquad i := 0,1..4 \qquad y_i := (x_i)^2$$

$$E := \begin{bmatrix} 0.25 \\ 0.45 \\ 0.8 \\ 1.16 \\ 3 \end{bmatrix}$$
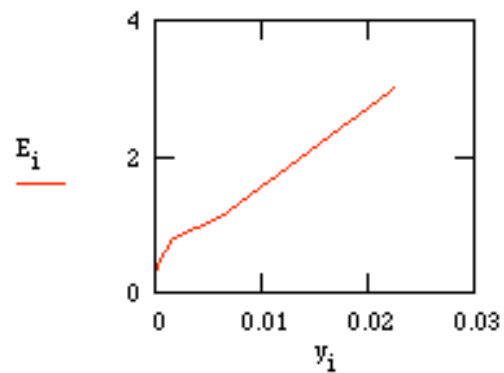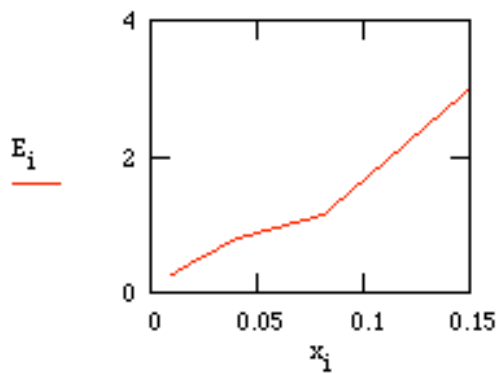
For the data given, a plot of ΔE vs X^2 yields a curve, while ΔE vs X yields a straight line. Therefore, the damping is likely <u>Coulomb</u> in nature

2.71   Calculate the equivalent loss factor for a system with Coulomb damping.

Solution:

Loss Factor:
$$\eta = \frac{\Delta E}{2\pi U_{max}}$$

For Coulomb damping:
$$\Delta E = 4\mu mgX$$

$$U_{max} = \frac{1}{2}kX^2$$

$$\eta = \frac{4\mu mgX}{2\pi\left(\frac{1}{2}kX^2\right)} = \frac{4\mu mg}{\pi kX}$$

Substituting for X (from Equation 2.109):

$$\eta = \frac{4\mu mg}{\pi F_o}\frac{\left|1-r^2\right|}{\sqrt{1-\left(\frac{4\mu mg}{\pi F_o}\right)^2}}$$

2.72   A spring-mass system ($m = 10$ kg, $k = 4\times10^3$ N/m) vibrates horizontally on a surface with coefficient of friction $\mu = 0.15$. When excited harmonically at 5 Hz, the steady-state displacement of the mass is 5 cm. Calculate the amplitude of the harmonic force applied.

Solution: Given: $m = 10$ kg, $k = 4\times10^3$ N/m, $\mu = 0.15$, $X = 5$ cm $= 0.05$ m,

$$\omega = 5(2\pi) = 10\pi \text{ rad/s}, \ \omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}$$

Equation (2.109)

$$X = \frac{\frac{F_0}{k}}{\sqrt{(1-r^2)^2 + \left(\frac{4\mu mg}{\pi kX}\right)^2}} \Rightarrow$$

$$F_o = kX\sqrt{(1-r^2)^2 + \left(\frac{4\mu mg}{\pi kX}\right)^2} = (0.05)(4\times10^3)\sqrt{\left(1-\left(\frac{10\pi}{20}\right)^2\right)^2 + \left(\frac{4(0.15)(10)(9.81)}{\pi(4x10^3)(0.05)}\right)^2}$$

$$F_o = 294 \text{ N}$$

2.73   Calculate the displacement for a system with air damping using the equivalent viscous damping method.

Solution:

The equivalent viscous damping for air is given by Equation (2.131):

$$c_{eq} = \frac{8}{3\pi}\alpha\omega X$$

From Equation 2.31:

$$X = \frac{F_o}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}} = \frac{F_o}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(\frac{c_{eq}}{m}\omega_n\right)^2}}$$

$$X = \frac{F_o}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(\frac{8}{3\pi m}\alpha\omega X\right)^2}} = \frac{F_o m}{k\sqrt{(1-r^2)^2 + \left(\frac{8}{3\pi m}\alpha r^2 X\right)^2}}$$

Solving for X and taking the real solution:

$$X = \frac{\sqrt{-\frac{1}{2}(1-r^2)^2 + \frac{1}{2}\sqrt{(1-r^2)^2 + \left(\frac{16 F_o\alpha r^2}{3\pi k m}\right)^2}}}{\left(\frac{8\alpha r^2}{3\pi m}\right)}$$

2.74   Calculate the semimajor and semiminor axis of the ellipse of equation (2.119). Then calculate the area of the ellipse. Use $c = 10$ kg/s, $\omega = 2$ rad/s and $X = 0.01$ m.

Solution: The equation of an ellipse usually appears when the plot of the ellipse is oriented along with the x axis along the principle axis of the ellipse. Equation (2.1109) is the equation of an ellipse rotated about the origin. If k is known, the angle of rotation can be computed from formulas given in analytical geometry. However, we know from the energy calculation that the stiffness does not effect the amount of energy dissipated. Thus only the orientation of the ellipse is effected by the stiffness, not its area or axis. Thus we can use this fact to answer the question. First re-write equation (2.119) with $k = 0$ to get:

$$F^2 + c^2\omega^2 x^2 = c^2\omega^2 X^2$$

$$\Rightarrow \left(\frac{F}{c\omega X}\right)^2 + \left(\frac{x}{X}\right)^2 = 1$$

This is the equation of an ellipse with major axis a and minor axis b given by

$$a = X = 0.01 \text{ m, and } b = c\omega X = 0.2 \text{ kg m/s}^2$$

The area, and hence energy lost per cycle through the damper then becomes

$$\pi c\omega_n X^2 = (3.14159)(10)(2)(.0001) = 0.006283 \text{ Joules.}$$

Alternately, realized that Equation 2.119 is that of ellipse rotated by an angle θ defined by $\tan 2\theta = -2k/(c^2\omega_n^2 + k^2 - 1)$. Then match the ellipse to standard form, read off the major and minor axis (say a and b) and calculate the area from πab. See the following web site for an elipse http://mathworld.wolfram.com/Ellipse.html

2.75   The area of a force deflection curve of Figure P2.28 is measured to be 2.5 N- m, and the maximum deflection is measured to be 8 mm. From the "slope" of the ellipse the stiffness is estimated to be 5×10^4 N/m. Calculate the hysteretic damping coefficient. What is the equivalent viscous damping if the system is driven at 10 Hz?

Solution:

Given: Area $= 2.5 \text{ N} \cdot \text{m}$, $k = 5\text{x}10^4$ N/m, $X = 8$ mm, $\omega = 10(2\pi) = 20\pi$ rad/s

Hysteric Damping Coefficient:

$$\Delta E \quad = \text{Area} = \pi k\beta X^2$$
$$2.5 = \pi(5\times10^4)\beta(0.008)^2$$
$$\beta = 0.249$$

Equivalent Viscous Damping:

$$c_{eq} = \frac{k\beta}{\omega} = \frac{(5\times10^4)(0.249)}{20\pi}$$
$$c_{eq} = 198 \text{ kg/s}$$

2.76 The area of the hysteresis loop of a hysterically damped system is measured to be 5 N • m and the maximum deflection is measured to be 1 cm. Calculate the equivalent viscous damping coefficient for a 20-Hz driving force. Plot c_{eq} versus ω for $2\pi \leq \omega \leq 100\pi$ rad/s.

Solution:

Given: Area $= 5\,\text{N} \bullet \text{m}$, $X = 1$ cm, $\omega = 20(2\pi) = 40\pi$ rad/s

Hysteric Damping Coefficient:
$$\Delta E \quad = \text{Area} = \pi k \beta X^2$$
$$5 = \pi k \beta (0.01)^2$$
$$k\beta = 15{,}915 \text{ N/m}$$

Equivalent Viscous Damping:
$$c_{eq} = \frac{k\beta}{\omega} = \frac{15915}{40\pi}$$
$$c_{eq} = 126.65 \text{ kg/s}$$

To plot, rearrange so that
$$\pi c_{eq}\omega X^2 = \Delta E$$
$$c_{eq} = \frac{\Delta E}{\pi\omega X^2} = \frac{5}{\pi\omega(.01)^2} = \frac{50{,}000}{\pi\omega}$$

$$c\left(\omega\right) := \frac{50000}{\pi \cdot \omega}$$
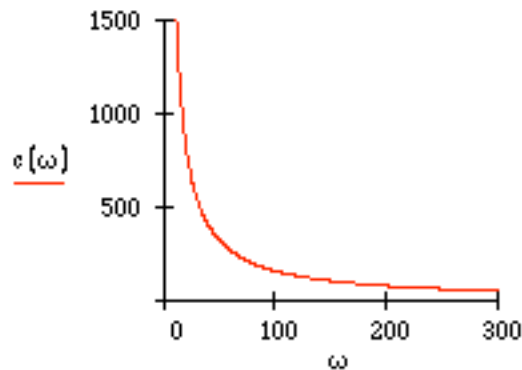
2.77 Calculate the nonconservative energy of a system subject to both viscous and hysteretic damping.

Solution:

$$\Delta E = \Delta E_{hys} + \Delta E_{visc}$$
$$\Delta E = \pi c \omega X^2 + k\pi\beta X^2$$
$$\Delta E = (c\omega + k\beta)\pi X^2$$

2.78 Derive a formula for equivalent viscous damping for the damping force of the form, $F_d = c(\dot{x})^n$ where n is an integer.

Solution:

Given: $\qquad F_d = c(\dot{x})^n$

Assume the steady-state response $x = X\sin\omega t$.

The energy lost per cycle is given by Equation (2.99) as:

$$\Delta E = \oint F_d dx = \int_0^{\frac{2\pi}{\omega}} c(\dot{x})^n \dot{x} dt = c \int_0^{\frac{2\pi}{\omega}} (\dot{x})^{n+1} dt$$

Substituting for $\dot{x}$:

$$\Delta E = \int_0^{\frac{2\pi}{\omega}} \left[\omega^{n+1} X^{n+1} \cos^{n+1}(\omega t) \right] dt$$

Let $u = \omega t$:

$$\Delta E = cX^{n+1}\omega^n \int_0^{2\pi} \left(\cos^{n+1} u \right) du$$

Equating this to Equation 2.91 yields:

$$\pi c_{eq}\omega X^2 = cX^{n+1}\omega^n \int_0^{2\pi} (\cos^{n+1} u) du$$

$$c_{eq} = \frac{cX^{n-1}\omega^{n-1}}{\pi} \int_0^{2\pi} (\cos^{n+1} u) du$$

2.79   Using the equivalent viscous damping formulation, determine an expression for the steady-state amplitude under harmonic excitation for a system with both Coulomb and viscous damping present.

Solution:

$$\Delta E = \Delta E_{visc} + \Delta E_{coul}$$
$$\Delta E = \pi c \omega X^2 + 4umgX$$

Equate to Equivalent Viscously Damped System

$$\pi c_{eq} \omega X^2 = \pi c \omega X^2 + 4\mu mg$$

$$c_{eq} = \frac{\pi c \omega X + 4\mu mg}{\pi \omega X} = c + \frac{4\mu mg}{\pi \omega X} = 2\zeta_{eq}\omega_n m$$

$$\zeta_{eq} = \zeta + \frac{2\mu g}{\pi \omega \omega_n X}$$

Amplitude:

$$X = \frac{\dfrac{F_o}{k}}{\sqrt{(1-r^2)^2 + (2\zeta_{eq}r)^2}} = \frac{\dfrac{F_o}{k}}{\sqrt{(1-r^2)^2 + \left(2\zeta r + \dfrac{4\mu mg}{\pi k X}\right)^2}}$$

Solving for X:

$$X = \frac{-\left(\dfrac{8\mu gcr^2}{\pi k \omega}\right) + \sqrt{\left(\dfrac{8\mu gcr^2}{\pi k \alpha \omega}\right)^2 - 4\left[(1-r^2)^2 + \dfrac{c^2 \, r^2}{km}\right]\left[\left(\dfrac{4\mu gr}{\pi \omega_n \omega}\right)^2 - \left(\dfrac{F_o}{k}\right)^2\right]}}{2\left[(1-r^2)^2 + \dfrac{c^2 r^2}{km}\right]}$$

Problems and Solutions Section 2.8 (2.80 through 2.86)

2.80*. Numerically integrate and plot the response of an underdamped system determined by $m = 100$ kg, $k = 20,000$ N/m, and $c = 200$ kg/s, subject to the initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s, and the applied force $F(t) = 150\cos5t$. Then plot the exact response as computed by equation (2.33). Compare the plot of the exact solution to the numerical simulation.

Solution: The solution is presented in Matlab:

First the m file containing the state equation to integrate is set up and saved as ftp2_72.m

```
function xdot=f(t, x)
xdot=[-(200/100)*x(1)-(20000/100)*x(2)+(150/100)*cos(5*t); x(1)];
% xdot=[x(1)'; x(2)']=[-2*zeta*wn*x(1)-wn^2*x(2)+fo*cos(w*t) ; x(1)]
% which is a state space form of
% x" + 2*zeta*wn*x' + (wn^2)*x = fo*cos(w*t)    (fo=Fo/m)

clear all;
```

Then the following m file is created and run:

```
%---- numerical simulation ---
x0=[0.1; 0.01];          %[xdot(0); x(0)]
tspan=[0 10];
[t,x]=ode45('fp2_72',tspan,x0);
plot(t, x(:,2), '.');
hold on;

%--- exact solution ----
t=0: .002: 10;
m=100; k=20000; c=200; Fo=150 ; w=5
wn=sqrt(k/m); zeta=c/(2*wn*m); fo=Fo/m; wd=wn*sqrt(1-zeta^2)
x0=0.01; v0= 0.1;
xe= exp(-zeta*wn*t) .* ( (x0-fo*(wn^2-w^2)/((wn^2-w^2)^2 ...
 +(2*zeta*wn*w)^2))*cos(wd*t) ...
 + (zeta*wn/wd*( x0-fo*(wn^2-w^2)/((wn^2-w^2)^2+(2*zeta*wn*w)^2)) ...
 - 2*zeta*wn*w^2*fo/(wd*((wd^2-w^2)^2  ...
 + (2*zeta*wn*w)^2))+v0/wd)*sin(wd*t) ) ...
 + fo/((wn^2-w^2)^2+(2*zeta*wn*w)^2)*((wn^2-w^2)*cos(w*t) ...
 + 2*zeta*wn*w*sin(w*t))

plot(t, xe, 'w');
hold off;
```

This produces the following plot:
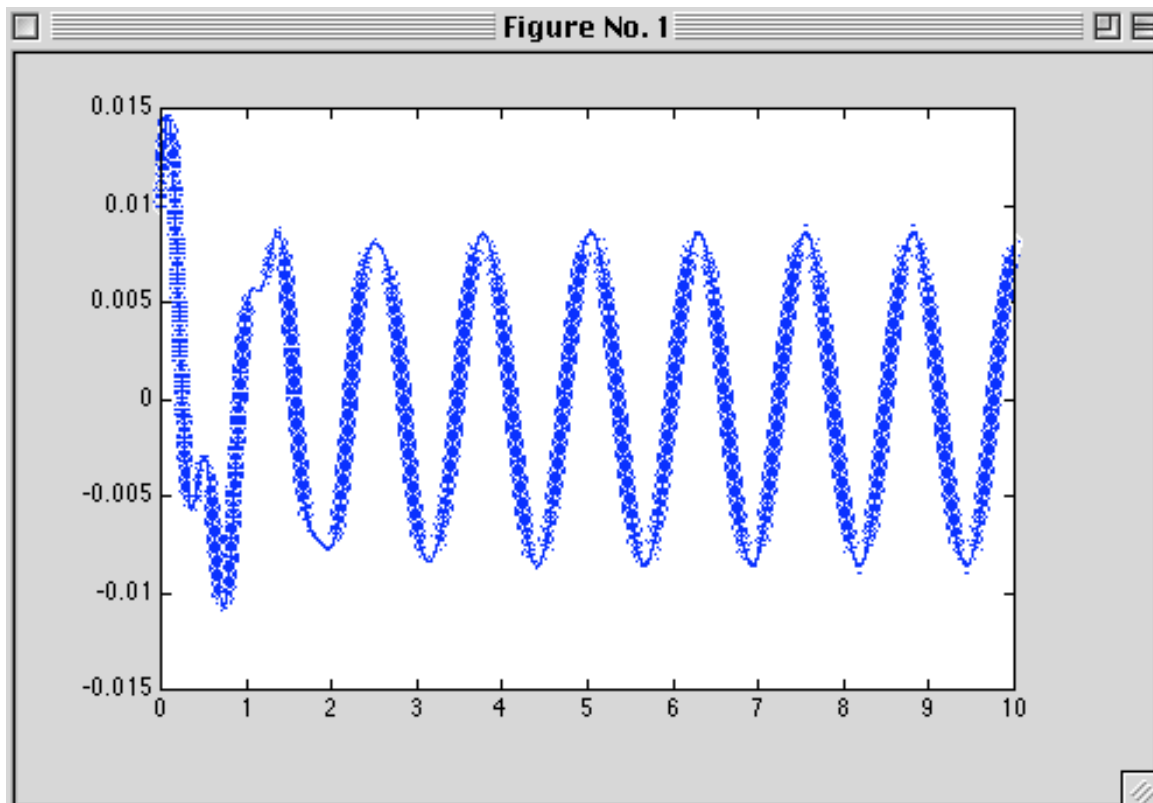
2.81*. Numerically integrate and plot the response of an underdamped system determined by $m = 150$ kg, and $k = 4000$ N/m subject to the initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s, and the applied force $F(t) = 15\cos 10t$, for various values of the damping coefficient. Use this "program" to determine a value of damping that causes the transient term to die out with in 3 seconds. Try to find the smallest such value of damping remembering that added damping is usually expensive.

Solution: The solution is given by the following Mathcad session. A value of $c = 350$ kg/s corresponding to $\zeta = 0.226$ gives the desired result.

Mathcad session window titled 2.73:

$m := 150 \qquad k := 4000 \qquad c := 350$

$\omega n := \sqrt{\dfrac{k}{m}} \qquad \zeta := \dfrac{c}{2 \cdot \sqrt{k \cdot m}} \qquad F0 := 15 \qquad \omega := 10$

$x := \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix} \qquad D(t,x) := \begin{bmatrix} x_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot x_1 - \omega n^2 \cdot x_0\right) + \dfrac{F0}{m} \cdot \sin(\omega \cdot t) \end{bmatrix}$

$Z := \text{rkfixed}(x, 0, 10, 4000, D)$

$t := Z^{<0>} \qquad\qquad xr := Z^{<1>} \qquad \zeta = 0.226$

$\omega n = 5.164$
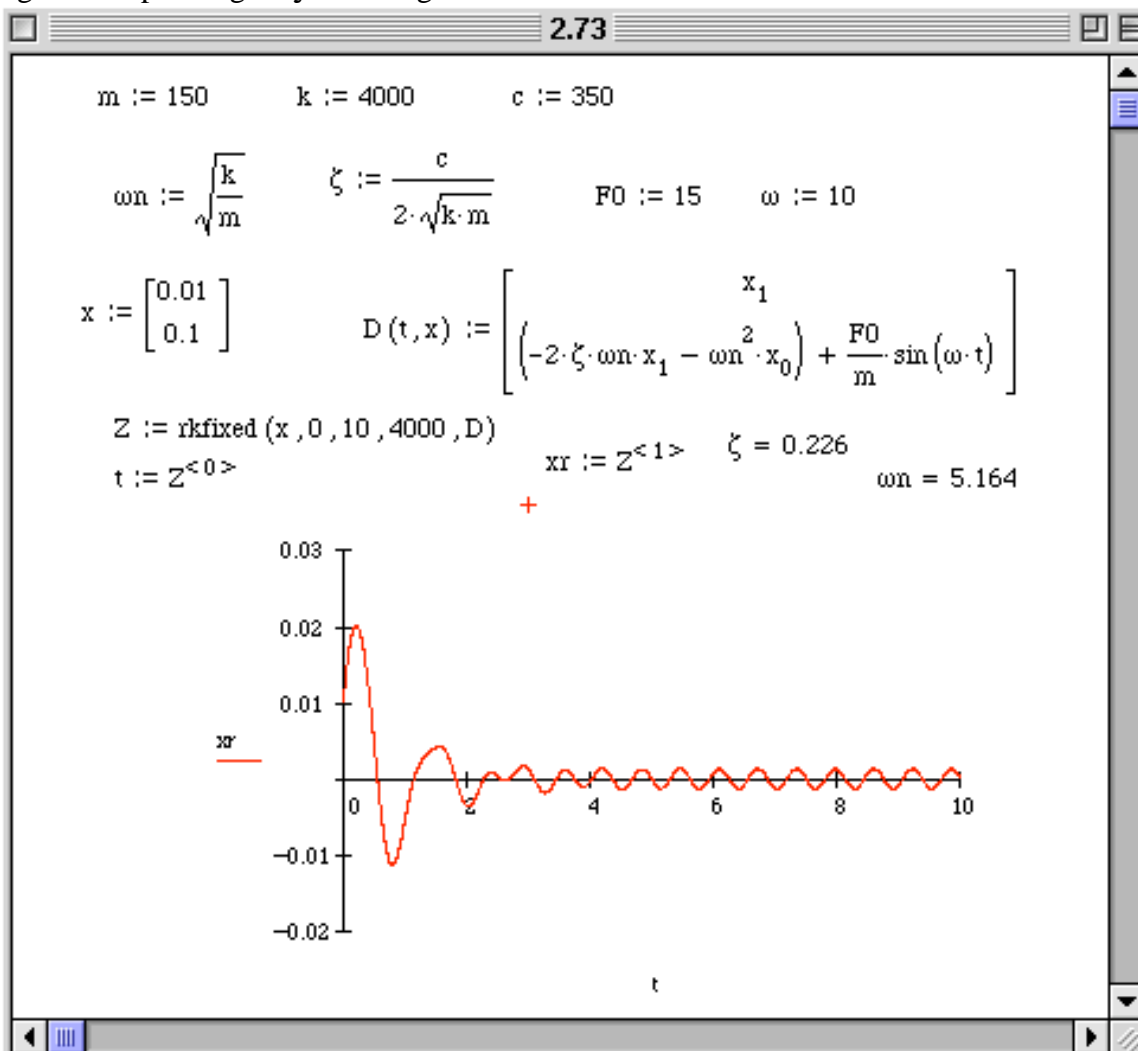
2.82*. Solve Problem 2.7 by numerically integrating rather than using analytical expressions.

Solution: The following session in Mathcad illustrates the solution:
a) zero initial conditions
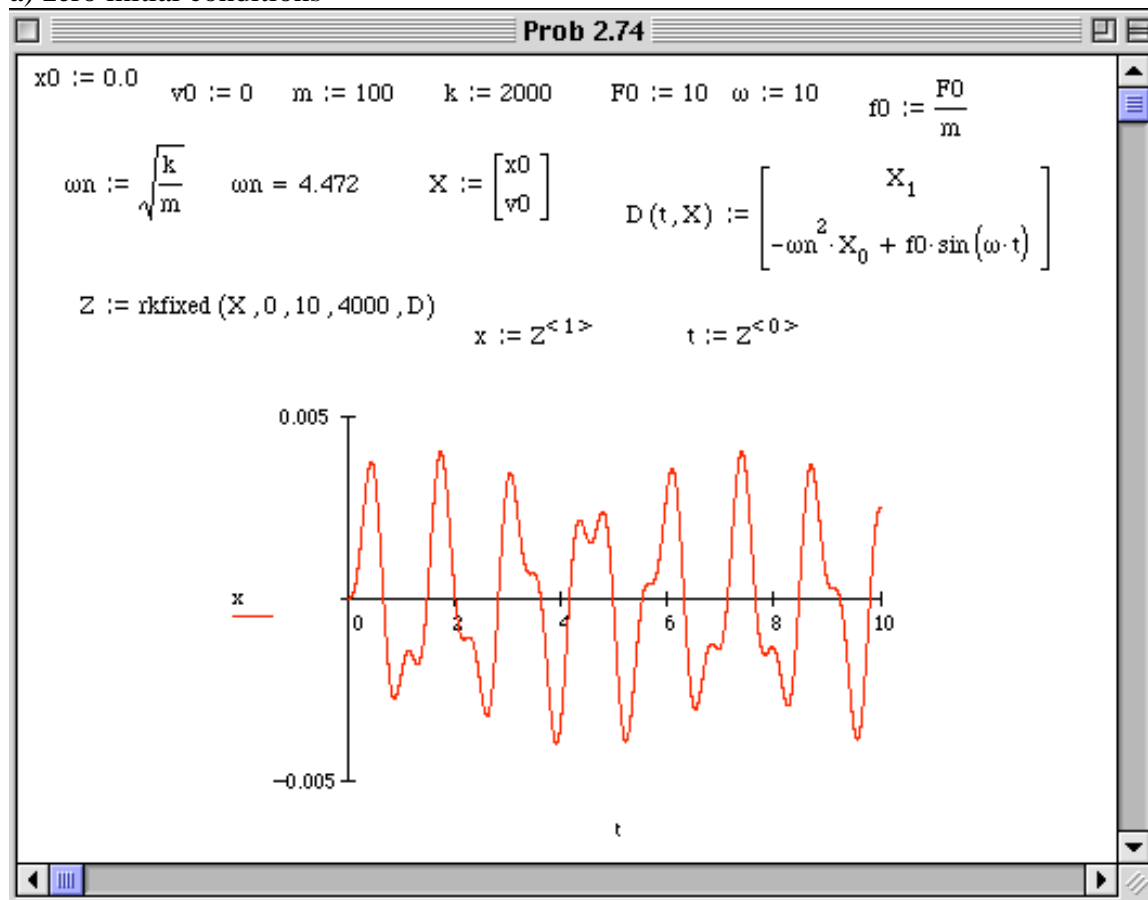
b) Using and initial condition of $x(0) = 0.05$ m.   Note the difference in the response.

Prob 2.74

$x0 := 0.05$ $v0 := 0$ $m := 100$ $k := 2000$ $F0 := 10$ $\omega := 10$ $f0 := \dfrac{F0}{m}$

$\omega n := \sqrt{\dfrac{k}{m}}$ $\omega n = 4.472$ $X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$ $D(t, X) := \begin{bmatrix} X_1 \\ -\omega n^2 \cdot X_0 + f0 \cdot \sin(\omega \cdot t) \end{bmatrix}$

$Z := rkfixed(X, 0, 10, 4000, D)$ $x := Z^{<1>}$ $t := Z^{<0>}$
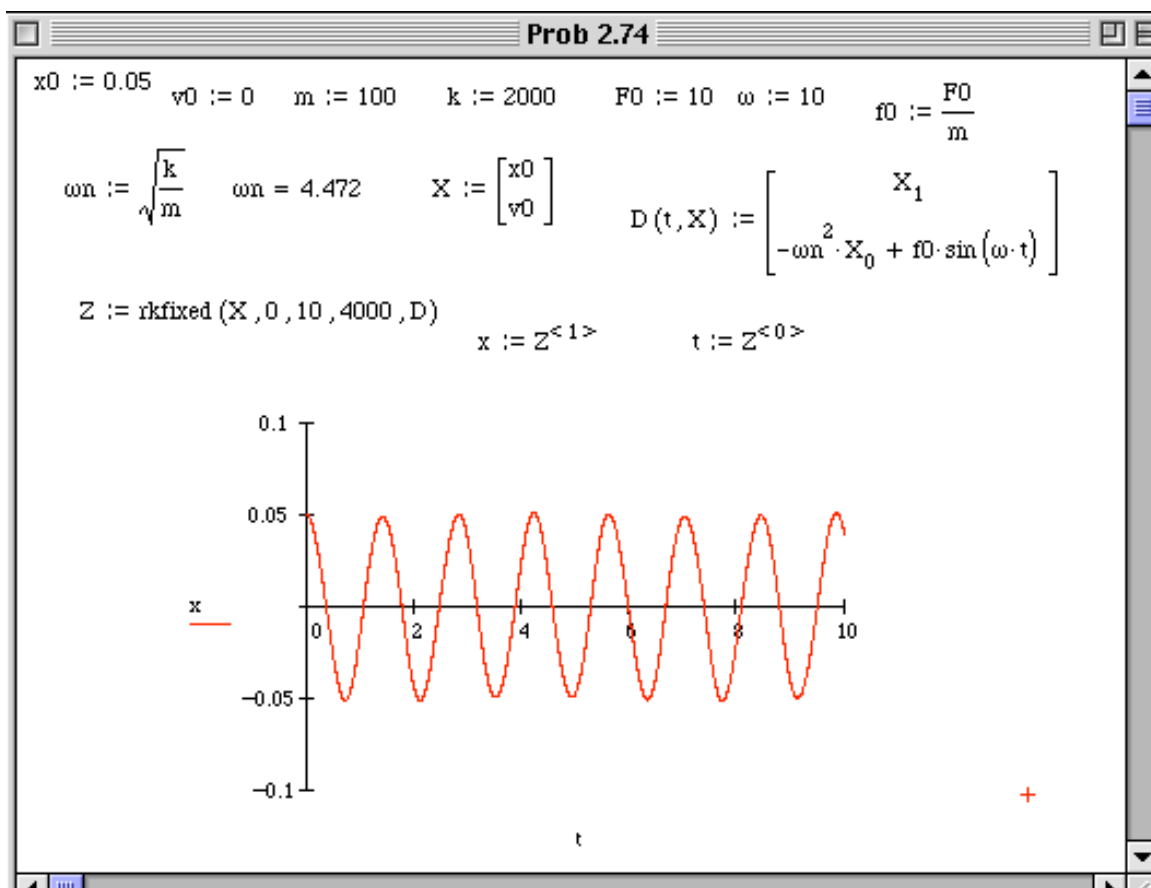
2.83*. Numerically simulate the response of the system of Problem 2.30.

> **Solution:** From problem 2.30, the equation of motion is
> $$9a^2\,m\ddot{\theta} + 4a^2c\cos\theta\dot{\theta} + a^2k\sin\theta = -3aF(t)$$
> where $k = 2000$ kg, $c = 25$ kg/s , $m = 25$ kg , $F(t) = 50\cos 2\pi t$, $a = 0.05$ m
> Placing the equation of motion in first order form and numerically integrating
> using Mathcad yields

$$k := 2000 \qquad\qquad c := 25 \quad m := 25 \quad a := 0.05 \qquad f0 := \dfrac{-50}{3\cdot a\cdot m}$$

$$X := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$D(t,X) := \begin{bmatrix} X_1 \\ \dfrac{-4\cdot c}{9\cdot m}\cdot X_1\cdot\cos\left(X_0\right) - \dfrac{k}{9\cdot m}\cdot\sin\left(X_0\right) + f0\cdot\cos\left(2\cdot\pi\cdot t\right) \end{bmatrix}$$

$$Z := \text{rkfixed}\,(X,0,10,4000,D)$$

$$t := Z^{<0>} \qquad\qquad \theta := Z^{<1>}$$
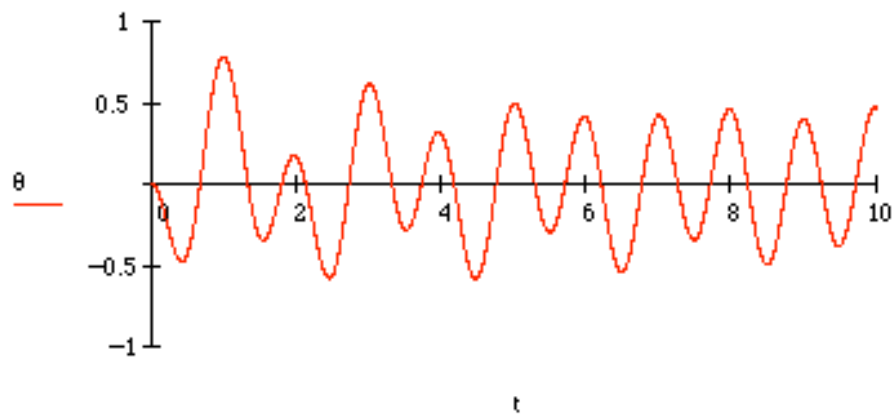
2.84*. Numerically integrate the system of Example 2.8.1 for the following sets of initial conditions:
a) $x_0 = 0.0$ m and $v_0 = 0.1$ m/s
b) $x_0 = 0.01$ m and $v_0 = 0.0$ m/s
c) $x_0 = 0.05$ m and $v_0 = 0.0$ m/s
d) $x_0 = 0.0$ m and $v_0 = 0.5$ m/s
Plot these responses on the same graph and note the effects of the initial conditions on the transient part of the response.

Solution: The following are the solutions in Mathcad. Of course the other codes and Toolbox will yield the same results.
a)

b)

c)

$x0 := 0.05$ $v0 := 0.$ $m := 100$ $k := 2000$ $c := 200$

$\omega n := \sqrt{\dfrac{k}{m}}$ $\zeta := \dfrac{c}{2\sqrt{k \cdot m}}$ $F0 := 150$ $f0 := \dfrac{F0}{m}$ $\omega := 10$ $\omega d := \omega n \cdot \sqrt{1 - \zeta^2}$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$ $D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$

$Z := rkfixed(X, 0, 6, 2000, D)$ $t := Z^{<0>}$ $x := Z^{<1>}$

d)

$x0 := 0.0$ $v0 := 0.5$ $m := 100$ $k := 2000$ $c := 200$

$\omega n := \sqrt{\dfrac{k}{m}}$ $\zeta := \dfrac{c}{2\sqrt{k \cdot m}}$ $F0 := 150$ $f0 := \dfrac{F0}{m}$ $\omega := 10$ $\omega d := \omega n \cdot \sqrt{1 - \zeta^2}$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$ $D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$

$Z := rkfixed(X, 0, 6, 2000, D)$ $t := Z^{<0>}$ $x := Z^{<1>}$
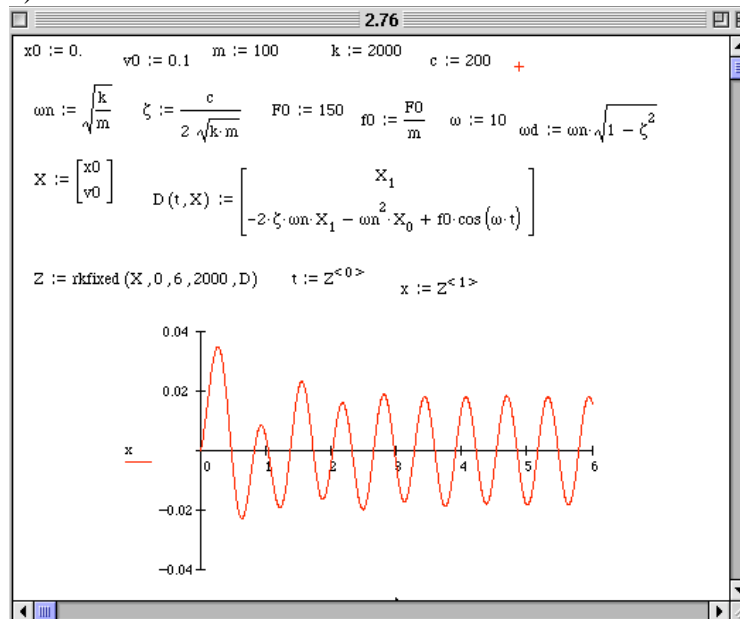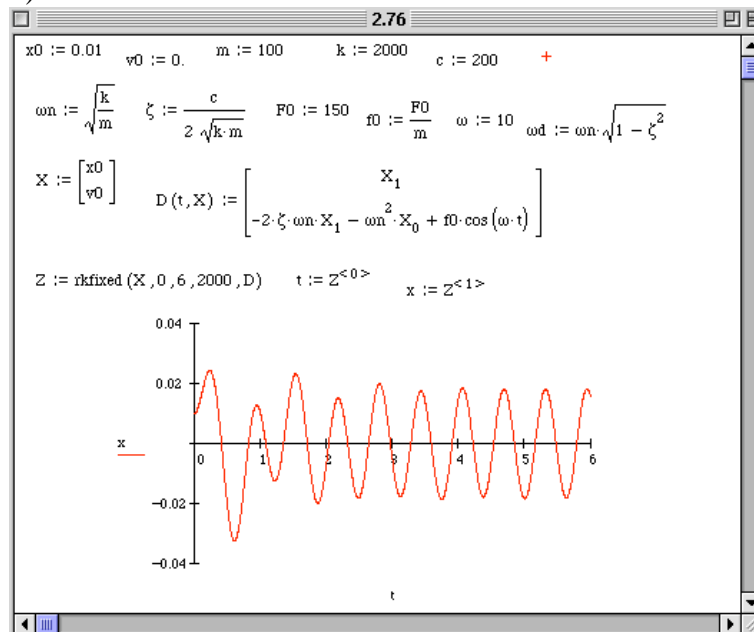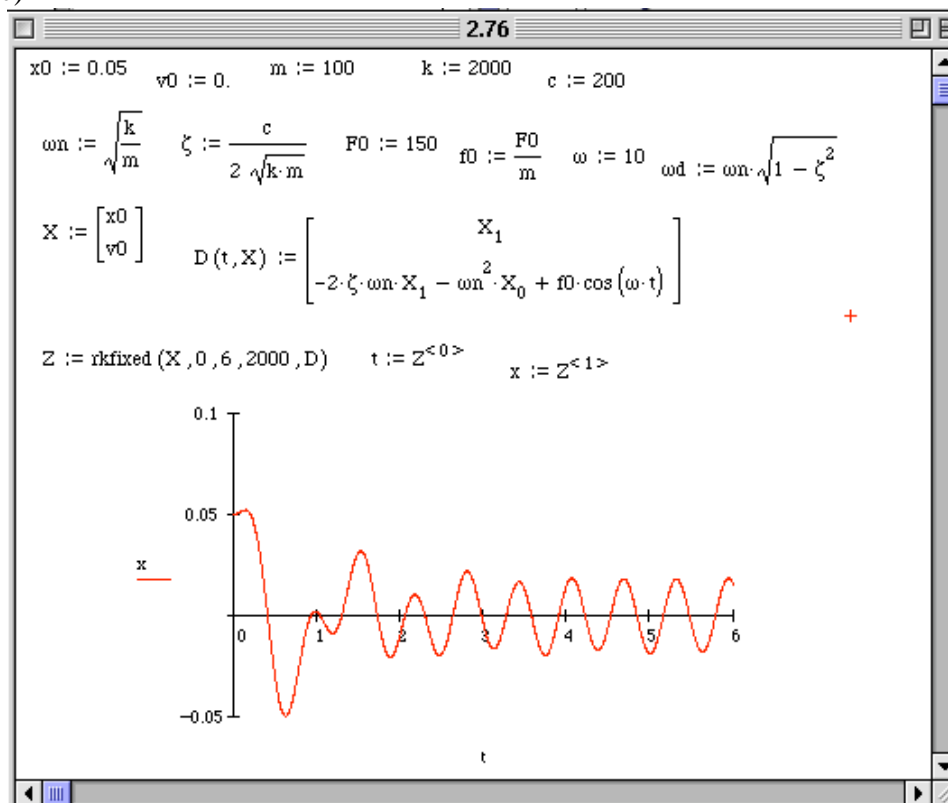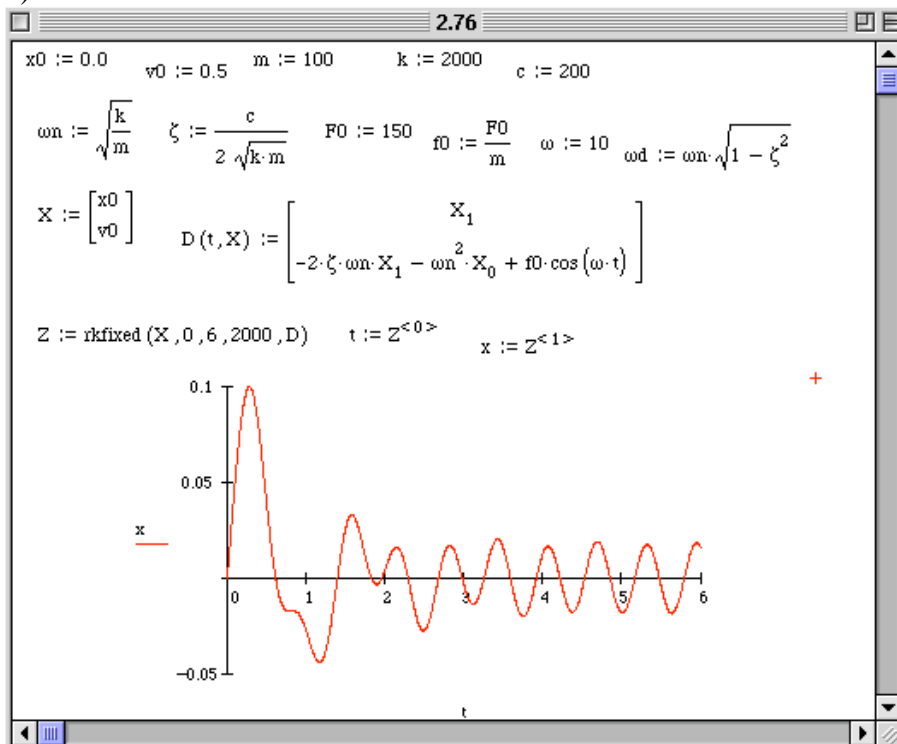
Note the profound effect on the transient, but of course no effect on the steady state.

2.85*. A DVD drive is mounted on a chassis and is modeled as a single degree-degree-of-freedom spring, mass and damper. During normal operation, the drive (having a mass of 0.4 kg) is subject to a harmonic force of 1 N at 10 rad/s. Because of material considerations and static deflection, the stiffness is fixed at 500 N/m and the natural damping in the system is 10 kg/s. The DVD player starts and stops during its normal operation providing initial conditions to the module of $x_0 = 0.001$ m and $v_0 = 0.5$ m/s. The DVD drive must not have an amplitude of vibration larger then 0.008 m even during the transient stage. First compute the response by numerical simulation to see if the constraint is satisfied. If the constraint is not satisfied, find the smallest value of damping that will keep the deflection less then 0.008 m.

Solution: The solution is given by the following Mathcad session:
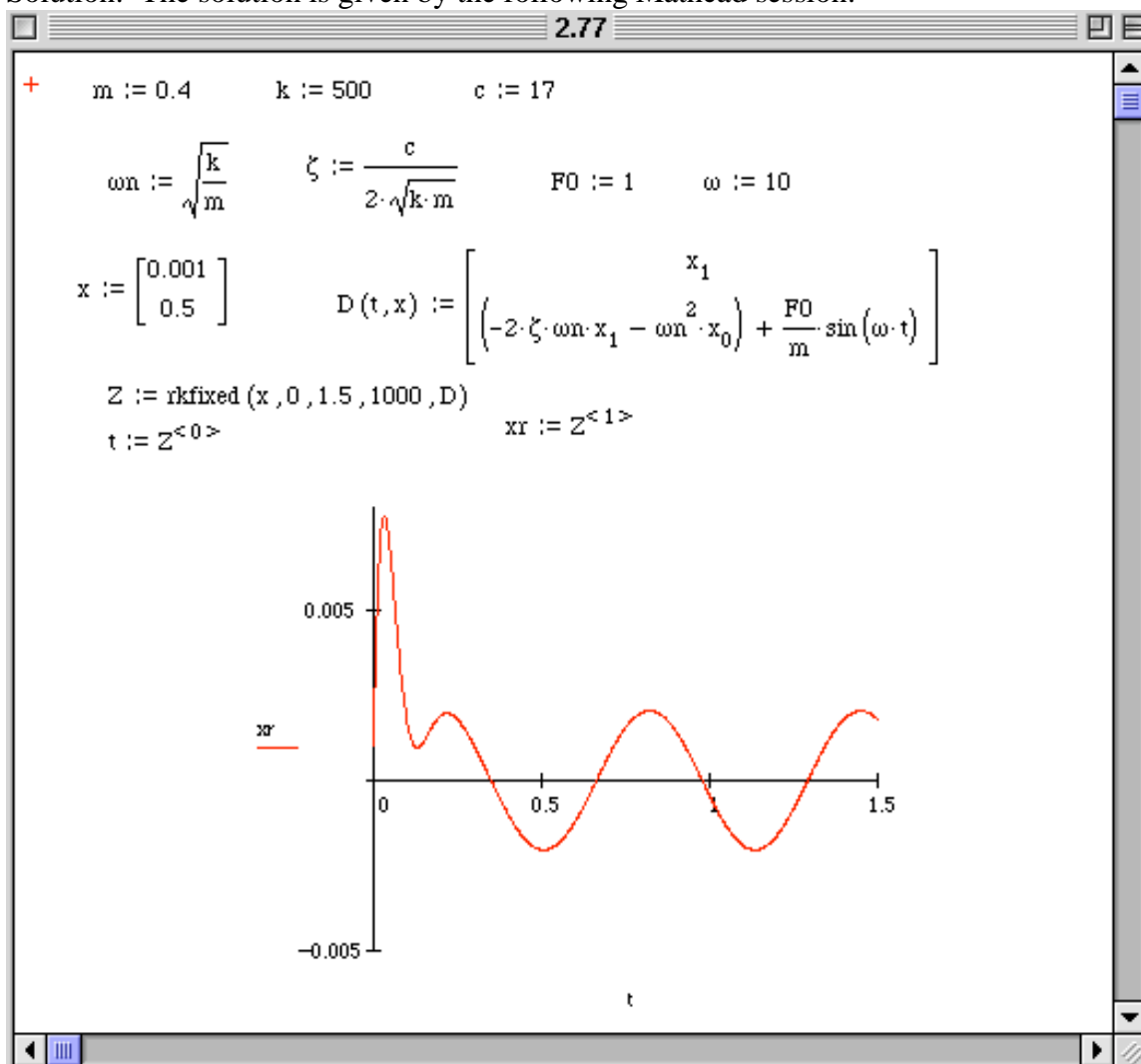
This yields $c = 17$ kg/s as a solution.

2.86   Use a plotting routine to examine the base motion problem of Figure 2.12 by
plotting the particular solution (for an undamped system) for the three cases $k =$
1500 N/m, and $k = 700$ N/m. Also note the values of the three frequency ratios
and the corresponding amplitude of vibration of each case compared to the input.
Use the following values: $\omega_b = 4.4$ rad/s, $m = 100$ kg, and $Y = 0.05$ m.

Solution; The following Mathcad worksheet shows the plotting:

$m := 100$ $k1 := 1500$ $\omega1 := \sqrt{\dfrac{k1}{m}}$

$k2 := 700$

$k3 := 2500$ $\omega2 := \sqrt{\dfrac{k2}{m}}$ $\omega3 := \sqrt{\dfrac{k3}{m}}$ $\omega1 = 3.873$ $\omega b := 4.4$

$r1 := \dfrac{\omega b}{\omega1}$ $r2 := \dfrac{\omega b}{\omega2}$ $r3 := \dfrac{\omega b}{\omega3}$ $\omega2 = 2.646$

$\omega3 = 5$

$r1 = 1.136$ $r2 = 1.663$ $r3 = 0.88$ $Y := 0.05$

$y(t) := Y \cdot \sin(\omega b \cdot t)$ then from equation 2.68 and 2.70 with zero damping:

$x1(t) := Y \cdot \dfrac{1}{\left|1 - r1^2\right|} \cdot \cos\left(\omega b \cdot t - \dfrac{\pi}{2}\right)$ $x2(t) := Y \cdot \dfrac{1}{\left|1 - r2^2\right|} \cdot \cos\left(\omega b \cdot t - \dfrac{\pi}{2}\right)$

$x3(t) := Y \cdot \dfrac{1}{\left|1 - r3^2\right|} \cdot \cos\left(\omega b \cdot t - \dfrac{\pi}{2}\right)$
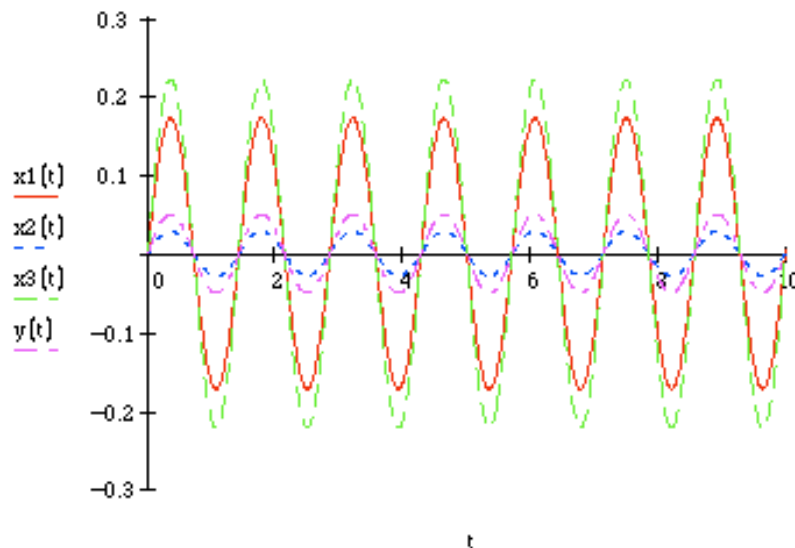
Note that $k2$, the softest system (smallest k) has the smallest amplitude, smaller
than the amplitude of the input as predicted by the magnitude plots in section 2.3.
Thus when $r > \sqrt{2}$, the amplitude is the smallest.

Problems and Solutions Section 2.9 (2.87 through 2.93)

2.87*. Compute the response of the system in Figure 2.34 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form

$$k(x) = kx - k_1 x^3$$

and the system is subject to a harmonic excitation of 300 N at a frequency of approximately one third the natural frequency ($\omega = \omega_n/3$) and initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s. The system has a mass of 100 kg, a damping coefficient of 170 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 10000 N/m^3. Compute the solution and compare it to the linear solution ($k_1 = 0$). Which system has the largest magnitude?

Solution: The following is a Mathcad simulation. The green is the steady state magnitude of the linear system, which bounds the linear solution, but is exceeded by the nonlinear solution. The nonlinear solution has the largest response.

Prop 2.78

$x0 := 0.01$

$v0 := 0.1$

$m := 100 \qquad k := 2000$

$k1 := 10000 \qquad F0 := 300$

$c := 170$

2.964
1.09

$\omega n := \sqrt{\dfrac{k}{m}}$

$\zeta := \dfrac{c}{2\sqrt{k \cdot m}}$

$f0 := \dfrac{F0}{m}$

$\omega := \dfrac{\omega n}{3}$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \qquad Y := X$

$D(t,X) := \begin{bmatrix} X_1 \\ -2\cdot\zeta\cdot\omega n\cdot X_1 - \omega n^2\cdot X_0 + \dfrac{k1}{m}\cdot(X_0)^3 + f0\cdot\cos(\omega\cdot t) \end{bmatrix}$

$Z := rkfixed(X, 0, 10, 4000, D)$

$L(t,Y) := \begin{bmatrix} Y_1 \\ \left(-2\cdot\zeta\cdot\omega n\cdot Y_1 - \omega n^2\cdot Y_0\right) + f0\cdot\cos(\omega\cdot t) \end{bmatrix}$

$t := Z^{<0>} \qquad x := Z^{<1>}$

$W := rkfixed(Y, 0, 10, 4000, L)$

$xL := W^{<1>}$

$d(t) := \dfrac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + (2\cdot\zeta\cdot\omega n\cdot\omega)^2}} \qquad \overrightarrow{F} := d(t)$

2.88*. Compute the response of the system in Figure 2.34 for the case that the damping is linear viscous and the spring is a nonlinear hard spring of the form

$$k(x) = kx + k_1 x^3$$

and the system is subject to a harmonic excitation of 300 N at a frequency equal to the natural frequency ($\omega = \omega_n$) and initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s. The system has a mass of 100 kg, a damping coefficient of 170 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 10000 N/m³. Compute the solution and compare it to the linear solution ($k_1 = 0$). Which system has the largest magnitude?

Solution: The Mathcad solution appears below. Note that in this case the linear amplitude is the largest!

Prop 2.79

$x0 := 0.01$

$v0 := 0.1$

$m := 100$ $k := 2000$ $c := 170$

$\alpha := 100$

$F0 := 300$

$\omega n := \sqrt{\dfrac{k}{m}}$ $\zeta := \dfrac{c}{2\sqrt{k \cdot m}}$ $f0 := \dfrac{F0}{m}$ $\omega := \dfrac{\omega n}{1}$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$ $Y := X$

$$D(t,X) := \begin{bmatrix} X_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0\right) - \alpha \cdot (X_0)^3 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 4000, D)$

$$L(t,Y) := \begin{bmatrix} Y_1 \\ -2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$t := Z^{<0>}$ $x := Z^{<1>}$ $W := \text{rkfixed}(Y, 0, 10, 4000, L)$

$xL := W^{<1>}$ $d(t) := \dfrac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + \left(2 \cdot \zeta \cdot \omega n \cdot \omega\right)^2}}$

$\alpha \cdot m = 1 \cdot 10^4$ $F := \overrightarrow{d(t)}$
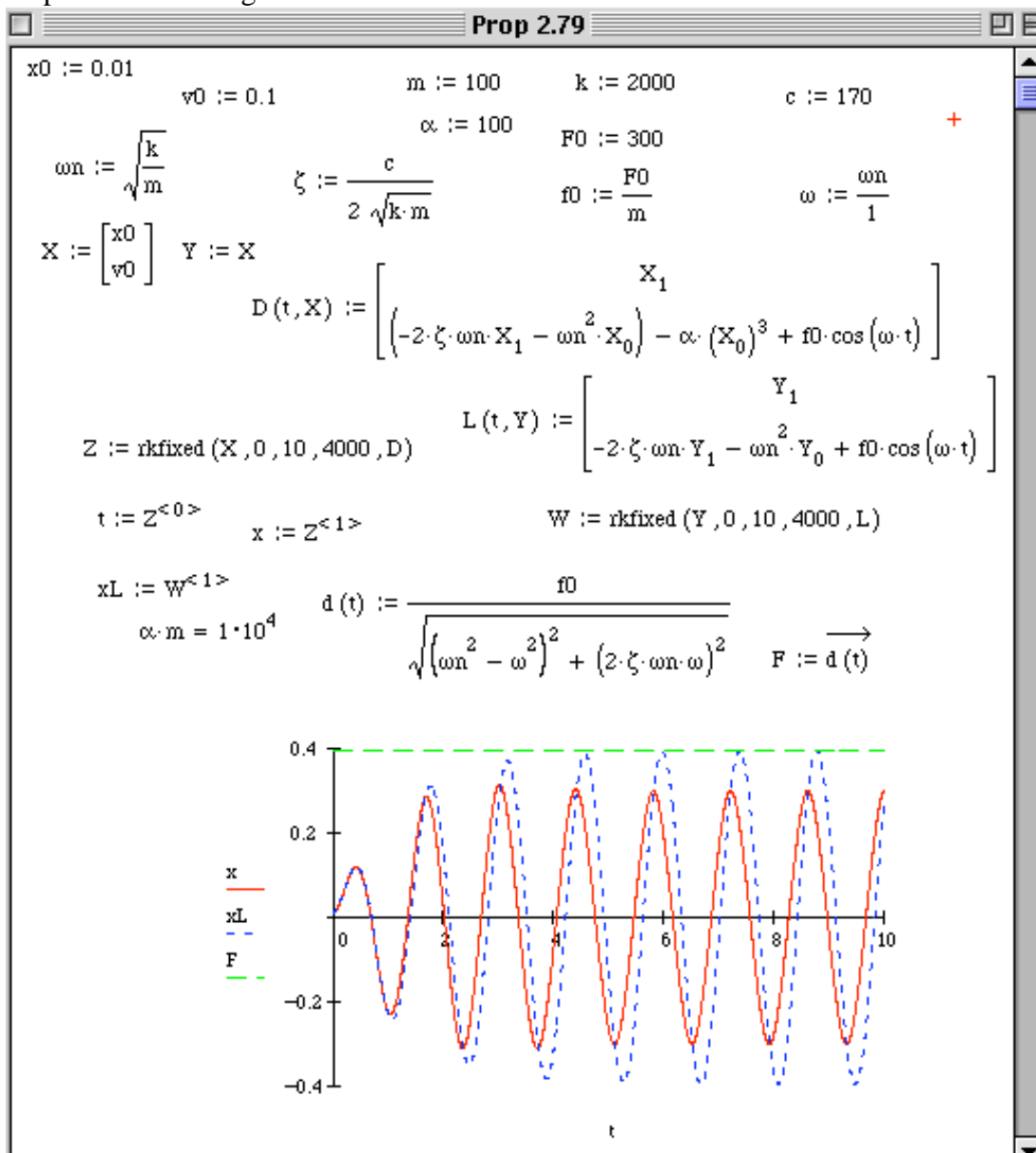
2.89*. Compute the response of the system in Figure 2.34 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form

$$k(x) = kx - k_1 x^3$$

and the system is subject to a harmonic excitation of 300 N at a frequency equal to the natural frequency ($\omega = \omega_n$) and initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s. The system has a mass of 100 kg, a damping coefficient of 15 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 100 N/m^3. Compute the solution and compare it to the hard spring solution ($k(x) = kx + k_1 x^3$).

Solution: The Mathcad solution is presented, first for a hard spring, then for a soft spring

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Prop 2.80

$x0 := 0.01$ $\quad v0 := 0.1$ $\quad m := 100$ $\quad k := 2000$ $\quad \alpha := 100$ $\quad c := 170$ $\quad F0 := 300$

$$\omega n := \sqrt{\frac{k}{m}} \quad \zeta := \frac{c}{2\sqrt{k \cdot m}} \quad f0 := \frac{F0}{m} \quad \omega := \frac{\omega n}{1} \quad X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \quad Y := X$$

$$D(t,X) := \begin{bmatrix} X_1 \\ \left(-2\cdot\zeta\cdot\omega n\cdot X_1 - \omega n^2 \cdot X_0\right) - \alpha\cdot(X_0)^3 + f0\cdot\cos(\omega\cdot t) \end{bmatrix}$$

$$L(t,Y) := \begin{bmatrix} Y_1 \\ -2\cdot\zeta\cdot\omega n\cdot Y_1 - \omega n^2 \cdot Y_0 + f0\cdot\cos(\omega\cdot t) \end{bmatrix}$$

$Z := \text{rkfixed}(X,0,10,4000,D)$

$t := Z^{<0>}$ $\qquad x := Z^{<1>}$ $\qquad\qquad W := \text{rkfixed}(Y,0,10,4000,L)$

$xL := W^{<1>}$ $\qquad\qquad d(t) := \dfrac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + \left(2\cdot\zeta\cdot\omega n\cdot\omega\right)^2}}$

$\alpha\cdot m = 1\cdot10^4$ $\qquad\qquad\qquad\qquad\qquad\qquad F := \overrightarrow{d(t)}$

\+
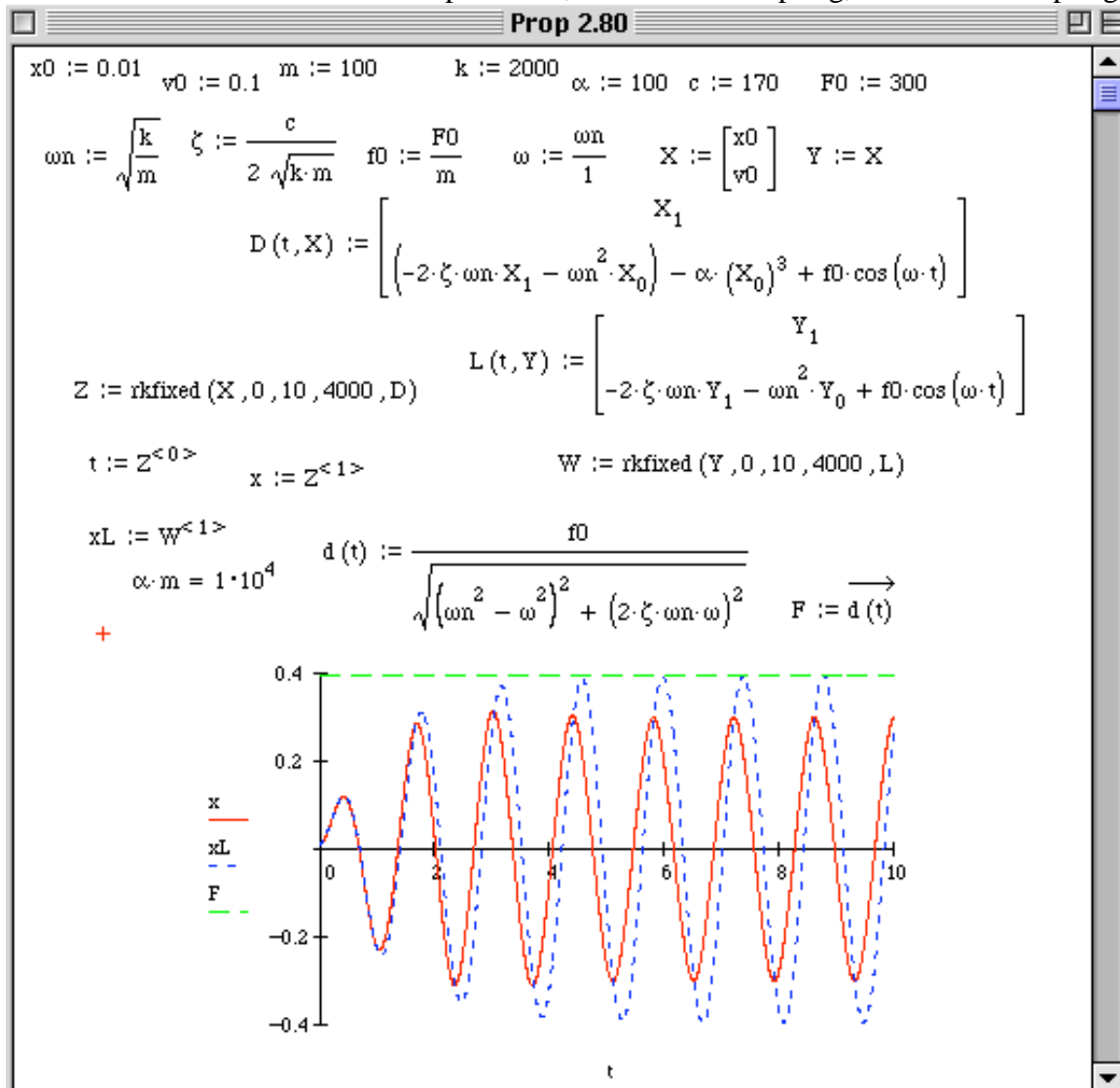
Next consider the result for the soft spring and note that the nonlinear response is higher in the transient then the linear case (opposite of the hardening spring), but nearly the same in steady state as the hardening spring.

Prop 2.80

$x0 := 0.01$ $v0 := 0.1$ $m := 100$ $k := 2000$ $\alpha := -100$ $c := 170$ $F0 := 300$

$$\omega n := \sqrt{\frac{k}{m}} \quad \zeta := \frac{c}{2\sqrt{k \cdot m}} \quad f0 := \frac{F0}{m} \quad \omega := \frac{\omega n}{1} \quad X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \quad Y := X \qquad +$$

$$D(t, X) := \begin{bmatrix} X_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0\right) - \alpha \cdot (X_0)^3 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ -2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$$Z := \text{rkfixed}(X, 0, 10, 4000, D)$$

$$t := Z^{<0>} \qquad x := Z^{<1>} \qquad W := \text{rkfixed}(Y, 0, 10, 4000, L)$$

$$xL := W^{<1>}$$

$$\alpha \cdot m = -1 \cdot 10^4 \qquad d(t) := \frac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + \left(2 \cdot \zeta \cdot \omega n \cdot \omega\right)^2}} \qquad F := \overrightarrow{d(t)}$$
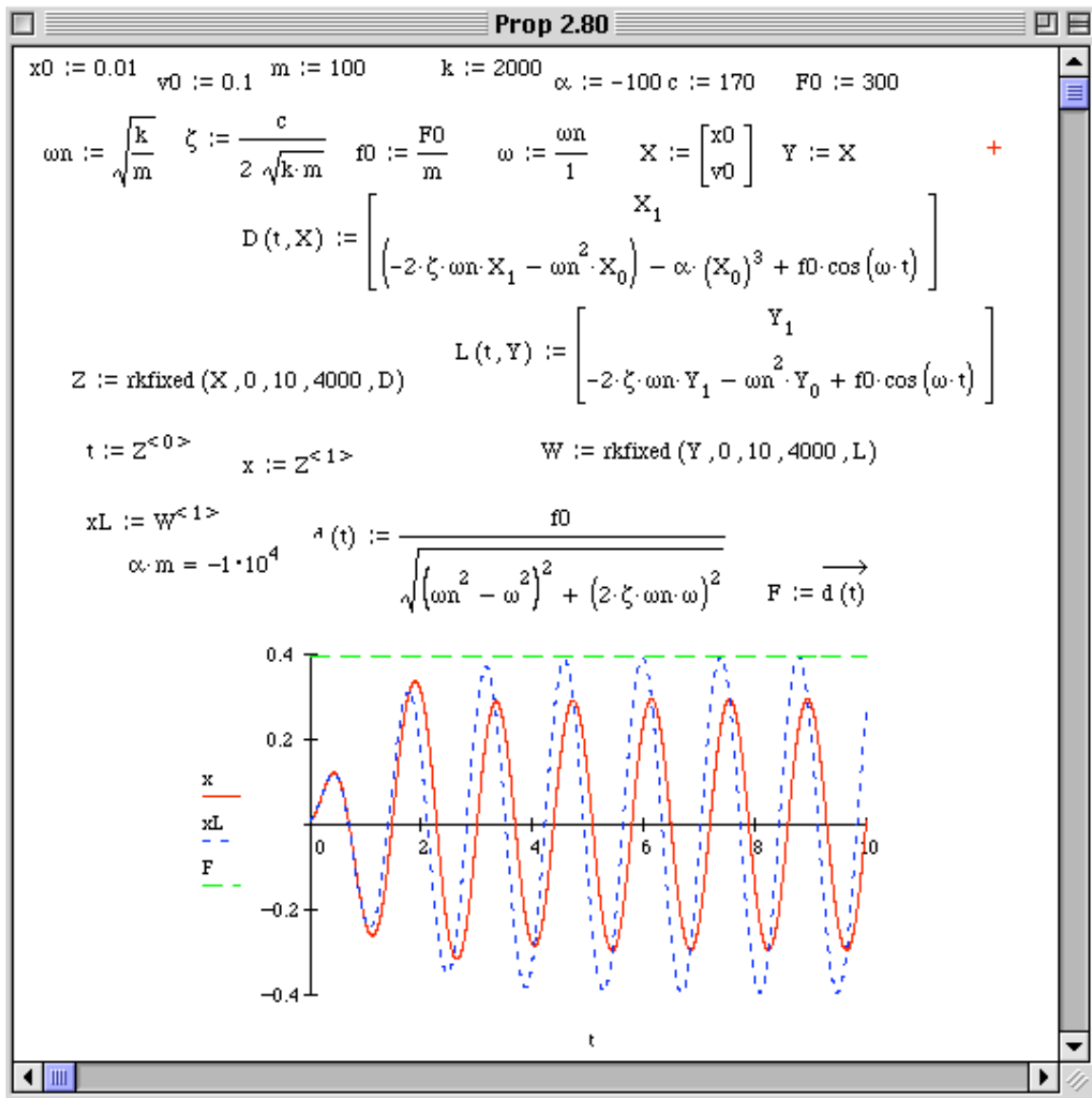
2.90*. Compute the response of the system in Figure 2.34 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form

$$k(x) = kx - k_1 x^3$$

and the system is subject to a harmonic excitation of 300 N at a frequency equal to the natural frequency ($\omega = \omega_n$) and initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s. The system has a mass of 100 kg, a damping coefficient of 15 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 1000 N/m³. Compute the solution and compare it to the quadratic soft spring ($k(x) = kx + k_1 x^2$).

Solution: The response to both the hardening and softening spring are given in the following Mathcad sessions. In each case the linear response is also shown for comparison. With the soft spring, the response is more variable, whereas the hardening spring seems to reach steady state.
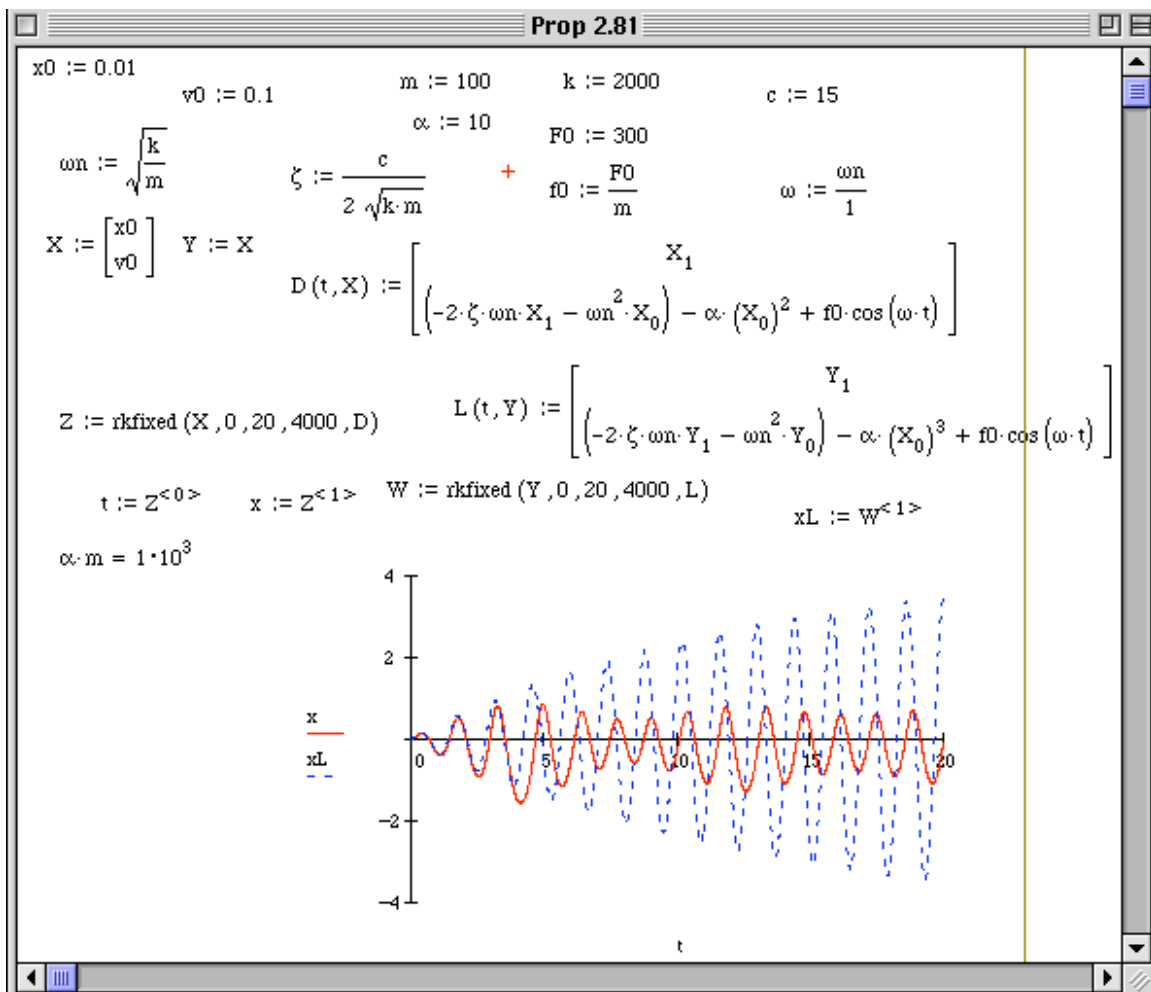
2.91*. Compare the forced response of a system with velocity squared damping as defined in equation (2.129) using numerical simulation of the nonlinear equation to that of the response of the linear system obtained using equivalent viscous damping as defined by equation (2.131). Use as initial conditions, $x_0 = 0.01$ m and $v_0 = 0.1$ m/s with a mass of 10 kg, stiffness of 25 N/m, applied force of 150 cos $(\omega_n t)$ and drag coefficient of $\alpha = 250$.

Solution:

Prop 2.82

$x0 := 1$ $v0 := 0.1$ $m := 10$ $k := 25$ $\alpha := 250$ $F0 := 150$

$\omega n := \sqrt{\dfrac{k}{m}}$ $f0 := \dfrac{F0}{m}$ $\omega := 1 \cdot \omega n$

$ceq := \sqrt{\dfrac{8 \cdot \alpha \cdot m}{3 \cdot \pi}} \cdot f0$ $\zeta := \dfrac{ceq}{2\sqrt{k \cdot m}}$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$ $Y := X$

$$D(t,X) := \begin{bmatrix} X_1 \\ -\omega n^2 \cdot X_0 - \dfrac{\alpha}{m} \cdot (X_1)^2 \cdot \dfrac{X_1}{|X_1|} + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$$L(t,Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$+$

$Z := rkfixed(X,0,30,2000,D)$

$t := Z^{<0>}$ $x := Z^{<1>}$ $W := rkfixed(Y,0,30,2000,L)$

$xL := W^{<1>}$ $d(t) := \dfrac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + \left(2 \cdot \zeta \cdot \omega n \cdot \omega\right)^2}}$ $\vec{F} := \vec{d(t)}$
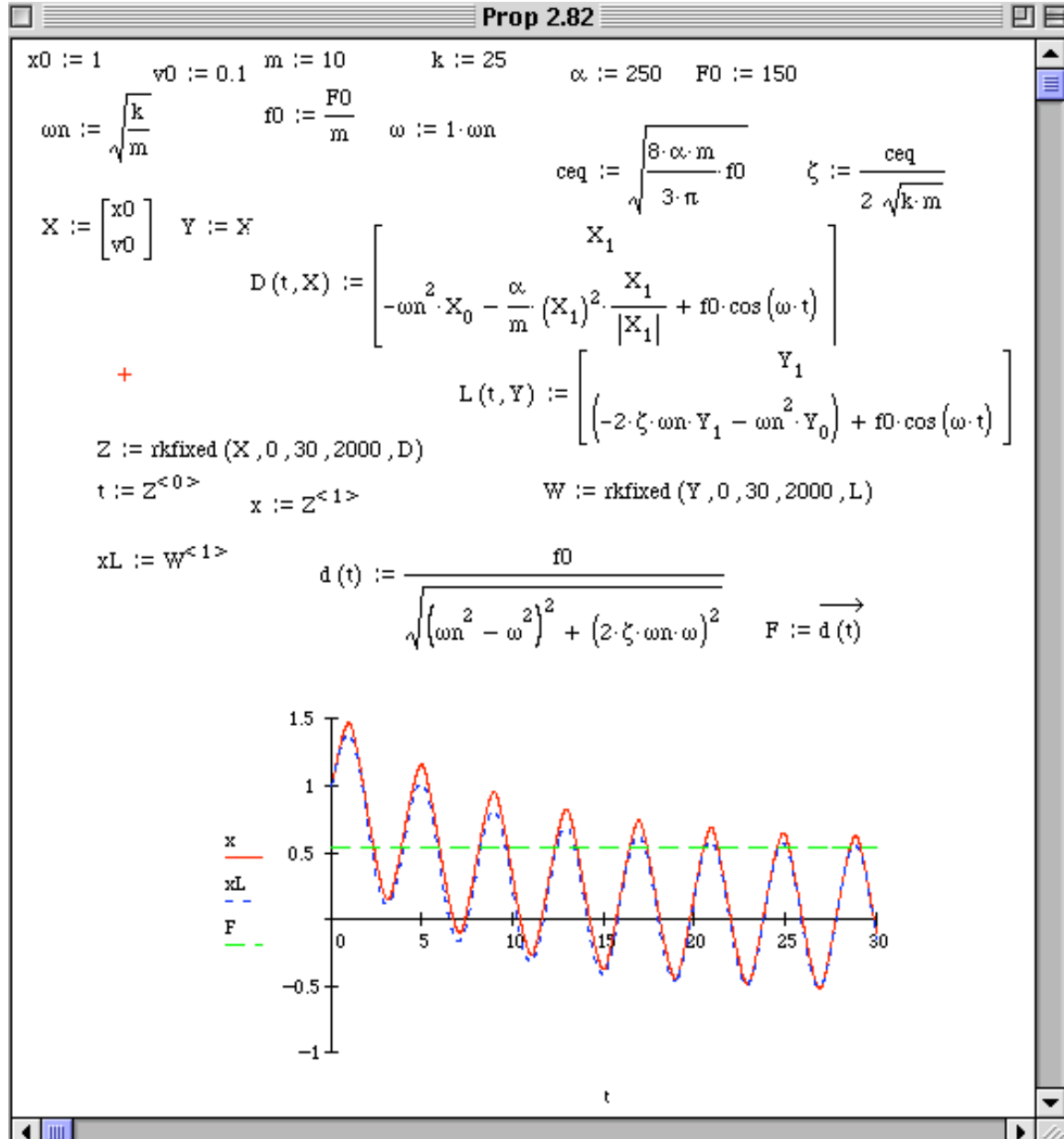
2.92*. Compare the forced response of a system with structural damping (see table 2.2) using numerical simulation of the nonlinear equation to that of the response of the linear system obtained using equivalent viscous damping as defined in Table 2.2. Use as initial conditions, $x_0 = 0.01$ m and $v_0 = 0.1$ m/s with a mass of 10 kg, stiffness of 25 N/m, applied force of 150 cos ($\omega_n t$) and solid damping coefficient of $b = 25$.
Solution: The solution is presented here in Mathcad

Prop 2.83

$x0 := 0.01$ $v0 := 0.1$ $m := 10$ $k := 25$ $b := 25$ $FO := 150$

$\omega n := \sqrt{\dfrac{k}{m}}$ $f0 := \dfrac{FO}{m}$ $\omega := 1 \cdot \omega n$

$ceq := 2 \cdot \dfrac{b}{\pi \cdot \omega}$ $\zeta := \dfrac{ceq}{2\sqrt{k \cdot m}}$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$ $Y := X$

$D(t,X) := \begin{bmatrix} X_1 \\ -\omega n^2 \cdot X_0 - \dfrac{b}{m} \cdot |X_0| \cdot \dfrac{X_1}{|X_1|} + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$

$L(t,Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$

+

$Z := \text{rkfixed}(X, 0, 25, 2500, D)$

$t := Z^{<0>}$ $x := Z^{<1>}$

$W := \text{rkfixed}(Y, 0, 25, 2500, L)$

$xL := W^{<1>}$
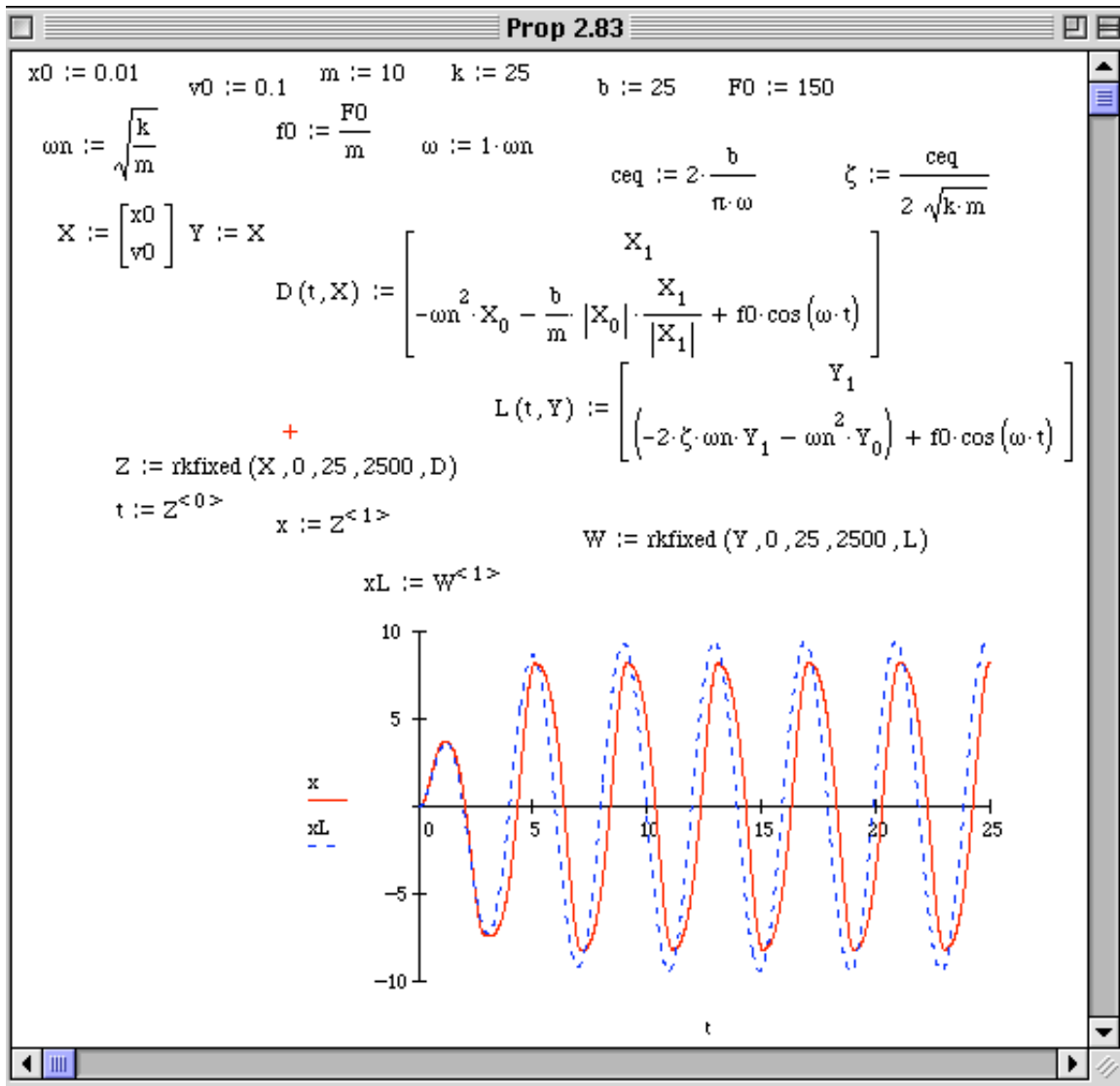
Chapter Three Solutions

Problem and Solutions for Section 3.1 (3.1 through 3.14)

3.1 Calculate the solution to

$$\ddot{x} + 2\dot{x} + 2x = \delta(t - \pi)$$
$$x(0) = 1 \quad \dot{x}(0) = 0$$

and plot the response.

Solution: Given: $\ddot{x} + 2\dot{x} + 2x = \delta(t - \pi)$ $\quad x(0) = 1, \quad \dot{x}(0) = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.414 \text{ rad/s}, \quad \zeta = \frac{c}{2\sqrt{km}} = 0.7071, \quad \omega_d = \omega_n\sqrt{1 - \zeta^2} = 1 \text{ rad/s}$$

Total Solution: $\qquad\qquad x(t) = x_h(t) + x_p(t)$

Homogeneous: From Equation (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \qquad \phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] = .785 \text{ rad}$$

$$\Rightarrow x_h(t) = 1.414e^{-t}\sin(t + .785)$$

Particular: From Equation. (3.9)

$$x_p(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t - \tau) = \frac{1}{(1)(1)}e^{-(t-\pi)}\sin(t - \pi)$$

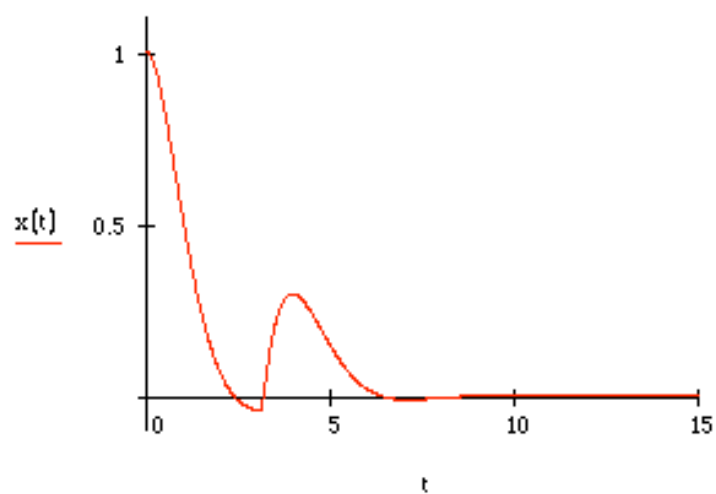$$\text{But,} \quad \sin(-t) = -\sin t \quad \text{So,} \quad x_p(t) = -e^{-(t-\pi)}\sin t \quad \Rightarrow$$

$$x(t) = 1.414e^{-t}\sin(t + 0.785) \qquad\qquad 0 < t < \pi$$

$$x(t) = 1.414e^{-t}\sin(t + 0.785) - e^{-(t-\pi)}\sin t \qquad t > \pi$$

This is plotted below using the Heaviside function.

$$x(t) := 1.414 \cdot e^{-t} \cdot \sin(t + 0.785) - e^{-(t - \pi)} \cdot \sin(t) \cdot \Phi(t - \pi)$$

3.2   Calculate the solution to

$$\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t - \pi)$$
$$x(0) = 0 \quad \dot{x}(0) = 1$$

and plot the response.

Solution: Given: $\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t - \pi)$, $x(0) = 0$, $\dot{x}(0) = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.732 \text{ rad/s}, \ \zeta = \frac{c}{2\sqrt{km}} = 0.5774, \ \omega_d = \omega_n\sqrt{1-\zeta^2} = 1.414 \text{ rad/s}$$

Total Solution:

$$x(t) = x_h + x_{p1} \qquad 0 < t < \pi$$
$$x(t) = x_h + x_{p1} + x_{p2} \qquad t > \pi$$

Homogeneous: Eq. (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) = Ae^{-t}\sin(1.414t + \phi)$$

Particular: #1 (Chapter 2)

$$x_{p1}(t) = X\sin(\omega t - \theta), \text{ where } \omega = 1 \text{ rad/s}. \text{ Note that } f_0 = \frac{F_0}{m} = 1$$

$$\Rightarrow X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = 0.3536, \text{ and } \theta = \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right] = 0.785 \text{ rad}$$

$$\Rightarrow x_{p1}(t) = 0.3536\sin(t - 0.7854)$$

Particular: #2 Equation 3.9

$$x_{p2}(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\pi)}\sin\omega_d(t - \tau) = \frac{1}{(1)(1.414)}e^{-(t-\pi)}\sin 1.414(t - \pi)$$

$$\Rightarrow x_{p2}(t) = 0.7071e^{-(t-\pi)}\sin 1.414(t - \pi)$$

The total solution for $0 < t < \pi$ becomes:

$$x(t) = Ae^{-t}\sin(1.414t + \phi) + 0.3536\sin(t - 0.7854)$$
$$\dot{x}(t) = -Ae^{-t}\sin(1.414t + \phi) + 1.414Ae^{-t}\cos(1.414t + \phi) + 0.3536\cos(t - 0.7854)$$
$$x(0) = 0 = A\sin\phi - 0.25 \Rightarrow A = \frac{0.25}{\sin\phi}$$

$$\dot{x}(0) = 1 = -A\sin\phi + 1.414A\cos\phi + 0.25 \Rightarrow 0.75 = 0.25 - 1.414(0.25)\frac{1}{\tan\phi}$$

$$\Rightarrow \phi = 0.34 \text{ and } A = 0.75$$

Thus for the first time interval, the response is

$$x(t) = 0.75e^{-t}\sin(1.414t + 0.34) + 0.3536\sin(t - 0.7854) \qquad 0 < t < \pi$$

Next consider the application of the impulse at $t = \pi$:

$$x(t) = x_h + x_{p1} + x_{p2}$$

$$x(t) = -0.433e^{-t}\sin(1.414t + 0.6155) + 0.3536\sin(t - 0.7854) - 0.7071e^{-(t-\pi)}\sin(1.414t - \pi) \quad t > \pi$$

The response is plotted in the following (from Mathcad):
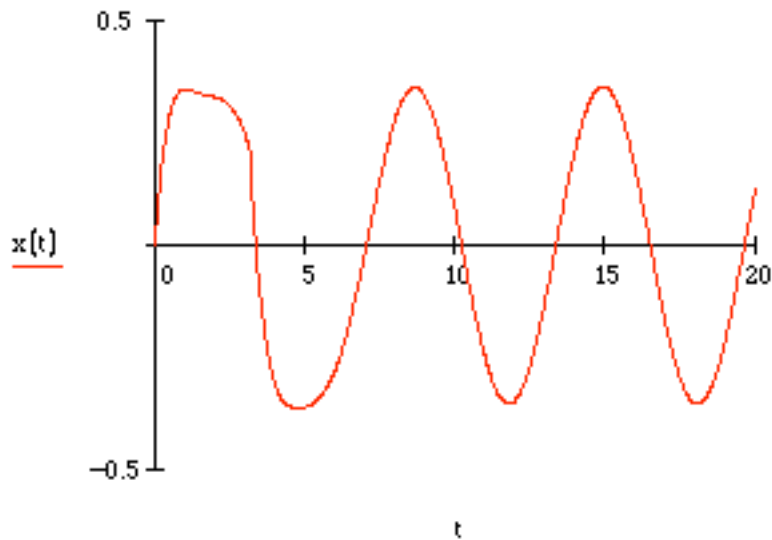
3.3 Calculate the impulse response function for a critically damped system.

Solution:

The change in the velocity from an impulse is $v_0 = \dfrac{\hat{F}}{m}$, while $x_0 = 0$. So for a critically damped system, we have from Eqs. 1.45 and 1.46 with $x_0 = 0$:

$$x(t) = v_0 t e^{-\omega_n t}$$

$$\Rightarrow x(t) = \frac{\hat{F}}{m} t e^{-\omega_n t}$$

3.4   Calculate the impulse response of an overdamped system.

Solution:

The change in velocity for an impulse $v_0 = \dfrac{\hat{F}}{m}$, while $x_0 = 0$. So, for an overdamped system, we have from Eqs. 1.41, 1.42 and 1.43:

$$x(t) = e^{-\zeta\omega_n t}\left[\frac{-v_0}{2\omega_n\sqrt{\zeta^2-1}}e^{-\omega(_n\sqrt{\zeta^2-1})t} + \frac{v_0}{2\omega_n\sqrt{\zeta^2-1}}e^{-\omega(_n\sqrt{\zeta^2-1})t}\right]$$

$$x(t) = \frac{\hat{F}}{2m\omega_n\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}\left[e^{-\omega(_n\sqrt{\zeta^2-1})t} - e^{-\omega(_n\sqrt{\zeta^2-1})t}\right]$$

3.5   Derive equation (3.6) from equations (1.36) and (1.38).

Solution:

Equation 1.36: $x(t) = Ae^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right)$

Equation 1.38: $A = \sqrt{\dfrac{\left(v_0 + \zeta\omega_n x_0\right)^2 + \left(x_0\omega_d\right)^2}{\omega_d^2}}$, $\phi = \tan^{-1}\left[\dfrac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right]$

Since $x_0 = 0$ and $v_0 = \dfrac{\hat{F}}{m}$, Equation 1.38 becomes

$$A = \frac{v_0}{\omega_d} = \frac{\hat{F}}{m\omega_d}$$

$$\phi = \tan^{-1}\left(0\right) = 0$$

So Equation 1.36 becomes

$$x(t) = \frac{\hat{F}}{m\omega_d}e^{-\zeta\omega_n t}\sin\left(\omega_d t\right) \text{ which is Equation 3.6}$$

3.6   Consider a simple model of an airplane wing given in Figure P3.6.   The wing is approximated as vibrating back and forth in its plane, massless compared to the missile carriage system (of mass *m*). The modulus and the moment of inertia of the wing are approximated by *E* and *I*, respectively, and *l* is the length of the wing. The wing is modeled as a simple cantilever for the purpose of estimating the vibration resulting from the release of the missile, which is approximated by the impulse funciton $F\delta(t)$. Calculate the response and plot your results for the case of an aluminum wing 2 m long with $m = 1000$ kg, $\zeta = 0.01$, and $I = 0.5$ m^4. Model F as 1000 N lasting over 10^{-2}s.
Modeling of wing vibration resulting from the release of a missile. (a) system of interest; (b) simplification of the detail of interest; (c) crude model of the wing: a cantilevered beam section (recall Figure 1.24); (d) vibration model used to calculate the response neglecting the mass of the wing.

Jet with missiles
(a)

Carriage

Missile

Top view with wing modeled as cantilevered
(b)

I

E

$k = \dfrac{3EI}{l^3}$

$F\delta(t)$

m

$x(t)$

Simplified beam model
(c)

c

k

m

$F\delta(t)$

$x(t)$

Vibration model
(d)
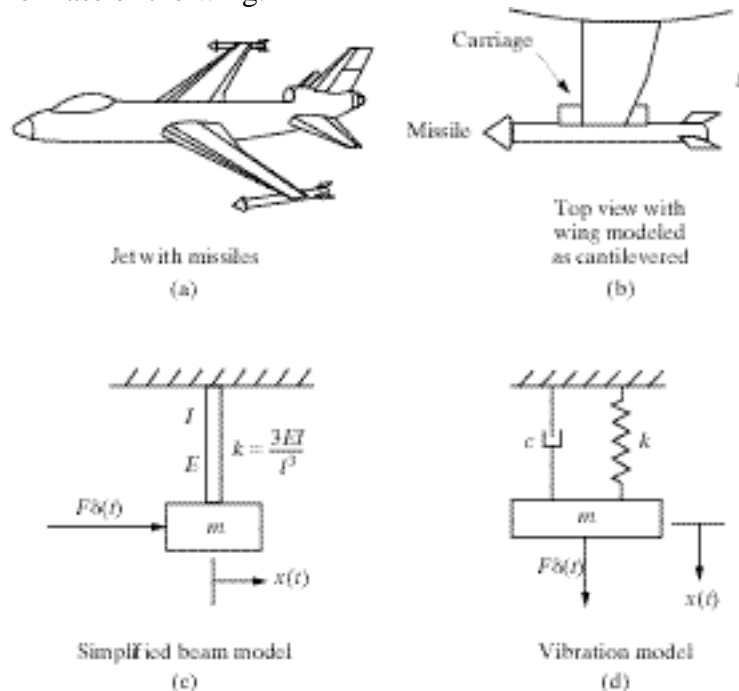
Solution: Given:

$$m = 1000 \text{ kg} \quad \zeta = 0.01$$
$$l = 4 \text{ m} \qquad I = 0.5 \text{ m}^4$$
$$F = 1000 \text{ N} \quad \Delta t = 10^{-2} \text{ s}$$

From Table 1.2, the modulus of Aluminum is $E = 7.1 \times 10^{10}$ N/m^2

The stiffness is

$$k = \frac{3EI}{\ell^3} = \frac{3\left(7.1 \times 10^{10}\right)\left(0.5\right)}{4^3} = 1.664 \times 10^9 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.29 \times 10^3 \text{ rad/s (205.4 Hz)}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 1.29 \times 10^3$$

Solution (Eq. 3.6):

$$x(t) = \frac{\left(F\Delta t\right)e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = 7.753\times10^{-6}e^{-12.9t}\sin\left(1290t\right) \text{ m}$$

The following m-file

```
t=(0:0.0001:0.5);
F=1000;dt=0.01;m=1000;zeta=0.01;E=7.1*10^10;I=0.5;L=4;
wn=sqrt((3*I*E/L^3)/m);
wd=wn*sqrt(1-zeta^2);
x=(F*dt/(m*wd))*exp(-zeta*wn*t).*sin(wd*t);
plot(t,x)
```

The solution worked out in Mathcad is given in the following:

$E := 7.1 \cdot 10^{10}$

$m := 1000$ $\qquad L := 4 \qquad I := 0.5 \quad F := 1000 \qquad \Delta t := 10^{-2} \quad$ sec

$\zeta := 0.01$

$k := \dfrac{3 \cdot I \cdot E}{L^3} \qquad \omega n := \sqrt{\dfrac{k}{m}} \qquad k = 1.664 \cdot 10^9 \qquad\qquad \omega n = 1.29 \cdot 10^3$

$fn := \dfrac{\omega n}{2 \cdot \pi} \qquad fn = 205.308 \qquad \omega d := \omega n \cdot \sqrt{1 - \zeta^2} \qquad \omega d = 1.29 \cdot 10^3$
$\qquad\qquad\qquad\qquad\qquad$ Hz

$\zeta \cdot \omega n = 12.9 \qquad\qquad\qquad\qquad \dfrac{F}{m \cdot \omega d} \cdot (\Delta t) = 7.752 \cdot 10^{-6}$

$t := 0 , 0.00001 .. 0.4$

$x(t) := \dfrac{F}{m \cdot \omega d} \cdot \left(\Delta t \cdot e^{-\zeta \cdot \omega n \cdot t} \right) \cdot \sin(\omega d \cdot t) \qquad$ meters

t
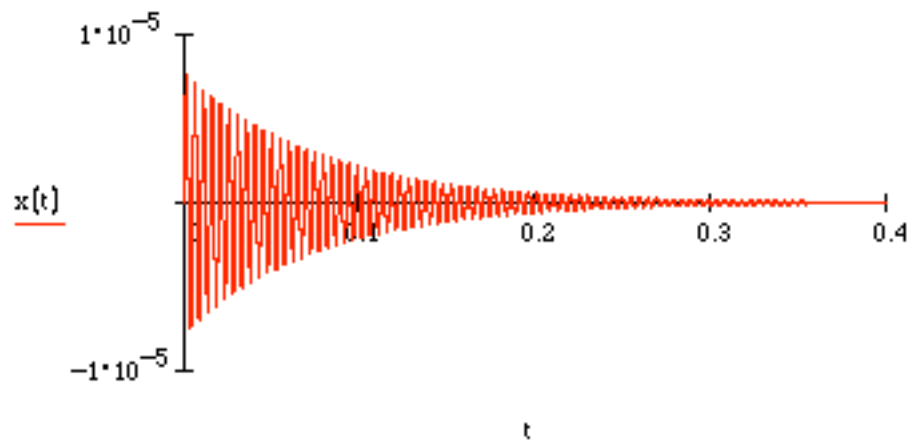
3.7   A cam in a large machine can be modeled as applying a 10,000 N-force over an interval of 0.005 s. This can strike a valve that is modeled as having physical parameters: $m = 10$ kg, $c = 18$ N•s/m, and stiffness $k = 9000$ N/m. The cam strikes the valve once every 1 s. Calculate the vibration response, $x(t)$, of the valve once it has been impacted by the cam. The valve is considered to be closed if the distance between its rest position and its actual position is less than 0.0001 m. Is the valve closed the very next time it is hit by the cam?

Solution: Given:

$$F = 10,000 \text{ N} \qquad \Delta t = 0.005 \text{ s}$$

$$m = 10 \text{ kg} \qquad c = 18 \text{ N} \cdot \text{s/m} \qquad k = 9000 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 30 \text{ rad/s} \quad \zeta = \frac{c}{2\sqrt{km}} = 0.03 \quad \omega_d = \omega_n \sqrt{1-\zeta^2} = 29.99 \text{ rad/s}$$

Solution Eq. (3.6):

$$x(t) = \frac{\left(F\Delta t\right)e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

$$x(t) = \frac{\left(10,000\right)\left(0.005\right)e^{-\left(0.03\right)\left(30\right)t}}{\left(10\right)\left(29.99\right)}\sin\left(29.99t\right)$$

$$x(t) = 0.1667e^{-0.9t}\sin\left(29.99t\right)\text{m}$$

At $t=1$ s: $x(1) = 0.1667e^{-0.9}\sin(29.99) = -.06707 \text{ m}$

Since $\left|x(1)\right| = 0.06707 > 0.0001$, the valve is not closed.

3.8   The vibration packages dropped from a height of h meters can be approximated by considering Figure P3.8 and modeling the point of contact as an impulse applied to the system at the time of contact. Calculate the vibration of the mass m after the system falls and hits the ground. Assume that the system is underdamped.
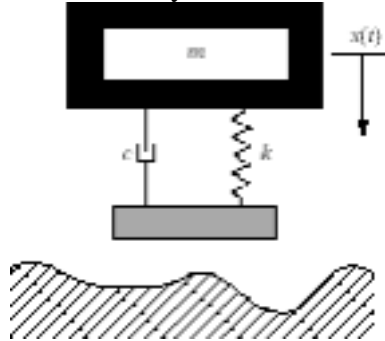
Solution: When the system hits the ground, it responds as if an impulse force acted on it.

From Equation (3.6): $x(t) = \dfrac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$ where $\dfrac{\hat{F}}{m} = v_0$

Calculate v_0:

For falling mass: $\qquad\qquad\qquad x = \dfrac{1}{2}at^2$

So, $v_0 = gt^*$, where t^* is the time of impact from height h

$$h = \frac{1}{2}gt^{*2} \Rightarrow t^* = \sqrt{\frac{2h}{g}}$$

$$v_0 = \sqrt{2gh}$$

Let $t = 0$ when the end of the spring hits the ground

The response is $\qquad\qquad x(t) = \dfrac{\sqrt{2gh}}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$

Where $\omega_n,\ \omega_d$, and ζ are calculated from m, c, k. Of course the problem could be solved as a free response problem with $x_0 = 0$, $v_0 = \sqrt{2gh}$ or an impulse response with impact model as the unit velocity given.

3.9   Calculate the response of

$$3\ddot{x}(t) + 12\dot{x}(t) + 12x(t) = 3\delta(t)$$

for zero initial conditions. The units are in Newtons. Plot the response.

Solution: Dividing the equation of motion by 3 reveals;

$$\omega_n = \sqrt{4} = 2 \text{ rad/s} \quad \zeta = \frac{12}{2(3)(2)} = 1 \Rightarrow \text{critically damped}$$

$$\hat{F} = 3 \qquad\qquad v_0 = \frac{F\Delta t}{m}, \qquad x_0 = 0$$

$$x = \left(a_1 + a_2 t\right)e^{-\omega_n t} \quad a_1 = 0 \qquad a_2 = \frac{F\Delta t}{m}$$

$$\Rightarrow x(t) = \frac{\hat{F}}{m}te^{-2t} = \frac{3}{3}te^{-2t} = te^{-2t}$$

$$x(t) := t \cdot e^{-2 \cdot t} \qquad +$$
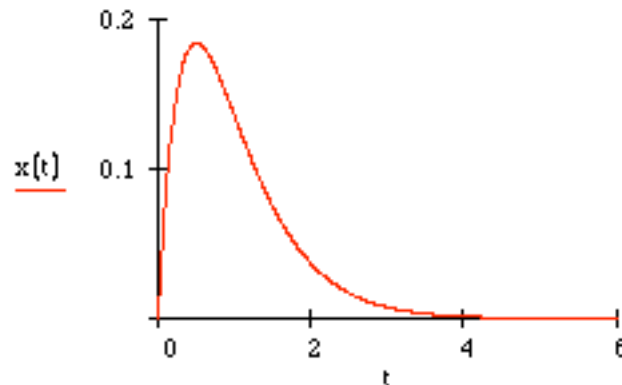
3.10 Compute the response of the system:

$$3\ddot{x}(t) + 12\dot{x}(t) + 12x(t) = 3\delta(t)$$

subject to the initial conditions $x(0) = 0.01$ m and $v(0) = 0$. The units are in Newtons. Plot the response.

Solution: From the previous problem the system is critically damped with a solution of the form

$$x(t) = (a_1 + a_2 t)e^{-2t}.$$

Applying the given initial conditions yields

$$x(0) = 0.01 = a_1 \quad \text{and} \quad \dot{x}(0) = 0 = -2(0.01 + a_2 0) + a_2$$

$$\Rightarrow x(t) = (0.01 + 0.02t)e^{-2t}$$

Next add to this the solution due to the unit impulse, which was calculated in Problem 3.9 to get:

$$x(t) = te^{-2t} + (0.01 + 0.02t)e^{-2t}$$

$$\Rightarrow \underline{x(t) = (0.01 + 1.02t)e^{-2t}}$$

$$x(t) := (0.01 + 1.02 \cdot t) \cdot e^{-2 \cdot t}$$

3.11   Calculate the response of the system

$$3\ddot{x}(t) + 6\dot{x}(t) + 12x(t) = 3\delta(t) - \delta(t-1)$$

subject to the initial conditions $x(0) = 0.01$ m and $v(0) = 1$ m/s. The units are in Newtons. Plot the response.

Solution: First compute the natural frequency and damping ratio:

$$\omega_n = \sqrt{\frac{12}{3}} = 2 \text{ rad/s}, \quad \zeta = \frac{6}{2\cdot 2\cdot 3} = 0.5, \quad \omega_d = 2\sqrt{1-0.5^2} = 1.73 \text{ rad/s}$$

so that the system is underdamped. Next compute the responses to the two impulses:

$$x_1(t) = \frac{\hat{F}}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t = \frac{3}{3(1.73)}e^{-(t-1)}\sin 1.73(t-1) = 0.577e^{-t}\sin 1.73t, t > 0$$

$$x_2(t) = \frac{\hat{F}}{m\omega_d}e^{-\zeta\omega_n (t-1)}\sin\omega_d(t-1) = \frac{1}{3(1.73)}e^{-t}\sin 1.73t = 0.193e^{-(t-1)}\sin 1.73(t-1), t > 1$$

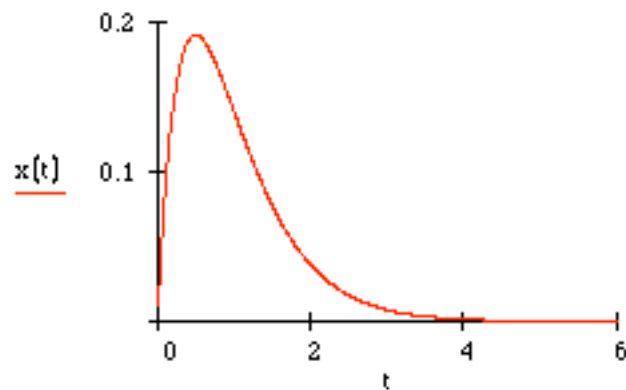Now compute the response to the initial conditions from Equation (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$A = \sqrt{\frac{\left(v_0 + \zeta\omega_n x_0\right)^2 + \left(x_0\omega_d\right)^2}{\omega_d^2}}, \quad \phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] = 0.071 \text{ rad}$$

$$\Rightarrow x_h(t) = 0.5775e^{-t}\sin(t + 0.017)$$

Using the Heaviside function the total response is

$$x(t) = 0.577e^{-t}\sin 1.73t + 0.583e^{-t}\sin(t + 0.017) + 0.193e^{-(t-1)}\sin 1.73(t-1)\Phi(t-1)$$

This is plotted below in Mathcad:

$$x(t) := \left[\frac{e^{-\zeta\cdot\omega n\cdot t}}{\omega d}\cdot\sin(\omega d\cdot t) + A\cdot e^{-\zeta\cdot\omega n\cdot t}\cdot\sin(\omega d\cdot t + \phi)\right] + \left[\frac{e^{-\zeta\cdot\omega n\cdot(t-1)}}{-3\cdot\omega d}\cdot\sin[\omega d\cdot(t-1)]\right]\cdot\Phi(t-1)$$
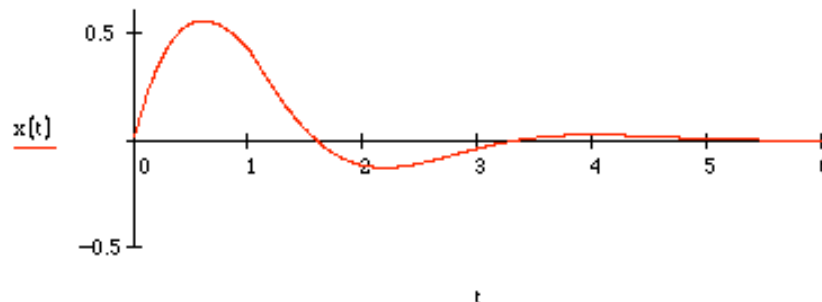
Note the slight bump in the response at $t = 1$ when the second impact occurs.

3.12   A chassis dynamometer is used to study the unsprung mass of an automobile as illustrated in Figure P3.12 and discussed in Example 1.4.1 and again in Problem 1.64. Compute the maximum magnitude of the center of the wheel due to an impulse of 5000 N

applied over 0.01 seconds. Assume the wheel mass is $m = 15$ kg, the spring stiffness is $k = 500,000$ N/m, the shock absorber provides a damping ratio of $\zeta = 0.3$, and the rotational inertia is $J = 2.323$ kg m^2. Compute and plot the response of the wheel system to an impulse of 5000 N over 0.01 s. Compare the undamped maximum amplitude to that of the maximum amplitude of the damped system (use $r = 0.457$ m).
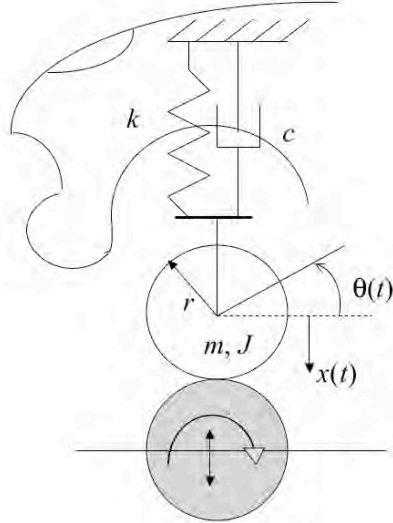
Figure P3.12 Simple model of an automobile suspension system mounted on a chassis dynamometer. The rotation of the car's wheel/tire assembly (of radius r) is given by $\theta(t)$ and is vertical deflection by $x(t)$.

Solution: With the values given the natural frequency, damped natural frequency, and impulse are calculated to be:

$$\omega_n = \sqrt{\frac{k}{m + J/r^2}} = 117.67 \text{ rad/s } = 18.73 \text{ Hz, } \omega_d = 112.25 \text{ rad/s, } X = \frac{F\Delta t}{(m + J/r^2)\omega_n} = 0.014 \text{ m}$$

The response is then plotted as

$$x(t) := \frac{F \cdot \Delta t}{\left(m + \dfrac{J}{r}\right) \cdot \omega n} \cdot e^{-\zeta \cdot \omega n \cdot t} \cdot \sin(\omega d \cdot t)$$
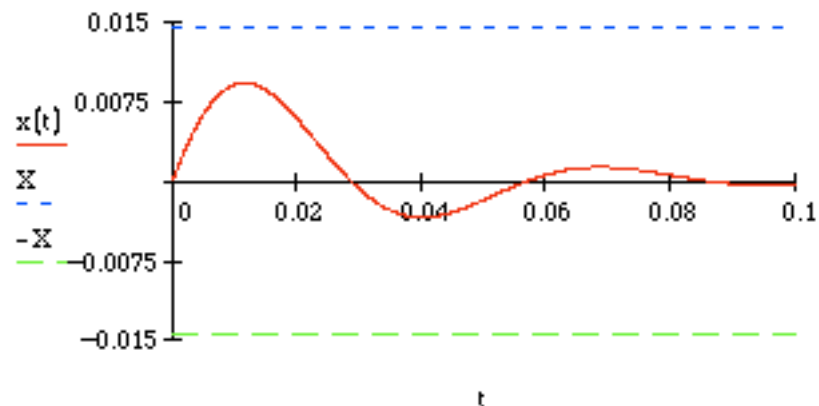
Note that the maximum amplitude of the undamped system, *X*, is not achieved.

3.13 Consider the effect of damping on the bird strike problem of Example 3.1.1. Recall from the example that the bird strike causes the camera to vibrate out of limits. Adding damping will cause the magnitude of the response to decrease but may not be able to keep the camera from vibrating past the 0.01 m limit. If the damping in the aluminum is modeled as $\zeta = 0.05$, approximately how long before the camera vibration reduces to the required limit? (Hint: plot the time response and note the value for time after which the oscillations remain below 0.01 m).

Solution: Using the values given in Example 3.1.1 and equations (3.7) and (3.8), the response has the form

$$x(t) = \frac{m_b v}{m\omega_n} e^{-\zeta\omega_n t} \sin\omega_d t = 0.026 e^{-13.07t} \sin 260.976t$$

Here m_b is the mass of the bird and m is the mass of the camera. This is plotted in Mathcad below

$$X := \frac{mb \cdot v}{mc \cdot \omega n} \qquad X = 0.026 \cdot m$$

$$\zeta := 0.05 \qquad \zeta \cdot \omega n = 13.065 \cdot \sec^{-1}$$

$$Y := 0.026 \qquad wn := 261.303 \qquad wd := wn\cdot\sqrt{1-\zeta^2} \qquad wd = 260.976$$

$$x(t) := Y\cdot e^{-\zeta\cdot wn\cdot t}\cdot\sin(wd\cdot t) \qquad XM(t) := 0.01$$
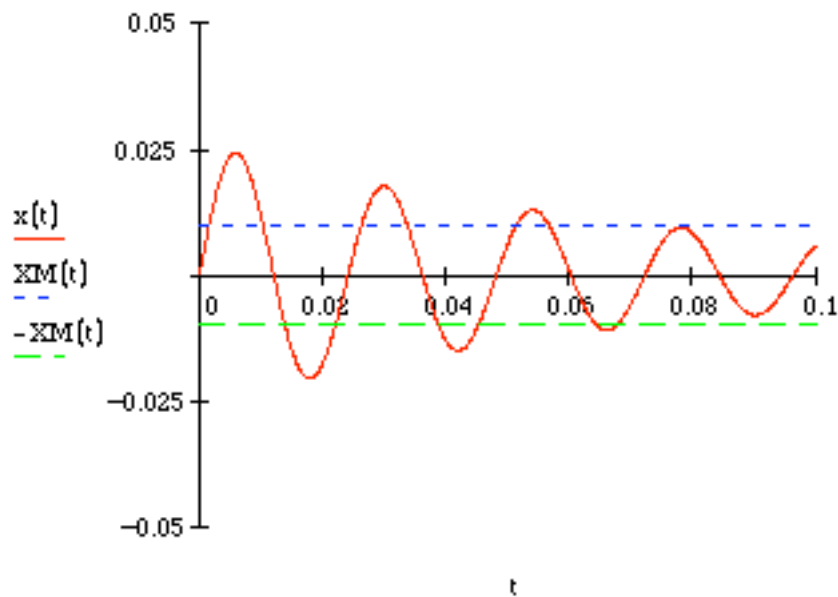
From the plot, the amplitude remains below 0.01 m after about 0.057 s.

3.14 Consider the jet engine and mount indicated in Figure P3.14 and model it as a mass on the end of a beam as done in Figure 1.24. The mass of the engine is usually fixed. Find a expression for the value of the transverse mount stiffness, k, as a function of the relative speed of the bird, v, the bird mass, the mass of the engine and the maximum displacement that the engine is allowed to vibrate.
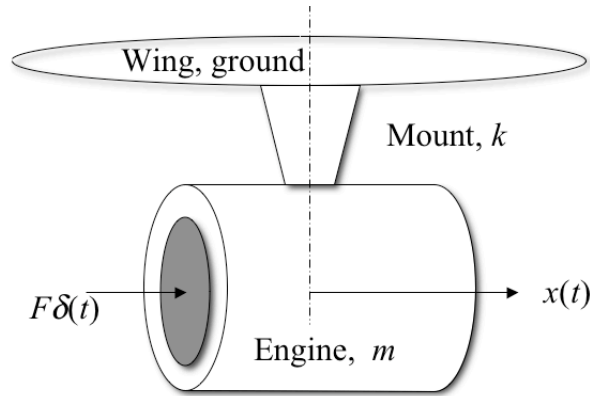
Figure P3.14 Model of a jet engine in transverse vibration due to a bird strike.

Solution: The equation of motion is

$$m\ddot{x}(t) + kx(t) = \hat{F}\delta(t)$$

From equations (3.7) and (3.8) the magnitude of the response is

$$\left|X\right| = \frac{\hat{F}}{m\omega_n}$$

for an undamped system. If the bird is moving with momentum $m_b v$ then:

$$\left|X\right| = \frac{m_b v}{m\omega_n} \Rightarrow \left|X\right| = \frac{m_b v}{\sqrt{mk}} \Rightarrow k = \frac{1}{m}\left(\frac{m_b v}{\left|X\right|}\right)^2$$

This can be used to provide some guidance in designing the engine mount.

Problems and Solutions for Section 3.2 (3.15 through 3.25)

3.15   Calculate the response of an overdamped single-degree-of-freedom system to an arbitrary non-periodic excitation.

Solution: From Equation (3.12): $x(t) = \int_0^t F(\tau) h(t - \tau) d\tau$

For an overdamped SDOF system (see Problem 3.4)

$$h(t - \tau) = \frac{1}{2m\omega_n \sqrt{\zeta^2 - 1}} e^{-\zeta\omega_n (t-\tau)} \left(e^{\omega_n \sqrt{\zeta^2 - 1}(t-\tau)} - e^{-\omega_n \sqrt{\zeta^2 - 1}(t-\tau)} \right) d\tau$$

$$x(t) = \int_0^t F(\tau) \frac{1}{2m\omega_n \sqrt{\zeta^2 - 1}} e^{-\zeta\omega_n (t-\tau)} \left(e^{\omega_n \sqrt{\zeta^2 - 1}(t-\tau)} - e^{-\omega_n \sqrt{\zeta^2 - 1}(t-\tau)} \right) d\tau$$

$$\Rightarrow x(t) = \frac{e^{-\zeta\omega_n}}{2m\omega_n \sqrt{\zeta^2 - 1}} \int_0^t F(\tau) e^{\zeta\omega_n \tau} \left(e^{\omega_n \sqrt{\zeta^2 - 1}(t-\tau)} - e^{-\omega_n \sqrt{\zeta^2 - 1}(t-\tau)} \right) d\tau$$

3.16 Calculate the response of an underdamped system to the excitation given in Figure P3.16.

Plot of a pulse input of the form $f(t) = F_0 \sin t$.

Figure P3.16

Solution:

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[F(\tau) e^{\zeta\omega_n \tau} \sin\omega_d (t - \tau) \right] d\tau$$

$$F(t) = F_0 \sin(t) \qquad t < \pi \quad \left(\text{From Figure P3.16}\right)$$

For $t \leq \pi$, $\qquad x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left(\sin\tau e^{\zeta\omega_n \tau} \sin\omega_d (t - \tau) \right) d\tau$

$$x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \times$$

$$\left[\frac{1}{2\left[1 + 2\omega_d + \omega_n^2\right]} \left\{ e^{\zeta\omega_n t} \left[\left(\omega_d - 1\right)\sin t - \zeta\omega_n \cos t\right] - \left(\omega_d - 1\right)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right.$$

$$\left. + \frac{1}{2\left[1 + 2\omega_d + \omega_n^2\right]} \left\{ e^{\zeta\omega_n t} \left[\left(\omega_d - 1\right)\sin t - \zeta\omega_n \cos t\right] + \left(\omega_d - 1\right)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right]$$

For $\tau > \pi,: \int_0^t f(\tau)h(t - \tau)d\tau = \int_0^\pi f(\tau)h(t - \tau)d\tau + \int_\pi^t (0)h(t - \tau)d\tau$

$$x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_0^{\pi} \left(\sin\tau e^{\zeta\omega_n\tau} \sin\omega_d(t-\tau) \right) d\tau$$

$$= \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \times$$

$$\left[\frac{1}{2\left[1 + 2\omega_d + \omega_n^2\right]} \left\{ \begin{array}{c} e^{\zeta\omega_n t}\left[(\omega_d - 1)\sin\left[\omega_d(t-\pi)\right] - \zeta\omega_n\cos\left[\omega_d(t-\pi)\right]\right] \\ -(\omega_d - 1)\sin\omega_d t - \zeta\omega_n\cos\omega_d t \end{array} \right\} \right.$$

$$\left. + \frac{1}{2\left[1 + 2\omega_d + \omega_n^2\right]} \left\{ \begin{array}{c} e^{\zeta\omega_n t}\left[(\omega_d + 1)\sin\left[\omega_d(t-\tau)\right] + \zeta\omega\cos\left[\omega_d(t-\pi)\right]\right] \\ +(\omega_d - 1)\sin\omega_d t - \zeta\omega_n\cos\omega_d t \end{array} \right\} \right]$$

Alternately, one could take a Laplace Transform approach and assume the under-damped system is a mass-spring-damper system of the form

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

The forcing function given can be written as

$$F(t) = F_0\left(H(t) - H(t-\pi)\right)\sin(t)$$

Normalizing the equation of motion yields

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = f_0\left(H(t) - H(t-\pi)\right)\sin(t)$$

where $f_0 = \dfrac{F_0}{m}$ and m, c and k are such that $0 < \zeta < 1$.

Assuming initial conditions, transforming the equation of motion into the Laplace domain yields

$$X(s) = \frac{f_0\left(1 + e^{-\pi s}\right)}{\left(s^2 + 1\right)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

The above expression can be converted to partial fractions

$$X(s) = f_0\left(1 + e^{-\pi s}\right)\left(\frac{As + B}{s^2 + 1}\right) + f_0\left(1 + e^{-\pi s}\right)\left(\frac{Cs + D}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

where A, B, C, and D are found to be

$$A = \frac{-2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$B = \frac{\omega_n^2 - 1}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$C = \frac{2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$D = \frac{\left(1-\omega_n^2\right) + \left(2\zeta\omega_n\right)^2}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

Notice that $X(s)$ can be written more attractively as

$$X(s) = f_0\left(\frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) + f_0 e^{-\pi s}\left(\frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

$$= f_0\left(G(s) + e^{-\pi s}G(s)\right)$$

Performing the inverse Laplace Transform yields

$$x(t) = f_0\left(g(t) + H(t - \pi)g(t - \pi)\right)$$

where $g(t)$ is given below

$$g(t) = A\cos(t) + B\sin(t) + Ce^{-\zeta\omega_n t}\cos(\omega_d t) + \left(\frac{D - C\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

ω_d is the damped natural frequency, $\omega_d = \omega_n\sqrt{1 - \zeta^2}$.

Let $m=1$ kg, $c=2$ kg/sec, $k=3$ N/m, and $F_0=2$ N. The system is solved numerically. Both exact and numerical solutions are plotted below
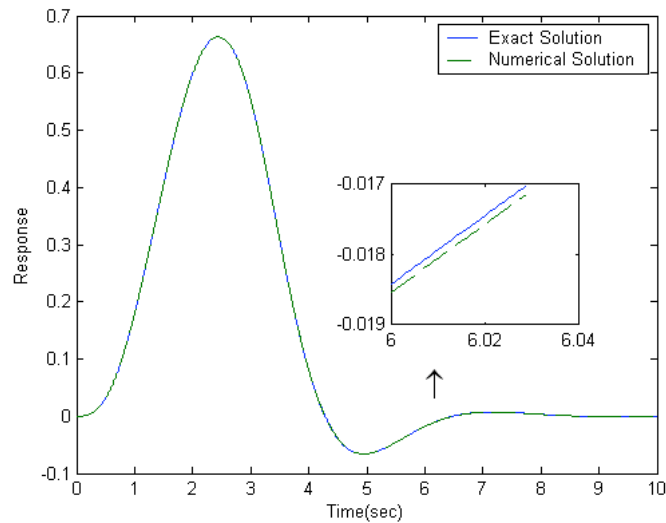
Figure 1 Analytical vs. Numerical Solutions

Below is the code used to solve this problem

```
% Establish a time vector
t=[0:0.001:10];

% Define the mass, spring stiffness and damping coefficient
m=1;
c=2;
k=3;

% Define the amplitude of the forcing function
F0=2;

% Calculate the natural frequency, damping ratio and normalized force amplitude
zeta=c/(2*sqrt(k*m));
wn=sqrt(k/m);
f0=F0/m;

% Calculate the damped natural frequency
wd=wn*sqrt(1-zeta^2);

% Below is the common denominator of A, B, C and D (partial fractions
% coefficients)
dummy=(1-wn^2)^2+(2*zeta*wn)^2;

% Hence, A, B, C, and D are given by
A=-2*zeta*wn/dummy;
B=(wn^2-1)/dummy;
C=2*zeta*wn/dummy;
```

```
D=((1-wn^2)+(2*zeta*wn)^2)/dummy;

% EXACT SOLUTION
%
************************************************************************
*
%
************************************************************************
*
for i=1:length(t)
   % Start by defining the function g(t)
   g(i)=A*cos(t(i))+B*sin(t(i))+C*exp(-zeta*wn*t(i))*cos(wd*t(i))+((D-
C*zeta*wn)/wd)*exp(-zeta*wn*t(i))*sin(wd*t(i));
   % Before t=pi, the response will be only g(t)
   if t(i)<pi
      xe(i)=f0*g(i);
      % d is the index of delay that will correspond to t=pi
      d=i;
   else
      % After t=pi, the response is g(t) plus a delayed g(t). The amount
      % of delay is pi seconds, and it is d increments
      xe(i)=f0*(g(i)+g(i-d));
   end;
end;

% NUMERICAL SOLUTION
%
************************************************************************
*
%
************************************************************************
*

% Start by defining the forcing function
for i=1:length(t)
   if t(i)<pi
      f(i)=f0*sin(t(i));
   else
      f(i)=0;
   end;
end;

% Define the transfer functions of the system
% This is given below
%        1
% --------------------------
```

```
% s^2+2*zeta*wn+wn^2

% Define the numerator and denominator
num=[1];
den=[1 2*zeta*wn wn^2];
% Establish the transfer function
sys=tf(num,den);

% Obtain the solution using lsim
xn=lsim(sys,f,t);

% Plot the results
figure;
set(gcf,'Color','White');
plot(t,xe,t,xn,'--');
xlabel('Time(sec)');
ylabel('Response');
legend('Forcing Function','Exact Solution','Numerical Solution');
text(6,0.05,'\uparrow','FontSize',18);
axes('Position',[0.55 0.3/0.8 0.25 0.25])
plot(t(6001:6030),xe(6001:6030),t(6001:6030),xn(6001:6030),'--');
```

3.17   Speed bumps are used to force drivers to slow down. Figure P3.17 is a model of a car going over a speed bump. Using the data from Example 2.4.1 and an undamped model of the suspension system ($k = 4 \times 10^5$ N/m, $m = 1007$ kg), find an expression for the maximum relative deflection of the car's mass versus the velocity of the car. Model the bump as a half sine of length 40 cm and height 20 cm. Note that this is a moving base problem.
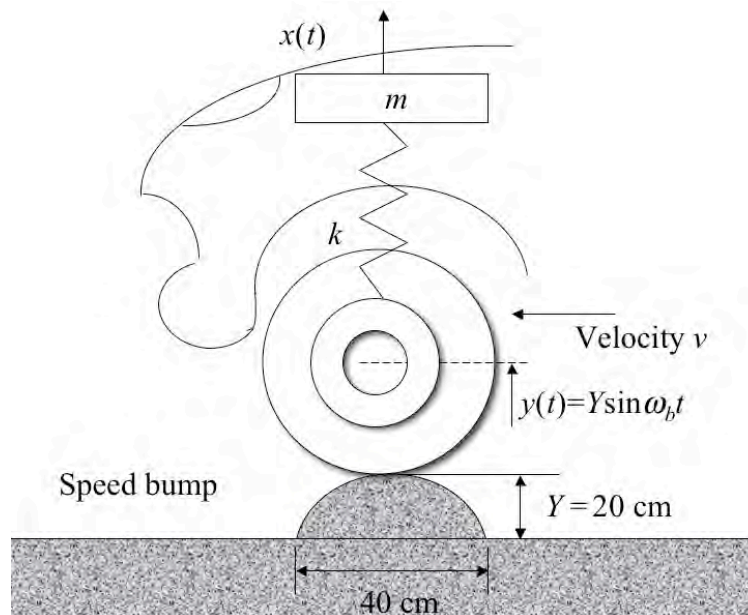
Figure P3.17 Model of a car driving over a speed bump.

Solution: This is a base motion problem, so the first step is to translate the equation of motion into a useable form. Summing forces yields in the vertical direction yields

$$m\ddot{x}(t) + k\left(x(t) - y(t)\right) = 0$$

were the displacement $y(t)$ is prescribed. Next defined the relative displacement to be $z(t) = x(t)-y(t)$, the relative motion between the car's wheel and body. The equation of motion becomes:

$$m\ddot{z}(t) + m\ddot{y}(t) + kz(t) = 0 \Rightarrow m\ddot{z}(t) + kz(t) = -m\ddot{y}(t)$$

Substitution of the form of $y(t)$ into this last expression yields:

$$m\ddot{z}(t) + kz(t) = mY\omega_b^2 \sin\omega_b t\left(\Phi(t) - \Phi(t - t_1)\right)$$

where Φ is the Heavyside step function and ω_b is the frequency associated with the bump. The relationship between the bump frequency and the car's constant velocity is

$$\omega_b = \frac{2\pi}{2\ell}v = \frac{\pi}{\ell}v$$

where v is the speed of the car in m/s. For constant velocity, the time $t_1 = v\ell$, when the car finishes going over the bump.

Here, $z(t)$ is From equation (3.13) with zero damping the solution is:

$$z(t) = \frac{1}{m\omega_n}\int_0^t f(t - \tau)\sin\omega_n\tau d\tau \qquad t < t_1$$

Substitution of $f(t) = y(t)$ yields:

$$z(t) = \frac{Y\omega_b^2}{\omega_n}\int_0^t \sin(\omega_b t - \omega_b\tau)\sin\omega_n\tau d\tau =$$

$$= \frac{Y\omega_b^2}{\omega_n}\frac{1}{2}\left[\frac{\sin\left(\omega_b t - (\omega_n + \omega_b)\tau\right)}{-(\omega_n + \omega_b)} - \frac{\sin\left(\omega_b t + (\omega_n - \omega_b)\tau\right)}{\omega_n - \omega_b}\right]_0^t$$

$$= \frac{Y\omega_b^2}{\omega_n}\frac{1}{\omega_n^2 - \omega_b^2}\left(\omega_n \sin\omega_b t - \omega_b \sin\omega_n t\right) \qquad t < t_1$$

where the integral has been evaluated symbolically. Clearly a resonance situation prevails. Consider two cases, high speed ($\omega_b \gg \omega_n$) and low speed ($(\omega_b \ll \omega_n)$) as when the two frequencies are near each other and obvious maximum occurs. For high speed, the amplitude can be approximated as

$$\frac{Y\omega_b^2}{\omega_n}\frac{\omega_b}{\omega_n^2 - \omega_b^2}\left((\omega_n / \omega_b)\sin\omega_b t - \sin\omega_n t\right) \approx \frac{Y\omega_b^2}{\omega_n}\frac{\omega_b}{\omega_n^2 - \omega_b^2}\sin\omega_n t$$

For the values given, this has magnitude:

$$\left|Z(v)\right| \approx \left| \frac{Y\left(\frac{\pi}{\ell}\right)^3 v^3}{\omega_n\left(\omega_n^2 - \omega_b^2\right)} \right|$$

This increases with the cube of the velocity. Thus the faster the car is going the more sever the bump is (larger relative amplitude of vibration), hence serving to slow motorist down. A plot of magnitude versus speed shows bump size is amplified by the suspension system.

$$k := 4 \cdot 10^5 \qquad m := 1007 \qquad \omega n := \sqrt{\frac{k}{m}} \qquad \omega n = 19.93$$

$$L := 0.4 \qquad Y := 0.2$$

$$Z(v) := \frac{Y \cdot \left(\frac{\pi}{L}\right)^2 \cdot v^2}{\omega n} \cdot \left[\frac{\left(\frac{\pi}{L}\right) \cdot v}{\left| \omega n^2 - \left(\frac{\pi}{L}\right)^2 \cdot v^2 \right|} \right]$$
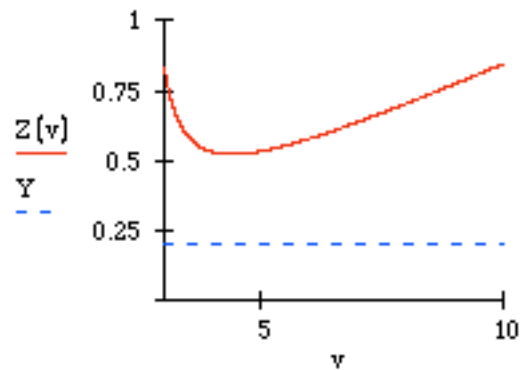
For slow speed, magnitude becomes

$$\left|Z(v)\right| \approx \left| \frac{Y\left(\frac{\pi}{\ell}\right)^2 v^2 \omega_n}{\omega_n\left(\omega_n^2 - \omega_b^2\right)} \right|$$

A plot of the approximate magnitude versus speed is given below

$$Z(v) := \frac{Y \cdot \left(\frac{\pi}{L}\right)^2 \cdot v^2}{\omega n} \cdot \frac{\omega n}{\left| \omega n^2 - \left(\frac{\pi}{L}\right)^2 \cdot v^2 \right|}$$
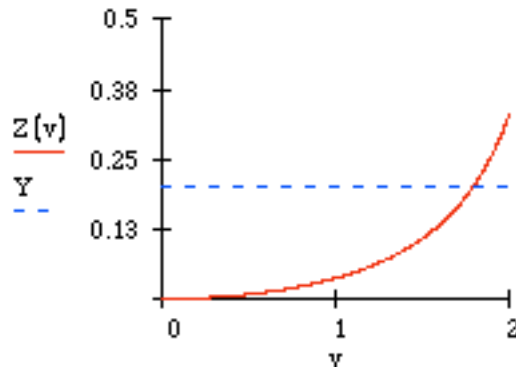
Clearly at speeds above the designed velocity there is strong amplification of the bump's magnitude, causing discomfort to the driver and passengers, encouraging a slow speed when passing over the bumb.

3.18 Calculate and plot the response of an undamped system to a step function with a finite rise time of t_1 for the case $m = 1$ kg, $k = 1$ N/m, $t_1 = 4$ s and $F_0 = 20$ N. This function is described by

$$F(t) = \begin{cases} \dfrac{F_0 t}{t_1} & 0 \le t \le t_1 \\ F_0 & t > t_1 \end{cases}$$

Solution: Working in Mathcad to perform the integrals the solution is:

3.14

$$m := 1 \qquad k := 1 \qquad F0 := 20 \qquad \omega n := \sqrt{\frac{k}{m}}$$

$$x1\,(t) := \frac{F0}{m \cdot \omega n \cdot 4} \cdot \left(\int_0^t \tau \cdot \sin(t - \tau)\, d\tau \right)$$

$$x1\,(t) := \frac{1}{4} \cdot \frac{F0}{(m \cdot \omega n)} \cdot (t - \sin(t))$$

$$x2\,(t) := \int_4^t \left[\frac{F0}{m \cdot \omega n} \cdot \left(1 - \frac{\tau}{4}\right) \cdot \sin(t - \tau) \right] d\tau$$

+

$$x2\,(t) := \frac{-1}{4} \cdot (t - 4) \cdot \frac{F0}{(m \cdot \omega n)} + \frac{1}{4} \cdot \sin(t - 4) \cdot \frac{F0}{(m \cdot \omega n)}$$

$$x\,(t) := x1\,(t) + x2\,(t) \cdot \Phi\,(t - 4)$$

$$\frac{1}{4} \cdot \frac{F0}{(m \cdot \omega n)} = 5 \qquad \frac{F0}{(m \cdot \omega n)} = 20$$
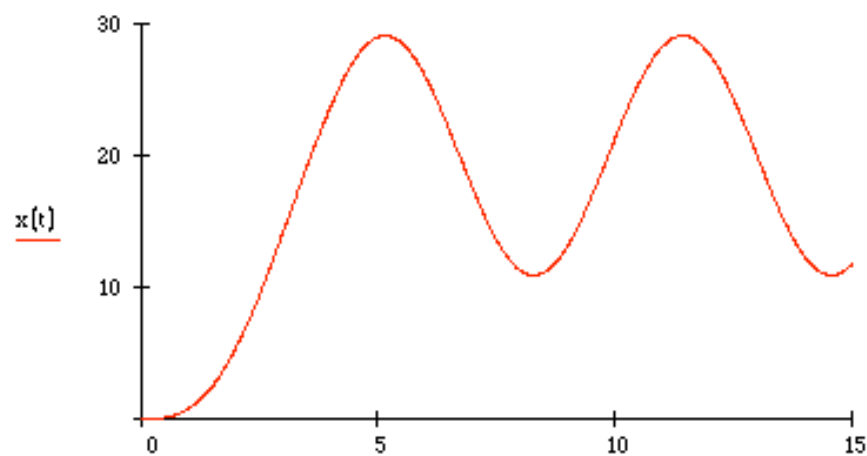
3.19 A wave consisting of the wake from a passing boat impacts a seawall. It is desired to calculate the resulting vibration. Figure P3.19 illustrates the situation and suggests a model. This force in Figure P3.19 can be expressed as

$$F(t) = \begin{cases} F_0\left(1 - \dfrac{t}{t_0}\right) & 0 \le t \le t_0 \\[2mm] 0 & t > t_0 \end{cases}$$

Calculate the response of the seal wall-dike system to such a load.

Physical setting — Model — Input model
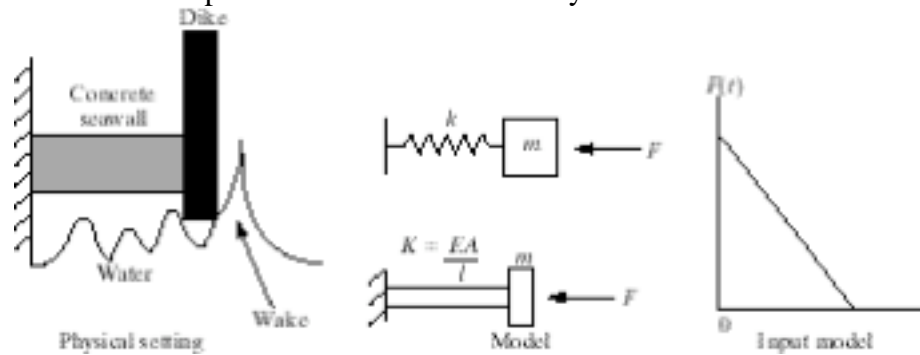
Solution: From Equation (3.12): $x(t) = \displaystyle\int_0^t F(\tau)h(t-\tau)\,d\tau$

From Problem 3.18, $h(t-\tau) = \dfrac{1}{m\omega_n}\sin\omega_n(t-\tau)$ for an undamped system

For $t < t_0$:

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^t F_0\left(1 - \frac{\tau}{t_0}\right)\sin\omega_n(t-\tau)\,d\tau\right]$$

$$x(t) = \frac{F_0}{m\omega_n}\left[\int_0^t \sin\omega_n(t-\tau)\,d\tau - \frac{1}{t_0}\int_0^t \tau\sin\omega_n(t-\tau)\,d\tau\right]$$

After integrating and rearranging,

$$x(t) = \frac{F_0}{kt_0}\left[\frac{1}{\omega_n}\sin\omega_n t - t\right] + \frac{F_0}{k}\left[1 - \cos\omega_n t\right] \quad t < t_0$$

For $t > t_0$: $\displaystyle\int_0^t f(\tau)h(t-\tau)\,d\tau = \int_0^{t_0} f(\tau)h(t-\tau)\,d\tau + \int_{t_0}^t (0)h(t-\tau)\,d\tau$

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^{t_0} F_0\left(1 - \frac{\tau}{t_0}\right)\sin\omega_n(t-\tau)\,d\tau\right]$$

$$x(t) = \frac{F_0}{m\omega_n}\left[\int_0^{t_0} \sin\omega_n(t-\tau)\,d\tau - \frac{1}{t_0}\int_0^{t_0} \tau\sin\omega_n(t-\tau)\,d\tau\right]$$

After integrating and rearranging,

$$x(t) = \frac{F_0}{kt_0\omega_n}\left[\sin\omega_n t - \sin\omega_n(t-t_0)\right] - \frac{F_0}{k}\left[\cos\omega_n t\right] \quad t > t_0$$

3.20 Determine the response of an undamped system to a ramp input of the form $F(t) = F_0 t$, where F_0 is a constant. Plot the response for three periods for the case $m = 1$ kg, $k = 100$ N/m and $F_0 = 50$ N.

Solution: From Eq. (3.12): $x(t) = \int_0^t F(\tau)h(t-\tau)d\tau$

From Problem 3.8, $h(t-\tau) = \frac{1}{m\omega_n}\sin\omega_n(t-\tau)$ for an undamped system.

Therefore,

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^t (F_0\tau)\sin\omega_n(t-\tau)d\tau\right] = \frac{F_0}{m\omega_n}\int_0^t \tau\sin\omega_n(t-\tau)d\tau$$

After integrating and rearranging,

$$x(t) = \frac{F_0}{m\omega_n}\left[\frac{\tau}{\omega_n} - \frac{1}{\omega_n^2}\sin\omega_n\tau\right] = \frac{F_0}{k}t - \frac{F_0}{k\omega_n}\sin\omega_n t$$

Using the values $m = 1$ kg, $k = 100$ kg, and $F_0 = 50$ N yields

$$x(t) = 0.5t - .05\sin(10t) \quad \text{m}$$

3. 21 A machine resting on an elastic support can be modeled as a single-degree-of-freedom, spring-mass system arranged in the vertical direction. The ground is subject to a motion $y(t)$ of the form illustrated in Figure P3.221. The machine has a mass of 5000 kg and the support has stiffness 1.5×10^3 N/m. Calculate the resulting vibration of the machine.
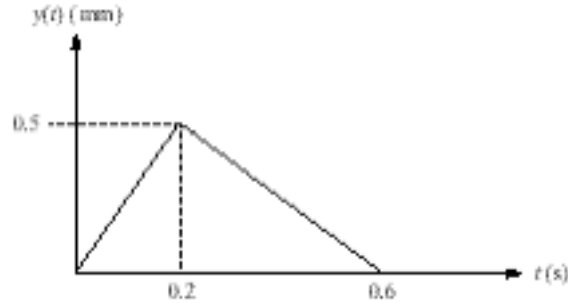
Solution: Given $m = 5000$ kg, $k = 1.5 \times 10^3$ N/m, $\omega_n = \sqrt{k/m} = 0.548$ rad/s and that the ground motion is given by:

$$y(t) = \begin{cases} 2.5t & 0 \le t \le 0.2 \\ 0.75 - 1.25t & 0.2 \le t \le 0.6 \\ 0 & t \ge 0.6 \end{cases}$$

The equation of motion is $m\ddot{x} + k(x - y) = 0$ or $m\ddot{x} + kx = ky = F(t)$ The impulse response function computed from equation (3.12) for an undamped system is

$$h(t - \tau) = \frac{1}{m\omega_n} \sin\omega_n(t - \tau)$$

This gives the solution by integrating a yh across each time step:

$$x(t) = \frac{1}{m\omega_n} \int_0^t ky(\tau)\sin\omega_n(t - \tau)d\tau = \omega_n \int_0^t y(\tau)\sin\omega_n(t - \tau)d\tau$$

For the interval $0 \le t \le 0.2$:

$$x(t) = \omega_n \int_0^t 2.5\tau \sin\omega_n(t - \tau)d\tau$$

$$\Rightarrow \underline{x(t) = 2.5t - 4.56\sin0.548t \ \ \text{mm} \ \ 0 \le t \le 0.2}$$

For the interval $0.2 \le t \le 0.6$:

$$x(t) = \omega_n \int_0^{0.2} 2.5\tau \sin\omega_n(t - \tau)d\tau + \omega_n \int_{0.2}^t (0.75 - 1.25\tau)\sin\omega_n(t - \tau)d\tau$$

$$= 0.75 - 0.5\cos0.548(t - 0.2) - 1.25t + 2.28\sin0.548(t - 0.2)$$

Combining this with the solution from the first interval yields:

$\underline{x(t) = 0.75 + 1.25t - 0.5\cos0.548(t - 0.2)}$

$\underline{+6.48\sin0.548(t - 0.2) - 4.56\sin0.548(t - 0.2) \ \ \text{mm} \ 0.2 \le t \le 0.6}$

Finally for the interval $t \ge 0.6$:

$$x(t) = \omega_n \int_0^{0.2} 2.5t \sin\omega_n(t-\tau)d\tau + \omega_n \int_{0.2}^{0.6} (0.75 - 1.25t)\sin\omega_n(t-\tau)d\tau + \omega_n \int_0^t (0)\sin\omega_n(t-\tau)d\tau$$

$$= -0.5\cos0.548(t-0.2) - 2.28\sin0.548(t-0.6) + 2.28\sin0.548(t-0.2)$$

Combining this with the total solution from the previous time interval yields:

$$x(t) = -0.5\cos0.548(t-0.2) + 6.84\sin0.548(t-0.2) - 2.28\sin0.548(t-0.6)$$

$$- 4.56\sin0.548t \quad \text{mm} \quad t \geq 0.6$$

3.22 Consider the step response described in Figure 3.7. Calculate t_p by noting that it occurs at the first peak, or critical point, of the curve.

Solution: Assume $t_0 = 0$. The response is given by Eq. (3.17):

$$x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t - \phi)$$

To find t_p, compute the derivative and let $\dot{x}(t) = 0$

$$\dot{x}(t) = \frac{-F_0}{k\sqrt{1-\zeta^2}} \left[-\zeta\omega_n e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) + e^{-\zeta\omega_n t}(-\omega_d)\sin(\omega_d t - \phi) \right] = 0$$

$$\Rightarrow -\zeta\omega_n \cos(\omega_d t - \phi) - \omega_d \sin(\omega_d t - \phi) = 0$$

$$\Rightarrow \tan(\omega_d t - \phi) = \frac{-\zeta\omega_n}{\omega_d}$$

$\omega_d t - \phi - \pi = \tan^{-1}\left(\dfrac{-\zeta\omega_n}{\omega_d} \right)$ (π can be added or subtracted without changing the

tangent of an angle)

$$t = \frac{1}{\omega_d} \left[\pi + \phi + \tan^{-1}\left(\frac{-\zeta\omega_n}{\omega_d} \right) \right]$$

But, $\phi = \tan^{-1}\left(\dfrac{\zeta}{\sqrt{1-\zeta^2}} \right)$

So,

$$t = \frac{1}{\omega_d} \left[\pi + \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}} \right) - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}} \right) \right]$$

$$t_p = \frac{\pi}{\omega_d}$$

3.23 Calculate the value of the overshoot (o.s.), for the system of Figure P3.7.

Solution:

The overshoot occurs at $t_p = \dfrac{\pi}{\omega_d}$

Substitute into Eq. (3.17):

$$x\left(t_p\right) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n\pi/\omega_d} \cos\left[\omega_d\left(\frac{\pi}{\omega_d}\right) - \theta\right]$$

The overshoot is

$$o.s. = x\left(t_p\right) - x_{ss}\left(t\right)$$

$$o.s. = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n\pi/\omega_d}\left(-\cos\theta\right) - \frac{F_0}{k}$$

Since $\theta = \tan^{-1}\left(\dfrac{\zeta}{\sqrt{1-\zeta^2}}\right)$, then $\cos\theta = \sqrt{1-\zeta^2}$

$$o.s. = -\frac{F_0}{k\sqrt{1-\zeta^2}}\left(e^{-\zeta\omega_n\pi/\omega_d}\right)\left(\sqrt{1-\zeta^2}\right)$$

$$o.s. = \frac{F_0}{k} e^{-\zeta\omega_n\pi/\omega_d}$$

3.24 It is desired to design a system so that its step response has a settling time of 3 s and a time to peak of 1 s. Calculate the appropriate natural frequency and damping ratio to use in the design.

Solution:

Given $t_s = 3$s, $t_p = 1$s

Settling time:

$$t_s = \frac{3.5}{\zeta\omega_n} = 3 \text{ s} \Rightarrow \zeta\omega_n = \frac{3.5}{3} = 1.1667 \text{ rad/s}$$

Peak time:

$$t_p = \frac{\pi}{\omega_d} = 1 \text{ s} \qquad \Rightarrow \omega_d = \omega_n\sqrt{1-\zeta^2} = \pi \text{ rad/s}$$

$$\Rightarrow \omega_n\sqrt{1-\left(\frac{1.1667}{\omega_n}\right)^2} = \pi \Rightarrow \omega_n^2\left[1-\left(\frac{1.1667}{\omega_n}\right)^2\right] = \pi^2$$

$$\Rightarrow \omega_n^2\left[1-\frac{1.3611}{\omega_n^2}\right] = \pi^2 \Rightarrow \omega_n^2 - 1.311 = \pi^2 \Rightarrow \underline{\omega_n = 3.35 \text{ rad/s}}$$

Next use the settling time relationship to get the damping ratio:

$$\zeta = \frac{1.1667}{\omega_n} = \frac{1.1667}{3.35} \Rightarrow \underline{\zeta = \mathbf{0.3483}}$$

3- 35

3.25 Plot the response of a spring-mass-damper system for this input of Figure 3.8 for the case that the pulse width is the natural period of the system (i.e., $t_1 = \pi/\omega_n$).

Solution:

The values from Figure 3.7 will be used to plot the response.

$$F_0 = 30 \text{ N}$$
$$k = 1000 \text{ N/m}$$
$$\zeta = 0.1$$
$$\omega = 3.16 \text{ rad/s}$$

From example 3.2.2 and Figure 3.7, with $t_1 = \dfrac{\pi}{\omega}$ we have for $t = 0$ to t_1,

$$x(t) = \frac{F_0}{k} - \frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi) \qquad \text{where } \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

$x(t) = .03 - .03015e^{-.316t}\cos(3.144t - .1002) \qquad 0 < t \le t_1$

For $t > t_1$,

$$x(t) = \frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\left\{ e^{\zeta\omega_n t_1}\cos\left[\omega_d\left(t - \frac{\pi}{\omega_n}\right) - \phi\right] - \cos(\omega_d t - \phi)\right\}$$

$x(t) = 0.0315e^{-.316t}\{1.3691\cos(3.144t - 3.026) - \cos(3.144t - .1002)\}\ t > t_1$

The plot in Mathcad follows:

$\omega := 3.144$

$x(t) := 0.03 - 0.301 \cdot e^{-.316 \cdot t} \cdot \cos(\omega \cdot t - .1002) + \left[(0.0315) \cdot e^{-.316 \cdot t} \cdot \left(1.3691 \cdot \cos(\omega \cdot t - 3.226) - \cos(\omega \cdot t - .1002)\right)\right] \cdot \Phi\left(t - \frac{\pi}{\omega}\right)$
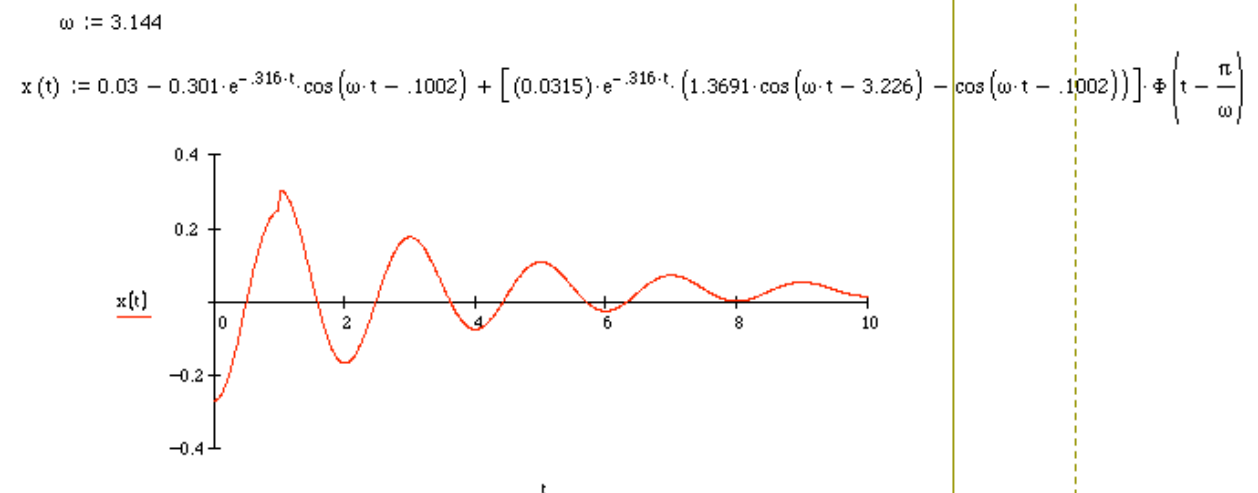
Problems and Solutions Section 3.3 (problems 3.26-3.32)

3.26 Derive equations (3.24). (3.25) and (3.26) and hence verify the equations for the Fourier coefficient given by equations (3.21), (3.22) and (3.23).

Solution: For $n \neq m$, integration yields:

$$\int_0^T \sin n\omega_T t \sin m\omega_T t \, dt = \left[\frac{\sin(n-m)\omega_T t}{\omega_T 2(n-m)} - \frac{\sin(n+m)\omega_T t}{\omega_T 2(n+m)} \right]_0^T$$

$$= \frac{\sin\left[(n-m)\left(\frac{2\pi}{T}\right)T \right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)\left(\frac{2\pi}{T}\right)T \right]}{2(n+m)\omega_T}$$

$$= \frac{\sin\left[(n-m)(2\pi) \right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)(2\pi) \right]}{2(n+m)\omega_T} = 0$$

Since m and n are integers, the sine terms are 0, so this is equal to 0.

Equation (3.24), for $m = n$:

$$\int_0^T \sin^2 n\omega_T t \, dt = \left[\frac{1}{2}t - \frac{1}{4n\omega_T}\sin(2n\omega_T t) \right]_0^T = \frac{T}{2} - \frac{T}{8n\pi}\sin\left[2\pi\left(\frac{2\pi}{T}\right)T \right]$$

$$= \frac{T}{2} - \frac{T}{8n\pi}\sin\left[4n\pi \right] = \frac{T}{2}$$

Since n is an integer, the sine term is 0, so this is equal to $T/2$.

So, $\displaystyle\int_0^T \sin n\omega_T t \sin m\omega_T t \, dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$

Equation (3.25), for $m \neq n$

$$\int_0^T \cos n\omega_T t \cos m\omega_T t\, dt = \left[\frac{\sin\left(n-m\right)\omega_T t}{2\left(n-m\right)\omega_T} - \frac{\sin\left(n+m\right)\omega_T t}{2\left(\eta+m\right)\omega_T} \right]_0^T$$

$$= \frac{\sin\left[\left(n-m\right)\left(\frac{2\pi}{T}\right)T\right]}{2\left(n-m\right)\omega_T} - \frac{\sin\left[\left(n+m\right)\left(\frac{2\pi}{T}\right)T\right]}{2\left(n+m\right)\omega_T}$$

$$= \frac{\sin\left[\left(n-m\right)\left(2\pi\right)\right]}{2\left(n-m\right)\omega_T} - \frac{\sin\left[\left(n+m\right)\left(2\pi\right)\right]}{2\left(n+m\right)\omega_T} = 0$$

Since m and n are integers, the sine terms are 0, so this is equal to 0.

Equation (3.25), for $m = n$ becomes:

$$\int_0^T \cos^2 n\omega_T t\, dt = \left[\frac{1}{2}t + \frac{1}{4n\omega_T}\sin\left(2n\omega_T t\right) \right]_0^T = \frac{T}{2} + \frac{T}{8n\pi}\sin\left[2n\left(\frac{2\pi}{T}\right)T\right]$$

$$= \frac{T}{2} + \frac{T}{8n\pi}\sin\left[4n\pi\right] = \frac{T}{2}$$

Since n is an integer, the sine term is 0, so this is equal to $T/2$.

So, $$\int_0^T \cos n\omega_T t \cos m\omega_T t\, dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

Equation (3.26), for $m \neq n$:

$$\int_0^T \cos n\omega_T t \sin m\omega_T t\, dt = \left[\frac{\cos\left(n-m\right)\omega_T t}{2\omega_T\left(n-m\right)} - \frac{\cos\left(n+m\right)\omega_T t}{2\omega_T\left(n+m\right)} \right]_0^T$$

$$= \frac{\cos\left[\left(n-m\right)\left(\frac{2\pi}{T}\right)T\right]}{2\left(n-m\right)\omega_T} - \frac{\cos\left[\left(n+m\right)\left(\frac{2\pi}{T}\right)T\right]}{2\left(n+m\right)\omega_T} - \frac{1}{2\left(m-n\right)\omega_T} + \frac{1}{2\left(m+n\right)\omega_T}$$

$$= \frac{\cos\left[\left(n-m\right)\left(2\pi\right)\right]}{2\left(n-m\right)\omega_T} - \frac{\cos\left[\left(n+m\right)\left(2\pi\right)\right]}{2\left(n+m\right)\omega_T} - \frac{1}{2\left(m-n\right)\omega_T} + \frac{1}{2\left(m+n\right)\omega_T} = 0$$

Since n is an integer, the cosine term is 1, so this is equal to 0.

So, $\displaystyle\int_0^T \cos n\omega_T t \sin m\omega_T t\,dt = 0$

Equation (3.26) for $n = m$ becomes:

$$\int_0^T \cos n\omega_T t \sin n\omega_T t\,dt = \left[\frac{1}{2n\omega_T}\sin^2 n\omega_T t\right]_0^T = \frac{T}{4n\pi}\sin^2 2\pi n = 0$$

$$\text{Thus } \int_0^T \cos n\omega_T t \sin n\omega_T t\,dt = 0$$

3.27 Calculate b_n from Example 3.3.1 and show that $b_n = 0$, $n = 1,2,\ldots,\infty$ for the triangular force of Figure 3.12. Also verify the expression a_n by completing the integration indicated. (*Hint*: Change the variable of integration from t to $x = 2\pi nt/T$.)

Solution: From Equation (3.23), $b_n = \dfrac{2}{T}\displaystyle\int_0^T F(t)\sin n\omega_T t\,dt$. Computing the integral yields:

$$b_n = \frac{2}{T}\left[\int_0^{T/2}\left(\frac{4}{T}t-1\right)\sin n\omega_T t\,dt + \int_{T/2}^T\left[1-\frac{4}{T}\left(t-\frac{T}{2}\right)\right]\sin n\omega_T t\,dt\right]$$

$$b_n = \frac{2}{T}\left[\frac{4}{T}\int_0^{T/2} t\sin n\omega_T t\,dt - \int_0^{T/2}\sin n\omega_T t\,dt + 3\int_{T/2}^T \sin n\omega_T t\,dt - \frac{4}{T}\int_{T/2}^T t\sin n\omega_T t\,dt\right]$$

Substitute $x = n\omega_T t = \dfrac{2\pi n}{T}t$

$$b_n = \frac{1}{\pi n}\left[\frac{2}{\pi n}\int_0^{\pi n} x\sin x\,dx - \int_0^{\pi n}\sin x\,dx + 3\int_{\pi n}^{2\pi n}\sin x\,dx - \frac{2}{\pi n}\int_{\pi n}^{2\pi n} x\sin x\,dx\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}(\sin x - x\cos x)\Big|_0^{\pi n} + \cos x\Big|_0^{\pi n} - 3\cos x\Big|_{\pi n}^{2\pi n} - \frac{2}{\pi n}(\sin x - x\cos x)\Big|_{\pi n}^{2\pi n}\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}(-\pi n\cos\pi n) + \cos\pi n - 1 - 3 + 3\cos\pi n - \frac{2}{\pi n}(-2\pi n + \pi n\cos\pi n)\right]$$

$$= \frac{1}{\pi n}\left[-2\cos\pi n + 4\cos\pi n - 4 + 4 - 2\cos\pi n\right] = \frac{1}{\pi n}[0] = 0$$

From equation (3.22), $a_n = \dfrac{2}{T}\displaystyle\int_0^T F(t)\cos n\omega_T t\,dt$

$$a_n = \frac{2}{T}\left[\int_0^{T/2}\left(\frac{4}{T}t-1\right)\cos n\omega_T t\,dt + \int_{T/2}^T\left[1-\frac{4}{T}\left(t-\frac{T}{2}\right)\right]\cos n\omega_T t\,dt\right]$$

$$a_n = \frac{2}{T}\left[\frac{4}{T}\int_0^{T/2} t\cos n\omega_T t\,dt - \int_0^{T/2}\cos n\omega_T t\,dt + 3\int_{T/2}^T \cos n\omega_T t\,dt - \frac{4}{T}\int_{T/2}^T t\cos n\omega_T t\,dt\right]$$

Substitute $x = n\omega_T t = \dfrac{2\pi n}{T}t$

$$a_n = \frac{1}{\pi n}\left[\frac{2}{\pi n}\int_0^{\pi n} x\cos x\,dx - \int_0^{\pi n} \cos x\,dx + 3\int_{\pi n}^{2\pi n} \cos x\,dx - \frac{2}{\pi n}\int_{\pi n}^{2\pi n} x\cos x\,dx\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}\big(\cos x + x\sin x\big)\Big|_0^{\pi n} - \sin x\Big|_0^{\pi n} + 3\sin x\Big|_{\pi n}^{2\pi n} - \frac{2}{\pi n}\big(\cos x - \sin x\big)\Big|_{\pi n}^{2\pi n}\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}\big(\cos \pi n - 1\big) - \frac{2}{\pi n}\big(1 - \cos \pi n\big)\right]$$

$$= \frac{2}{\pi^2 n^2}\Big[\cos \pi n - 1 - 1 + \cos \pi n\Big]$$

$$= \frac{4}{\pi^2 n^2}\Big[\cos \pi n - 1\Big] = \begin{cases} 0 & n \text{ even} \\ \dfrac{-8}{\pi^2 n^2} & n \text{ odd} \end{cases}$$

3.28 Determine the Fourier series for the rectangular wave illustrated in Figure P3.28.
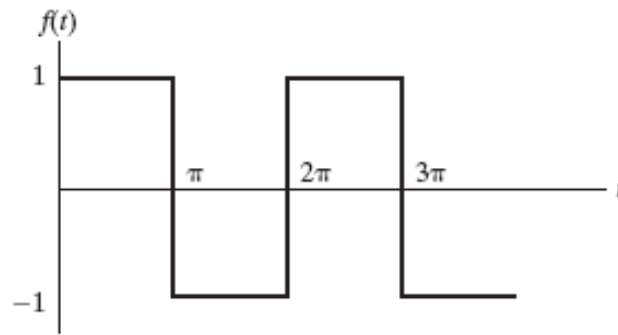
Solution: The square wave of period T is described by

$$F(t) = \begin{cases} 1 & 0 \le t \le \pi \\ -1 & \pi \le t \le 2\pi \end{cases}$$

Determine the coefficients a_0, a_n, b_n from direct integration:

$$a_0 = \frac{2}{T} \int_0^T F(t)\, dt$$

$$= \frac{2}{2\pi} \left[\int_0^\pi (1)\, dt + \int_\pi^{2\pi} (-1)\, dt \right]$$

$$= \frac{1}{\pi} \left[t \Big|_0^\pi - t \Big|_\pi^{2\pi} dt \right]$$

$$= \frac{1}{\pi} \left[\pi - 2\pi + \pi \right] = \frac{1}{\pi}(0) \qquad \Rightarrow a_0 = 0$$

$$a_n = \frac{2}{T} \int_0^T F(t) \cos n\omega_T t\, dt, \text{ where } \omega_T = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$= \frac{2}{2\pi} \left[\int_0^\pi \cos ntdt - \int_\pi^{2\pi} \cos ntdt \right] = \frac{1}{\pi} \left[\frac{1}{n} \sin nt \Big|_0^\pi - \frac{1}{n} \sin nt \Big|_\pi^{2\pi} \right]$$

$$= \frac{1}{\pi n} \left[\sin(n\pi) - \sin(n2\pi) + \sin(n\pi) \right] = 0$$

$$b_n = \frac{2}{T} \int_0^T F(t) \sin \omega_T t\, dt = \frac{2}{2\pi} \left[\int_0^\pi \sin ntdt - \int_\pi^{2\pi} \sin ntdt \right]$$

$$= \frac{1}{\pi} \left[\frac{-1}{n} \cos nt \Big|_0^\pi - \frac{1}{n} \cos nt \Big|_\pi^{2\pi} \right] = \frac{1}{\pi n} \left[-\cos n\pi + 1 - 1 - \cos n\pi \right] = \frac{2}{\pi n} \left[1 - \cos n\pi \right]$$

If n is even, $\cos n\pi = 1$. If n is odd, $\cos n\pi = -1$

So, $b_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{\pi n} & n \text{ odd} \end{cases}$

Thus the Fourier Series collapses to a sine series of the form

$$F\left(t\right) = \sum_{n=1}^{\infty} b_n \sin nt = \sum_{n=1,3,\ldots}^{\infty} \frac{4}{n\pi} \sin nt$$

The Vibration Toolbox can also be used:
t=0:pi/100:2*pi-pi/100;
f=-2*floor(t/pi)+1;
vtb3_3(f',t',100)
[a,b]=vtb3_3(f',t',100)

Note that vtb3_3 always gives some error on the order of delta t (.01 in this case). Using a smaller delta t reduced the error.

3.29 Determine the Fourier series representation of the sawtooth curve illustrated in Figure P3.29.
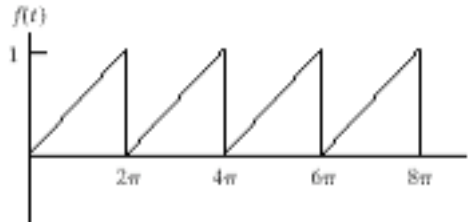
Solution: The sawtooth curve of period T is

$$F(t) = \frac{1}{2\pi}t \qquad 0 \le t \le 2\pi$$

Determine coefficients a_0, a_n, b_n :

$$a_0 = \frac{2}{T}\int_0^T F(t)\,dt = \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)dt = \left(\frac{1}{2\pi^2}\right)\frac{1}{2}t^2\bigg|_0^{2\pi}$$

$$= \frac{1}{4\pi^2}\left[4\pi^2 - 0\right] = 1$$

$$a_n = \frac{2}{T}\int_0^T F(t)\cos n\omega_T t\,dt, \text{ where } \omega_T = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$= \frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\cos nt\,dt\right] = \frac{1}{2\pi^2}\left[\int_0^{2\pi} t\cos nt\,dt\right]$$

$$= \frac{1}{2\pi^2}\left[\frac{1}{n^2}\cos nt + \frac{1}{n}t\sin nt\right]_0^{2\pi} = \frac{1}{2\pi^2}\left[\frac{1}{n^2}(1-1) + \frac{1}{n}(0-0)\right] = 0$$

$$b_n = \frac{2}{T}\int_0^T F(t)\sin n\omega_T t\,dt = \frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\sin nt\,dt\right] = \frac{1}{2\pi^2}\left[\int_0^{2\pi} t\sin nt\,dt\right]$$

$$= \frac{1}{2\pi^2}\left[\frac{1}{n^2}\sin nt - \frac{1}{n}t\cos nt\right]_0^{2\pi} = \frac{1}{2\pi^2}\left[\frac{1}{n^2}(0-0) - \frac{1}{n}(2\pi-0)\right]$$

$$= \frac{1}{2\pi^2}\left(\frac{-2\pi}{n}\right) = \frac{-1}{\pi n}$$

Fourier Series

$$F(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{-1}{\pi n}\right)\sin nt$$

$$F(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin nt$$

3.30 Calculate and plot the response of the base excitation problem with base motion specified by the velocity

$$\dot{y}(t) = 3e^{-t/2}\Phi(t) \text{ m/s}$$

where $\Phi(t)$ is the unit step function and $m = 10$ kg, $\zeta = 0.01$, and $k = 1000$ N/m. Assume that the initial conditions are both zero.

Solution: Given:

$$\dot{y}(t) = 3e^{-t/2}\mu(t) \text{ m/s}$$
$$m = 10 \text{ kg}, \ \zeta = 0.01, \ k = 1000 \text{ N/m}$$
$$x(0) = \dot{x}(0) = 0$$

From Equation (2.61):

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$
$$m\ddot{x} + c\ddot{x} + kx = c\dot{y} + ky$$

Integrate by parts to find $y(t)$:

$$y(t) = \int \dot{y}(t)\,dt = 3e^{-t/2}\mu(t)\,dt$$

Let

$$u = \mu(t) \qquad dv = 3e^{-t/2}\,dt$$
$$du = \delta(t)\,dt \qquad v = -6e^{-t/2}$$

When

$$t > 0, \mu(t) = 1, \ \text{ so } \ y(t) = 6\left(1 - e^{-1/2}\right)$$

So, $m\ddot{x} + c\dot{x} + kx = c\left(3e^{-t/2}\right) + 6k\left(1 - e^{t/2}\right)$

Since $c = 2\zeta\sqrt{km} = 2$ kg/s,

$$10\ddot{x} + 2\dot{x} + 1000x = 6000 - 5994e^{-t/2}$$

The solution is given by equation (3.13):

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[F(\tau) e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right] d\tau$$

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 10 \text{ rad/s}$$

$$F(t) = 6000 - 5994 e^{-t/2}$$

$$x(t) = \frac{1}{100} e^{-0.1t} \int_0^t \left[\left(6000 - 5994 e^{-\tau/2} \right) e^{0.1\tau} \sin\left(10(t-\tau)\right) \right] d\tau$$

$$x(t) = 60 e^{-0.1t} \left\{ \int_0^t e^{0.1\tau} \sin\left[10(t-\tau)\right] d\tau - \int_0^t e^{-0.4\tau} \sin\left[10(t-\tau)\right] d\tau \right\}$$

After integrating and rearranging

$$x(t) = 6 - 5.979 e^{-t/2} - 0.0295\cos 10t - 0.2990\sin 10t \text{ m}$$
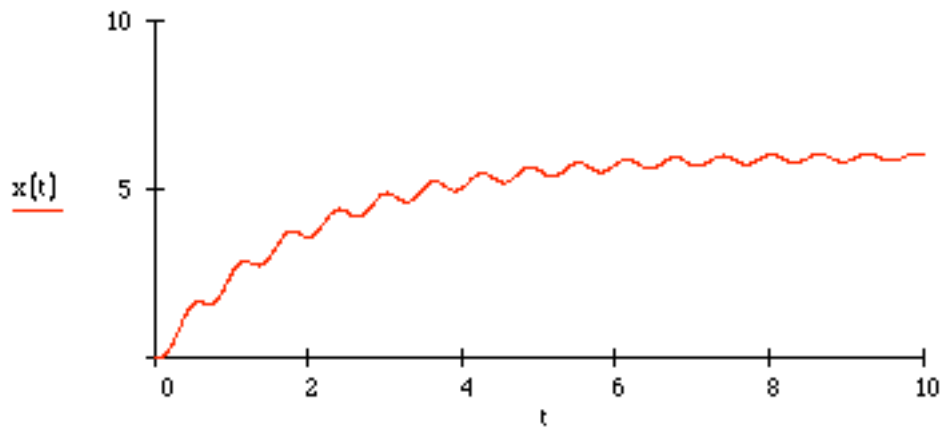
3.31 Calculate and plot the total response of the spring-mass-damper system of Figure 2.1 with $m = 100$ kg, $\zeta = 0.1$ and $k = 1000$ N/m to the signal of Figure 3.12, with maximum force of 1 N. Assume that the initial conditions are zero and let $T = 2\pi$ s.

Solution: Given:

$m = 100$ kg, $k = 1000$ N/m, $\zeta = 0.1$, $T = 2\pi$ s, $F_{max} = 1N$,

$$x(0) = \dot{x}(0) = 0, \quad \omega_n = \sqrt{\frac{k}{m}} = 3.16 \text{ rad/s}, \quad \omega_d = \omega\sqrt{1 - \zeta^2} = 3.15 \text{ rad/s}, \quad \omega_T = \frac{2\pi}{T} = 1 \text{ rad/s}$$

From example 3.3.1 and Figure 3.10,

$$F(t) \sum_{n=1}^{\infty} a_n \cos nt, \qquad a_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{-8}{\pi^2 n^2} & n \text{ odd} \end{cases}$$

So, $m\ddot{x} + c\dot{x} + kx = \displaystyle\sum_{n=1}^{\infty} a_n \cos nt \qquad (n \text{ odd})$

The total solution is

$$x(t) = x_h(t) + \sum_{n=1}^{\infty} x_{cn}(t) \qquad (n \text{ odd})$$

From equation (3.33),

$$x_{cn}(t) = \frac{a_n / m}{\left[\left[\omega_n^2 - (n\omega_T)^2\right]^2 + \left[2\zeta\omega_n n\omega_T\right]^2\right]^{1/2}} \cos\left(n\omega_T t - \phi_n\right)$$

$$\phi_n = \tan^{-1}\left(\frac{2\zeta\omega_n n\omega_T}{\omega_n^2 - n^2\omega_T^2}\right) = \tan^{-1}(\frac{0.6325n}{10 - n^2})$$

$$x_{cn}(t) = \frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}} \cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]$$

So,

$$x(t) = Ae^{\zeta\omega_n t}\sin(\omega_d t - \theta) + \sum_{n=1}^{\infty}\left[\frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]\right] \quad (n \text{ odd})$$

$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t - \theta)$$

$$+ \omega_d Ae^{-\zeta\omega_n t}\cos(\omega_d t - \theta) + \sum_{n=1}^{\infty}\left[\frac{0.00811}{n\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\sin nt - \tan^{-1}\frac{0.6325n}{10 - n^2}\right] \quad (n \text{ odd})$$

$$x(0) = 0 = -A\sin\theta + \sum_{n=1}^{\infty}\left[\frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]\right] \quad (n \text{ odd})$$

$$0 = -A\sin\theta - 0.00110$$

$$\dot{x}(0) = 0 = \zeta\omega_n A\sin\theta + \omega_d A\cos\theta$$

$$+ \sum_{n=1}^{\infty}\left[\frac{-0.000569}{\left[n^4 - 19.6n^2 + 100\right]^{1/2}\left[0.00493n^2 + 1\right]}\right] \quad (n \text{ odd})$$

$$0 = \zeta\omega_n A\sin\theta + \omega_d A\cos\theta - 0.001186$$

So $A = 0.00117$ m and $\theta = -1.232$ rad.

The total solution is:

$$x(t) = 0.00117e^{-0.316t}\sin(3.15t + 1.23)$$

$$+ \sum_{n=1}^{\infty}\left[\frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]\right] \text{ m} \quad (n \text{ odd})$$

3.32 Calculate the total response of the system of Example 3.3.2 for the case of a base motion driving frequency of $\omega_b = 3.162$ rad/s.

Solution: Let $\omega_b = 3.162$ rad/s. From Example 3.3.2,

$$F(t) = cY\omega_b \cos\omega_b t + kY \sin\omega_b t = 1.581\cos(3.162t) + 50\sin(3.162t)$$

Also,

$$\omega_n = \sqrt{\frac{k}{m}} = 31.62 \text{ rad/s and } \zeta = \frac{c}{2\sqrt{km}} = 0.158$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 31.22 \text{ rad/s}$$

The solution is

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \theta) + \omega_n Y\left[\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}\right]^{1/2}\cos(\omega_b t - \phi_1 - \phi_2)$$

$$x(t) = Ae^{-5t}\sin(31.22t + \theta) + 0.0505\cos(3.162t - \phi_1 - \phi_2)$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right) = 0.0319 \text{ rad}$$

$$\phi_2 = \tan^{-1}\left(\frac{\omega_n}{2\zeta\omega_b}\right) = 1.54 \text{ rad}$$

So,

$$x(t) = Ae^{-5t}\sin(31.22t + \theta) + 0.0505\cos(3.162t - 1.57)$$

$$\dot{x}(t) = -5Ae^{-5t}\sin(31.22t + \theta) + 31.22Ae^{-5t}\cos(31.22t + \theta) - 0.16\sin(3.162t - 1.57)$$

$$\Rightarrow x(0) = 0.01 = A\sin\theta + 0.0505(0)$$

$$\Rightarrow \dot{x}(0) = 3 - 5A\sin\theta + 31.22A\cos\theta + 0.16(1)$$

So, $A = 0.0932$ m and $\theta = 0.107$ rad

The total solution is

$$x(t) = 0.0932e^{-5t}\sin(31.22t + 0.107) + 0.0505\cos(3.162t - 1.57) \text{ m}$$

Problems and Solutions for Section 3.4 (3.35 through 3.38)

3.35 Calculate the response of
$$m\ddot{x} + c\dot{x} + kx = F_0\Phi(t)$$
where $\Phi(t)$ is the unit step function for the case with $x_0 = v_0 = 0$. Use the Laplace transform method and assume that the system is underdamped.

Solution:

Given:
$$m\ddot{x} + c\dot{x} + kx = F_0\mu(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\mu(t) \qquad (\zeta < 1)$$

Take Laplace Transform:

$$s^2 X(s) + 2\zeta\omega_n s X(s) + \omega_n^2 X(s) = \frac{F_0}{m}\left(\frac{1}{s}\right)$$

$$X(s) = \frac{F_0 / m}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)s} = \left(\frac{F_0}{m\omega_n^2}\right)\frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

Using inverse Laplace tables,

$$x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \cos^{-1}(\zeta)\right)$$

3.36 Using the Laplace transform method, calculate the response of the system of Example 3.4.4 for the overdamped case ($\zeta > 1$). Plot the response for $m = 1$ kg, k = 100 N/m, and $\zeta = 1.5$.

Solution:

From example 3.4.4,

$$m\ddot{x} + c\dot{x} + kx = \delta(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}\delta(t) \qquad (\zeta > 1)$$

Take Laplace Transform:

$$s^2 X(s) + 2\zeta\omega_n s X(s) + \omega_n^2 X(s) = \frac{1}{m}$$

$$X(s) = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1/m}{(s + a)(s + b)}$$

Using inverse Laplace tables, $a = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$, $b = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

$$x(t) = \frac{e^{-\zeta\omega_n t}}{2m\omega_n\sqrt{\zeta^2 - 1}}\left[e^{\omega_n\sqrt{\zeta^2 - 1}t} - e^{-\omega_n\sqrt{\zeta^2 - 1}t} \right]$$

Inserting the given values yields: $x(t) = \dfrac{e^{-15t}}{22.36}\left[e^{11.18t} - e^{-11.18t} \right]$ m

$$x(t) := \frac{e^{-15 \cdot t}}{22.36}\cdot\left(e^{11.18\cdot t} - e^{-11.18\cdot t}\right)$$
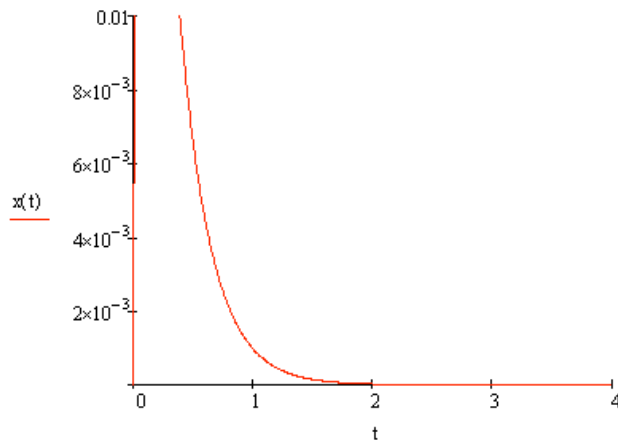
3.37   Calculate the response of the underdamped system given by

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{-at}$$

using the Laplace transform method. Assume $a > 0$ and that the initial conditions are all zero.

Solution:

Given:

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{-at} \qquad a > 0, \ \text{initial conditions} = 0$$

Rewrite:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m} e^{-at}$$

Take Laplace Transform:

$$s^2 X(s) + 2\zeta\omega_n s X(s) + \omega_n^2 X(s) = \frac{F_0}{m}\left(\frac{1}{s+a}\right)$$

$$X(s) = \frac{F_0 / m}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s+a)}$$

For an underdamped system, the inverse Laplace Transform is

$$x(t) = \left(\frac{F_0}{m\left(2\zeta\omega_n a - \omega_n^2 - a^2\right)}\right)\left\{ e^{-\zeta\omega_n t}\left[\frac{\zeta\omega_n - a}{\omega_d}\sin(\omega_d t) + \cos(\omega_d t)\right] - e^{-at}\right\}$$

3.38 Solve the following system for the response $x(t)$ using Laplace transforms:
$$100\ddot{x}(t) + 2000x(t) = 50\delta(t)$$
where the units are in Newtons and the initial conditions are both zero.

Solution:

First divide by the mass to get

$$\ddot{x} + 20x(t) = 0.5\delta(t)$$

Take the Laplace Transform to get

$$(s^2 + 20)X(s) = 0.5$$

So

$$X(s) = \frac{0.5}{s^2 + 20}$$

Taking the inverse Laplace Transform using entry 5 of Table 3.1 yields

$$X(s) = \frac{0.5}{\sqrt{20}} \cdot \frac{\omega}{s^2 + \omega^2} \quad \text{where } \omega = \sqrt{20}$$

$$\Rightarrow x(t) = \frac{1}{4\sqrt{5}} \sin\sqrt{20}t$$

Problems and Solutions Section 3.5 (3.39 through 3.42)

3.39   Calculate the mean-square response of a system to an input force of constant PSD, S_0, and frequency response function $H(\omega) = \dfrac{10}{\left(3 + 2\,j\omega\right)}$

Solution:

Given: $S_{ff} = S_0$ and $H(\omega) = \dfrac{10}{3 + 2\,j\omega}$

The mean square of the response can be found from Eqs (3.66) and (3.68):

$$\overline{x}^2 = E\left[x^2\right] = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{ff}(\omega)\,d\omega$$

$$\overline{x}^2 = S_0 \int_{-\infty}^{\infty} \left|\frac{10}{3 + 2\,j\omega}\right|^2 d\omega$$

Using Eq. (3.67) yields

$$\overline{x}^2 = \frac{50\pi S_0}{3}$$

3.40 Consider the base excitation problem of Section 2.4 as applied to an automobile model of Example 2.4.1 and illustrated in Figure 2.16. In this problem let the road have a random stationary cross section producing a PSD of S_0. Calculate the PSD of the response and the mean-square value of the response.

Solution: Given: $S_{ff} = S_0$

From example 2.4.1: $m = 1007$ kg, $c = 2000$ kg/s, $k = 40,000$ N/m

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{2000}{2\sqrt{40000 \cdot 1007}} = 0.157 \quad \text{(underdamped)}$$

So,

$$H(\omega) = \frac{1}{k - m\omega^2 + jc\omega} = \frac{1}{4 \times 10^4 - 1007\omega^2 + 2000\,j\omega}$$

$$\left|H(\omega)\right|^2 = \frac{1}{\left(4 \times 10^4 - 1007\omega^2\right)^2 + \left(2000\right)^2 j\omega^2}$$

$$\left|H(\omega)\right|^2 = \frac{1}{1.01 \times 10^6 \omega^4 - 4.06 \times 10^7 \omega^2 + 1.6 \times 10^9}$$

The PSD is found from equation (3.62):

$$S_{xx}(\omega) = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

$$S_{xx}(\omega) = \frac{1}{1.01 \times 10^6 \omega^4 - 8.46 \times 10^7 \omega^2 + 1.6 \times 10^9}$$

The mean square value is found from equation (3.68):

$$\bar{x}^2 = E\left[x^2\right] = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{ff}(\omega)\,d\omega$$

$$\bar{x}^2 = S_0 \int_{-\infty}^{\infty} \left|\frac{1}{4 \times 10^4 - 1007\omega^2 + 2000\,j\omega}\right|^2 d\omega$$

Using equation (3.70) yields

$$\bar{x}^2 = \frac{\pi S_0}{8 \times 10^{10}}$$

3.41 To obtain a feel for the correlation functions, compute autocorrelation $R_{xx}(\tau)$ for the deterministic signal $A\sin\omega_n t$.

Solution: The autocorrelation is found from

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} A\sin(\omega_n t)A\sin(\omega_n(t+\tau))dt$$

$$= \lim_{T\to\infty} \frac{A^2}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} \sin(\omega_n t)\sin(\omega_n t)\cos(\omega_n \tau)dt$$

$$+ \underbrace{\lim_{T\to\infty} \frac{A^2}{T} \int_{0}^{T} \sin(\omega_n t)\cos(\omega_n t)\sin(\omega_n \tau)dt}_{\to 0}$$

Simplifying yields:

$$R_{xx}(\tau) = \frac{A^2 \cos(\omega_n \tau)}{2}$$

3.42 Verify that the average $x - \bar{x}$ is zero by using the definition given in equation (3.47).

Solution:

The definition is $\bar{f} = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} f(t)dt$. Let

$$f(t) = x(t) - \bar{x},$$

so that $\quad \bar{f} = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} \left(x(t) - \bar{x}\right)dt$

$$\bar{f} = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} x(t)dt - \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} \bar{x}dt$$

$$= \bar{x} - \bar{x} \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} dt = \bar{x} - \bar{x} = 0$$

Problems and Solutions Section 3.6 (3.43 through 3.44)

3.43 A power line pole with a transformer is modeled by

$$m\ddot{x} + kx = -\ddot{y}$$

where x and y are as indicated in Figure 3.23. Calculate the response of the relative displacement $(x - y)$ if the pole is subject to an earthquake base excitation of (assume the initial conditions are zero)

$$\ddot{y}(t) = \begin{cases} A\left(1 - \dfrac{t}{t_0}\right) & 0 \le t \le 2t_0 \\ 0 & t > 2t_0 \end{cases}$$

$m =$ effective mass

$x(t)$

$k =$ effective stiffness

$y(t)$
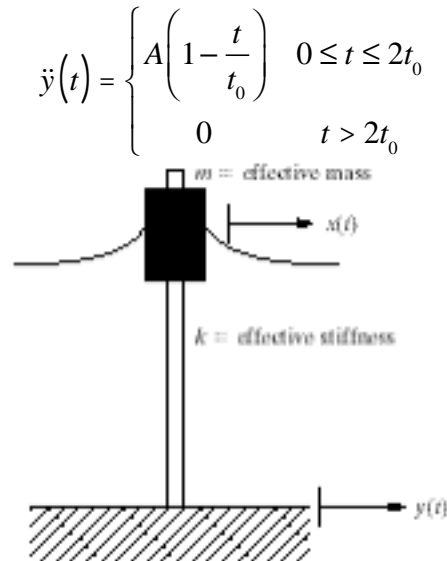
Solution: Given: $m\ddot{x} + kx = -\ddot{y}$

$$\ddot{y} = \begin{cases} A\left(1 - \dfrac{t}{t_0}\right) & 0 \le t \le 2t_0 \\ 0 & t > 2t_0 \end{cases}$$

$$x(0) = \dot{x}(0) = 0$$

The response $x(t)$ is given by Eq. (3.12) as

$$x(t) = \int_0^t F(\tau) h(t - \tau) d\tau$$

where $h(t - \tau) = \dfrac{1}{m\omega_n} \sin \omega_n (t - \tau)$ for an undamped system

For $0 \le t \le 2t_0$,

$$x(t) = \int_0^t A\left(1 - \frac{\tau}{t_0}\right)\left(\frac{1}{m\omega_n}\right)\sin\omega_n(t-\tau)d\tau$$

$$x(t) = \frac{A}{m\omega_n^2}\left[1 - \frac{t}{t_0} + \frac{1}{t_0\omega_n}\sin\omega_n t - \cos\omega_n t\right]$$

For $t > 2t_0$,

$$x(t) = \int_0^{2t_0} A\left(1 - \frac{\tau}{t_0}\right)\left(\frac{1}{m\omega_n}\right)\sin\omega_n(t-\tau)d\tau$$

$$x(t) = \frac{A}{m\omega_n^2}\left[\frac{1}{t_0\omega_n}\left(\sin\omega_n t - \sin\omega_n(t - 2t_0)\right) - \cos\omega_n t - \cos\omega_n(t - 2t_0)\right]$$

Find $y(t)$ when $0 \le t \le 2t_0$,

$$\ddot{y}(t) = A\left(1 - \frac{t}{t_0}\right)$$

$$\dot{y}(t) = At - \frac{A}{2t_0}t^2 + C_1$$

$$y(t) = \frac{A}{2}t^2 - \frac{A}{6t_0}t^3 + C_1 t + C_2$$

Using IC's yields $C_1 = C_2 = 0$. Find $y(t)$ when $t > wt_0$:

$$\ddot{y}(t) = 0$$

$$\dot{y}(t) = C_3$$

$$y(t) = C_3 t + C_4$$

Using IC's yields $C_3 = C_4 = 0$. The relative displacement $x(t) - y(t)$ is therefore:

For $0 \le t \le 2t_0$

$$x(t) - y(t) = \frac{A}{m\omega_n^2}\left[1 - \frac{t}{t_0} + \frac{1}{t_0\omega_n}\sin\omega_n t - \cos\omega_n t\right] - \frac{A}{2}t^2 + \frac{A}{6t_0}t^3$$

For $t > 2t_0$,

$$x(t) - y(t) = \frac{A}{m\omega_n^2}\left[\frac{1}{t_0\omega_n}\left(\sin\omega_n t - \sin\omega_n(t - 2t_0)\right) - \cos\omega_n t - \cos\omega_n(t - 2t_0)\right]$$

3.44 Calculate the response spectrum of an undamped system to the forcing function

$$F(t) = \begin{cases} F_0 \sin \dfrac{\pi t}{t_1} & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases}$$

assuming the initial conditions are zero.

Solution: Let $\omega = \pi / t_1$. The solution is the homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$ or $x(t) = x_h(t) + x_p(t)$. Thus

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\sin\omega t$$

where A and B are constants and ω_n is the natural frequency of the system:
Using the initial conditions $x(0) = \dot{x}(0) = 0$ the constants A and B are

$$A = 0, \quad B = \frac{-F_0\omega}{\omega_n\left(k - m\omega^2\right)}$$

so that $x(t) = \dfrac{F_0 / k}{1 - \left(\omega / \omega_n\right)^2}\left\{\sin\omega t - \dfrac{\omega}{\omega_n}\sin\omega_n t\right\}$, $0 \le t \le t_1$

Which can be written as (where $\delta = F_0 / k$ the static deflection)

$$\frac{x(t)}{\delta} = \frac{1}{1 - \left(\dfrac{\tau}{2t_1}\right)^2}\left\{\sin\frac{\pi t}{t_1} - \frac{\tau}{2t_1}\sin\frac{2\pi t}{\tau}\right\}, 0 \le t \le t_1$$

and where $\tau = 2\pi / \omega_n$. After t_1 the solution is a free response

$$x(t) = A'\cos\omega_n t + B'\sin\omega_n t, t > t_1$$

where the constants A' and B' can be found by using the values of $x(t = t_1)$ and $\dot{x}(t = t_1), t > t_0$. This gives

$$x(t = t_1) = a\left[-\frac{\tau}{2t_1}\sin\frac{2\pi t_1}{\tau}\right] = A'\cos\omega_n t_1 + B'\sin\omega_n t_1$$

$$\dot{x}(t = t_1) = a\left\{-\frac{\pi}{t_1} - \frac{\pi}{t_1}\cos\frac{2\pi t_1}{\tau}\right\} = -\omega_n A'\sin\omega_n t_1 + \omega_n B'\cos\omega_n t$$

where

$$a = \frac{\delta}{1 - \left(\dfrac{\tau}{2t_1}\right)^2}$$

These are solved to yield

$$A' = \frac{a\pi}{\omega_n t_1} \sin\omega_n t_1, \quad B' = -\frac{a\pi}{\omega_n t_1}\left[1 + \cos\omega_n t_1\right]$$

So that after t_1 the solution is

$$\frac{x(t)}{\delta} = \frac{(\tau/t_1)}{2\left\{1-(\tau/2t_1)^2\right\}}\left[\sin 2\pi\left(\frac{t_1}{\tau}-\frac{t}{\tau}\right) - \sin 2\pi \frac{t}{\tau}\right], \quad t \geq t_1$$

Problems and Solutions for Section 3.7 (3.45 through 3.52)

3.45 Using complex algebra, derive equation (3.89) from (3.86) with $s = j\omega$.

Solution: From equation (3.86):

$$H(s) = \frac{1}{ms^2 + cs + k}$$

Substituting $s = j\omega$ yields

$$H(j\omega) = \frac{1}{m(j\omega)^2 + c(j\omega) + k} = \frac{1}{k - m\omega^2 - cj\omega}$$

The magnitude is given by

$$\left|H(j\omega_{dr})\right| = \left[\left(\frac{1}{m(j\omega)^2 + (cj\omega) + k}\right)\left(\frac{1}{k - m\omega^2 - cj\omega}\right)\right]^{1/2}$$

$$\left|H(j\omega)\right| = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \text{ which is Eq. (3.89)}$$

3.46 Using the plot in Figure 3.20, estimate the system's parameters m, c, and k, as well as the natural frequency.

Solution: From Fig. 3.20

$$\frac{1}{k} = 2 \Rightarrow k = 0.5$$

$$\omega = \omega_n = 0.25 = \sqrt{\frac{k}{m}} \Rightarrow m = 8$$

$$\frac{1}{c\omega} \approx 4.6 \Rightarrow c = 0.087$$

3.47 Using the values determined in Problem 3.46 plot the inertance transfer function's magnitude and phase for this system.

Solution: From Problem 3.46

$$\frac{1}{k} = 2 \Rightarrow k = 0.5, \omega = \omega_n = 0.25 = \sqrt{\frac{k}{m}} \Rightarrow m = 8, \frac{1}{c\omega} \approx 4.6 \Rightarrow c = 0.087$$

The inertance transfer function is given by Eq. (3.88):

$$s^2 H(s) = \frac{s^2}{ms^2 + cs + k}$$

Substitute $s = j\omega$ to get the frequency response function. The magnitude is given by:

$$\left|(j\omega)^2 H(j\omega)\right| = \frac{\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}} = \frac{\omega^2}{\sqrt{\left(0.5 - 8\omega^2\right)^2 + \left(0.087\omega\right)^2}}$$

The phase is given by

$$\phi = \tan^{-1}\left(\frac{\text{Imaginary part of frequency response function}}{\text{Real part of frequency response function}}\right)$$

Multiply the numerator and denominator of $(j\omega)^2 H(j\omega)$ by $(k - m\omega^2) - cj\omega$ to get

$$(j\omega)^2 H(j\omega) = \frac{-\omega^2 \left(k - m\omega\right) + cj\omega^3}{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}$$

So, $\phi = \tan^{-1}\left(\dfrac{c\omega^3}{-\omega^2\left(k - m\omega^2\right)}\right) = \tan^{-1}\left(\dfrac{0.087\omega}{8\omega^2 - 0.5}\right)$

The magnitude and phase plots are shown on a semilog scale. The plots are given in the following Mathcad session

$$\phi(\omega) := atan\left(\frac{0.087 \cdot \omega}{8 \cdot \omega^2 - 0.5}\right) \qquad H(\omega) := \frac{\omega^2}{\sqrt{\left(0.5^2 - 8 \cdot \omega^2\right)^2 + (0.087 \cdot \omega)^2}}$$

$H(\omega)$

$\phi(\omega)$
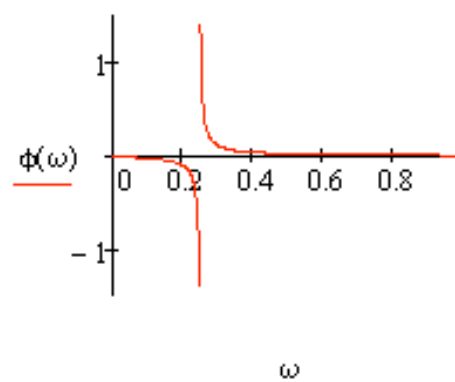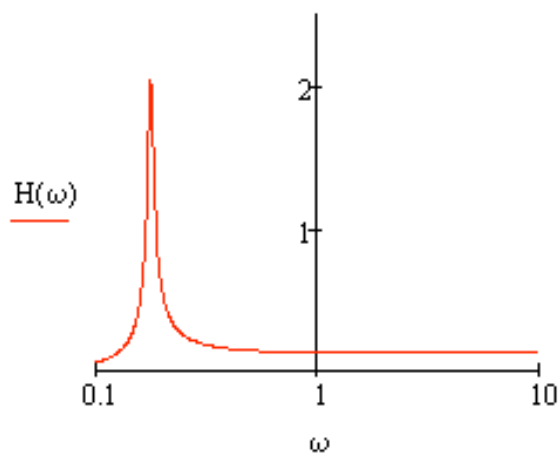
3.48 Using the values determined in Problem 3.46 plot the mobility transfer function's magnitude and phase for the system of Figure 3.20.

Solution: From Problem 3.46

$$\frac{1}{k} = 2 \Rightarrow k = 0.5, \omega = \omega_n = 0.25 = \sqrt{\frac{k}{m}} \Rightarrow m = 8, \frac{1}{c\omega} \approx 4.6 \Rightarrow c = 0.087$$

The mobility transfer function is given by equation (3.87):

$$sH(s) = \frac{s}{ms^2 + cs + k}$$

Substitute $s = j\omega$ to get the frequency response function. The magnitude is given by

$$\left|(j\omega)H(j\omega)\right| = \frac{\omega}{\sqrt{\left(k - j\omega^2\right)^2 + \left(c\omega\right)^2}} = \frac{\omega}{\sqrt{\left(0.5 - 8\omega^2\right)^2 + \left(0.087\omega\right)^2}}$$

The phase is given by

$$\phi = \tan^{-1}\left(\frac{\text{Imaginary part of frequency response function}}{\text{Real part of frequency response function}}\right)$$

Multiply the numerator and denominator of $j\omega H(j\omega)$ by j and by $-\left(k - m\omega^2\right)j - c\omega$ to get

$$(j\omega)H(j\omega) = \frac{j\omega\left(k - m\omega^2\right) + c\omega^2}{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}$$

So, $\phi = \tan^{-1}\left(\dfrac{\omega\left(k - m\omega^2\right)}{c\omega^2}\right) = \tan^{-1}\left(\dfrac{0.5 - 8\omega^2}{0.087\omega}\right)$

The magnitude and phase plots are shown on a semilog scale.

$$\phi(\omega) := atan\left(\frac{-8 \cdot \omega^2 + 0.5}{0.087 \cdot \omega}\right) \qquad H(\omega) := \frac{\omega^2}{\sqrt{\left(0.5^2 - 8 \cdot \omega^2\right)^2 + (0.087 \cdot \omega)^2}}$$
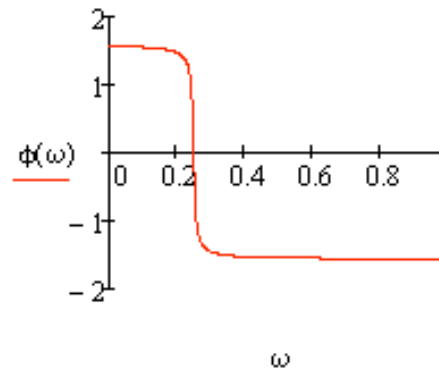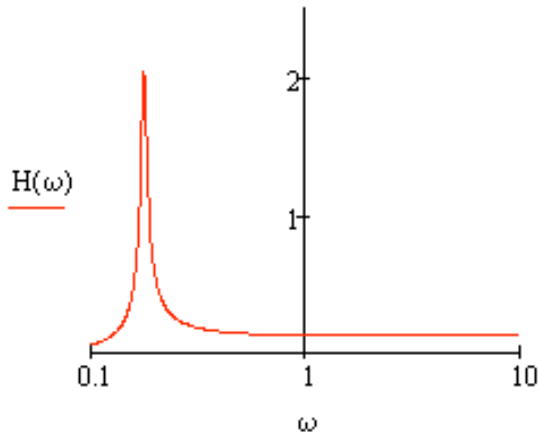
3.49 Calculate the compliance transfer function for a system described by

$$a\ddddot{x} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = f(t)$$

where $f(t)$ is the input force and $x(t)$ is a displacement.

Solution:

The compliance transfer function is $\dfrac{X(s)}{F(s)}$.

Taking the Laplace Transform yields

$$\left(as^4 + bs^3 + cs^2 + ds + e\right)X(s) = F(s)$$

So, $\dfrac{X(s)}{F(s)} = \dfrac{1}{as^4 + bs^3 + cs^2 + ds + e}$

3.50   Calculate the frequency response function for the compliance of Problem 3.49.

Solution: From problem 3.49,

$$H(s) = \frac{1}{as^4 + bs^3 + cs^2 + ds + e}$$

Substitute $s = j\omega$ to get the frequency response function:

$$H(j\omega) = \frac{1}{a(j\omega)^4 + b(j\omega)^3 + c(j\omega)^2 + d(j\omega) + e}$$

$$H(j\omega) = \frac{a\omega^4 - c\omega^2 + e - j(-b\omega^3 + d\omega)}{(a\omega^4 - c\omega^2 + e)^2 + (-b\omega^3 + d\omega)^2}$$

3.51   Plot the magnitude of the frequency response function for the system of Problem 3.49 for
$a = 1, b = 4, c = 11, d = 16,$ and $e = 8.$

Solution: From Problem 3.50

$$H(j\omega) = \frac{a\omega^4 - c\omega^2 + e - j(-b\omega^3 + d\omega)}{(a\omega^4 - c\omega^2 + e)^2 + (-b\omega^3 + d\omega)^2}$$

The magnitude is

$$\left| H(j\omega) \right| = \frac{1}{\sqrt{(\omega^4 - 11\omega^2 + 8)^2 + (-4\omega^3 + 16\omega)^2}}$$

This is plotted in the following Mathcad session:

$$H(\omega) := \frac{1}{\sqrt{\left(\omega^4 - 11\cdot\omega^2 + 8\right)^2 + \left(16\cdot\omega - 4\cdot\omega^3\right)^2}}$$
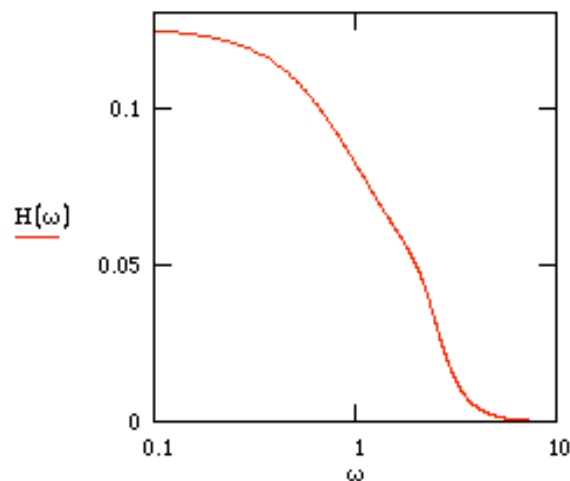
3.52 An experimental (compliance) magnitude plot is illustrated in Fig. P3.52. Determine $\omega, \zeta, c, m,$ and k. Assume that the units correspond to m/N along the vertical axis.

Mag H(jω)
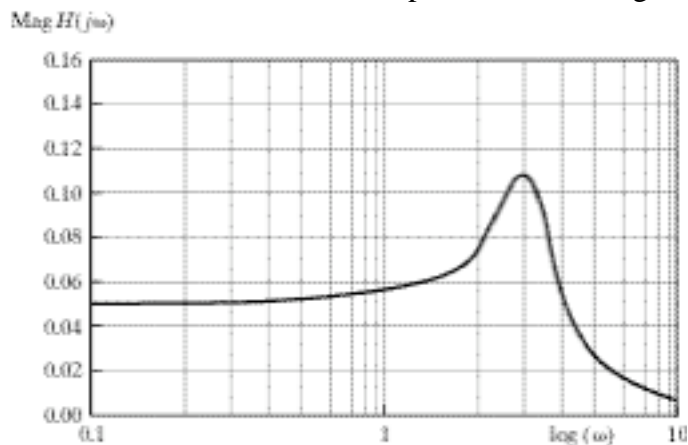
Solution: Referring to the plot, it starts at

$$\left|H(\omega j)\right| = \frac{1}{k}$$

Thus:
$$0.05 = \frac{1}{k} \Rightarrow k = 20 \text{ N/m}$$

At the peak, $\omega_n = \omega = 3$ rad/s. Thus the mass can be determined by

$$m = \frac{k}{\omega_n^2} \Rightarrow m = 2.22 \text{ kg}$$

The damping is found from

$$\frac{1}{c\omega} = 0.11 \Rightarrow c = 3.03 \text{ kg/s} \Rightarrow \zeta = \frac{c}{2\sqrt{km}} = \frac{3.03}{2\sqrt{20 \cdot 2.22}} = 0.227$$

Problems and Solutions Section 3.8 (3.53 through 3.56)

3.53   Show that a critically damped system is BIBO stable.

Solution:

For a critically damped system

$$h(t - \tau) = \frac{1}{m}(t - \tau)e^{-\omega_n(t-\tau)}$$

Let $f(t)$ be bounded by the finite constant M. Using the inequality for integrals and Equation (3.96) yields:

$$\left|x(t)\right| \leq \int_0^t f(\tau)\left|h(t - \tau)\right|d\tau = \int_0^t M\frac{1}{m}\left|(t - \tau)e^{-\omega_n(t-\tau)}\right|d\tau$$

The function $h(t - \tau)$ decays exponentially and hence is bounded by some constant times $1/t$, say M_1/t. This is just a statement the exponential decays faster then "one over t" does. Thus the above expression becomes;

$$\left|x(t)\right| < M\int_0^t \frac{M_1}{t}d\tau = MM_1$$

This is bounded, so a critically damped system is BIBO stable.

3.54 Show that an overdamped system is BIBO stable.

Solution: For an overdamped system,

$$h(t-\tau) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}} e^{-\zeta\omega_n(t-\tau)}\left(e^{\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)} - e^{-\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}\right)$$

Let $f(t)$ be bounded by M,

From equation (3.96),

$$|x(t)| \le M\int_0^t |h(t-\tau)|d\tau$$

$$|x(t)| \le M\int_0^t \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\left| e^{-\zeta\omega_n(t-\tau)}\left(e^{\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)} - e^{-\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}\right)\right|d\tau$$

$$|x(t)| \le \frac{M}{2m\omega_n\sqrt{\zeta^2-1}}\left[\left(\frac{-1}{\omega_n\sqrt{\zeta^2-1}-\zeta\omega}\right)\left(1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}\right)\right.$$
$$\left. -\left(\frac{-1}{\omega_n\sqrt{\zeta^2-1}+\zeta\omega_n}\right)\left(1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}\right)\right]$$

Since $\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n < 0$, then $1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}$ is bounded.

Also, since $-\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n < 0$, then $1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}$ is bounded.

Therefore, an overdamped system is BIBO stable.

3.55   Is the solution of $2\ddot{x} + 18x = 4\cos 2t + \cos t$ Lagrange stable?

Solution: Given

$$2\ddot{x} + 18x = 4\cos 2t + \cos t$$

$$\omega_n = \sqrt{\frac{k}{m}} = 3$$

The total solution will be

$$x(t) = x_h(t) + x_{P1}(t) + x_{P2}(t)$$

From Eq. (1.3):

$$x_h(t) = A\sin(\omega_n t + \phi)$$

From Eq. (2.7):

$$x_{P1}(t) = \frac{f_{0_1}}{\omega_n^2 - 2^2}\cos 2t$$

and

$$x_{P2}(t) = \frac{f_{0_2}}{\omega_n^2 - 1^2}\cos t$$

Adding the solutions yields

$$\left|x(t)\right| = \left|A\sin(3t + \phi) + \frac{f_{0_1}}{3^2 - 2^2}\cos 2t + \frac{f_{0_1}}{3^2 - 1^2}\cos t\right| < M$$

Since $3 \neq 2, 3 \neq 1$, and the homogeneous solution is marginally stable, this system is Lagrange stable.

3.56 Calculate the response of equation (3.99) for $x_0 = 0, v_0 = 1$ for the case that $a = 4$ and $b = 0$. Is the response bounded?

Solution: Given: $x_0 = 0, v_0 = 1, a = 4, b = 0$. From Eq. (3.99),

$$\ddot{x} + \dot{x} + 4x = ax + b\dot{x} = 4x$$

So, $\ddot{x} + \dot{x} = 0$

Let

$$x(t) = Ae^{\lambda t}$$
$$\dot{x}(t) = \lambda Ae^{\lambda t}$$
$$\ddot{x}(t) = \lambda^2 Ae^{\lambda t}$$

Substituting,

$$\lambda^2 Ae^{\lambda t} + \lambda Ae^{\lambda t} = 0$$
$$\lambda^2 + \lambda = 0$$

So, $\lambda_{1,2} = 0, -1$

The solution is

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 + A_2 e^{-t}$$
$$\dot{x}(t) = -A_2 e^{-t}$$
$$x(0) = 0 = A_1 + A_2$$
$$\dot{x}(0) = 1 = -A_2$$

So, $A_1 = 1$ and $A_2 = -1$

Therefore,

$$x(t) = 1 - e^{-t}$$

Since $|x(t)| = |1 - e^{-t}| 1$, the response is bounded.

Problems and Solutions from Section 3.9 (3.57-3.64)

3.57*. Numerically integrate and plot the response of an underdamped system determined by $m = 100$ kg, $k = 1000$ N/m, and $c = 20$ kg/s, subject to the initial conditions of $x_0 = 0$ and $v_0 = 0$, and the applied force $F(t) = 30\Phi(t - 1)$. Then plot the exact response as computed by equation (3.17). Compare the plot of the exact solution to the numerical simulation.

Solution: First the solution is presented in Mathcad:

Prob 3.50

$m := 100$ $k := 1000$ $\omega n := \sqrt{\dfrac{k}{m}}$ $c := 20$

$F0 := 30$ $\omega n = 3.162$ $\zeta := \dfrac{c}{2 \cdot \sqrt{k \cdot m}}$ $t0 := 1$ $\zeta = 0.032$

$\theta := \text{atan}\left[\dfrac{\zeta}{\sqrt{1 - \zeta^2}}\right]$ $\dfrac{F0}{k} = 0.03$ $\dfrac{F0}{k \cdot \sqrt{1 - \zeta^2}} = 0.03002$ $\zeta \cdot \omega n = 0.1$

$\omega d := \omega n \cdot \sqrt{1 - \zeta^2}$ $\omega d = 3.1607$ $\theta = 0.032$ $m := \left(\dfrac{k}{\omega n^2}\right)$

$xa(t) := \left[\left[\dfrac{F0}{k} - \dfrac{F0}{k \cdot \sqrt{1 - \zeta^2}} \cdot e^{-\zeta \cdot \omega n \cdot (t - t0)} \cdot \cos\left[\omega d \cdot (t - t0) - \theta\right]\right] \right] \cdot \Phi(t - t0)$

$X := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \dfrac{F0}{m} \cdot \Phi(t - t0) \end{bmatrix}$

$Z := \text{rkfixed}(X, 0, 30, 2000, D)$

$t := Z^{<0>}$ $\overrightarrow{x := xa(t)}$ $xn := Z^{<1>}$
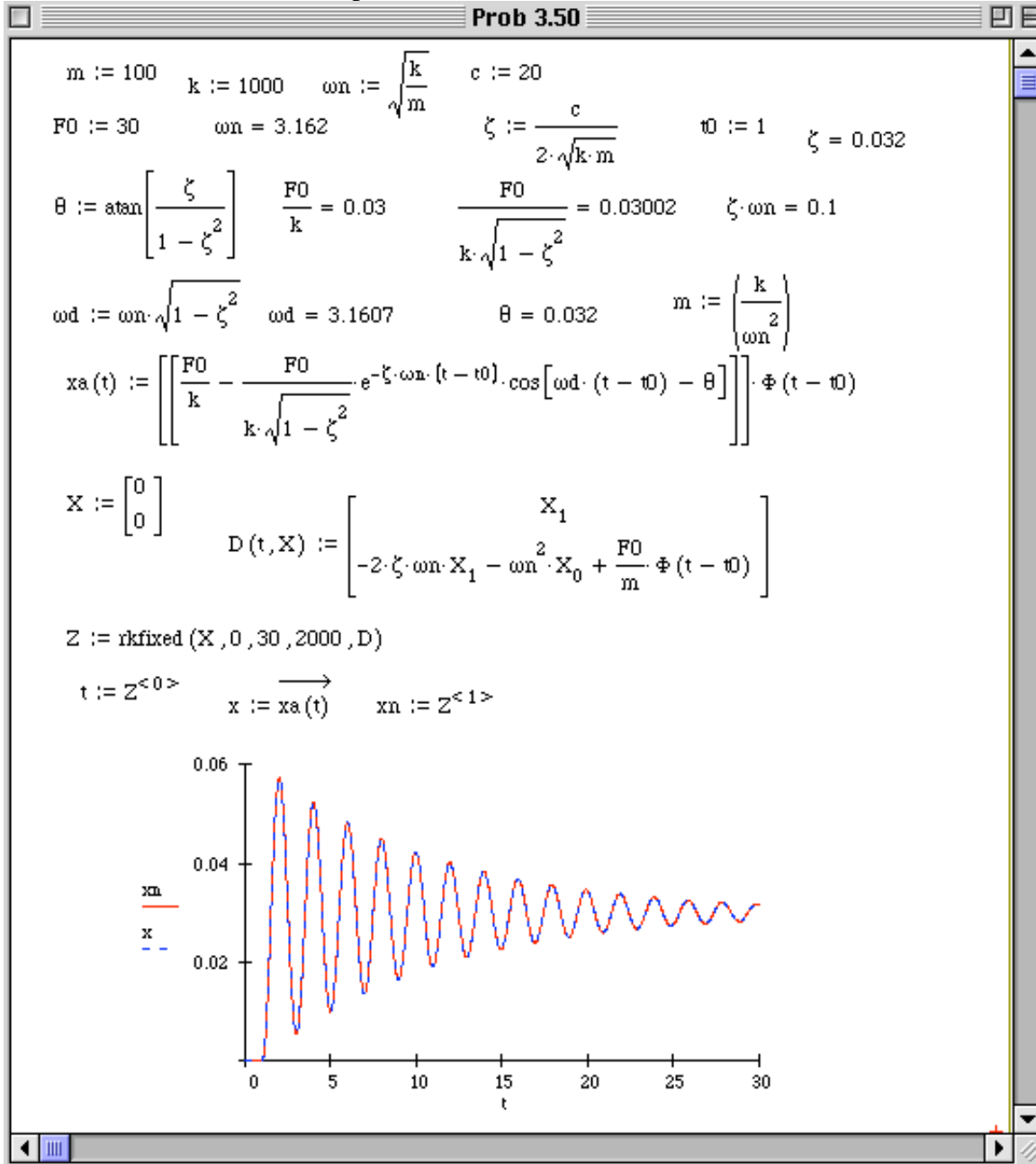
The Matlab code to provide similar plots is given next:

```
%Numerical Solutions
%Problem #57
clc
clear
close all
%Numerical Solution
x0=[0;0];
tspan=[0 15];

[t,x]=ode45('prob57a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #57');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution
m=100;
c=20;
k=1000;
F=30;
w=sqrt(k/m);
d=c/(2*w*m);
wd=w*sqrt(1-d^2);
to=1;
phi=atan(d/sqrt(1-d^2));

%for t<to
t=linspace(0,1,3);
x=0.*t;
plot(t,x,'*');

%for t>=to
t=linspace(1,15);
x=F/k-F/(k*sqrt(1-d^2)).*exp(-d.*w.*(t-to)).*cos(wd.*(t-to)-phi);
plot(t,x,'*');
legend('Numerical', 'Analytical')

%M-file for Prob #50

function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);
m=100;
c=20;
k=1000;
F=30;

if t<1
    dx==0;
    else
        dx(1)=x(2);
    dx(2)=-c/m*x(2) - k/m*x(1) + F/m;
end
```

3.58*. Numerically integrate and plot the response of an underdamped system determined by $m = 150$ kg, and $k = 4000$ N/m subject to the initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s, and the applied force $F(t) = F(t) = 15\Phi(t - 1)$, for various values of the damping coefficient. Use this "program" to determine a value of damping that causes the transient term to die out with in 3 seconds. Try to find the smallest such value of damping remembering that added damping is usually expensive.

Solution: First the solution is given in Mathcad followed by the equivalent Matlab code.

$$m := 150 \quad k := 4000 \quad \omega n := \sqrt{\frac{k}{m}} \quad c := 710 \quad x0 := 0.01 \quad v0 := 0.1$$

$$F0 := 15 \quad \omega n = \quad \zeta := \frac{c}{2 \cdot \sqrt{k \cdot m}} \quad t0 := 1 \quad \zeta =$$

$$\theta := \text{atan}\left(\frac{\zeta}{1-\zeta^2}\right) \quad \theta = \quad \omega d := \omega n \cdot \sqrt{1-\zeta^2}$$

$$A := \sqrt{\frac{(v0 + \zeta \cdot \omega n \cdot x0)^2 + (x0 \cdot \omega d)^2}{\omega d^2}}$$

$$\omega d = \quad \phi := \text{atan}\left(\frac{x0 \cdot \omega d}{v0 + \zeta \cdot \omega n \cdot x0}\right) \quad xh(t) := A \cdot e^{-\zeta \cdot \omega n \cdot t} \cdot \sin(\omega d \cdot t + \phi)$$

$$xa(t) := \left[\left[\frac{F0}{k} - \frac{F0}{k \cdot \sqrt{1-\zeta^2}} \cdot e^{-\zeta \cdot \omega n \cdot (t-t0)} \cdot \cos[\omega d \cdot (t - t0) - \theta]\right] \cdot \Phi(t - t0) + xh(t)\right.$$

$$xa(t) := \left[\left[\frac{F0}{k} - \frac{F0}{k \cdot \sqrt{1-\zeta^2}} \cdot e^{-\zeta \cdot \omega n \cdot (t-t0)} \cdot \cos[\omega d \cdot (t - t0) - \theta]\right] \cdot \Phi(t - t0) + xh(t)\right.$$

$$X := \begin{pmatrix} x0 \\ v0 \end{pmatrix} \quad D(t, X) := \begin{pmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \frac{F0}{m} \cdot \Phi(t - t0) \end{pmatrix} \quad F(t) := \frac{F0}{k}$$

$$Z := \text{rkfixed}(X, 0, 30, 2000, D)$$

$$t := Z^{\langle 0 \rangle} \quad x := \overrightarrow{xa(t)} \quad xn := Z^{\langle 1 \rangle} \quad S := \overrightarrow{F(t)}$$
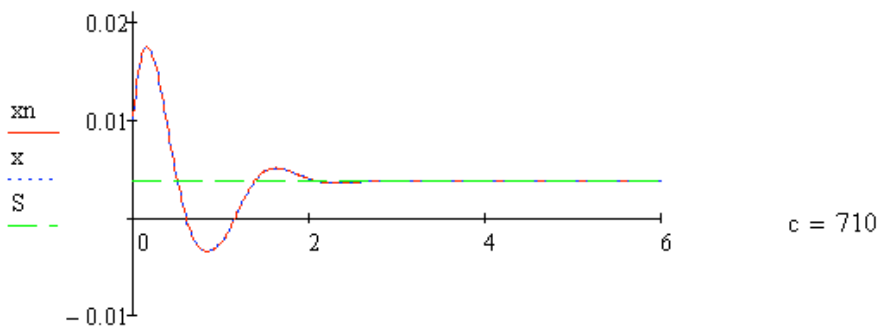
$$c = 710$$

A value of $c = 710$ kg/s will do the job.

```
%Vibrations
%Numerical Solutions
%Problem #51

clc
clear
close all

%Numerical Solution

x0=[0.01;0];
tspan=[0 15];

[t,x]=ode45('prob51a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #51');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

m=150;
c=0;
k=4000;
F=15;
w=sqrt(k/m);
d=c/(2*w*m);
wd=w*sqrt(1-d^2);
to=1;
phi=atan(d/sqrt(1-d^2));




%for t<to
t=linspace(0,1,10);
x0=0.01;
v0=0;
A=sqrt(v0^2+(x0*w)^2)/w;
theta=pi/2;
x=A.*sin(w.*t + theta);
plot(t,x,'*')

%for t>=to
t=linspace(1,15);
x2=F/k-F/(k*sqrt(1-d^2)).*exp(-d.*w.*(t-to)).*cos(wd.*(t-to)-phi);
x1=A.*sin(w.*t + theta);

x=x1+x2;
plot(t,x,'*');
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions
```

```
%M-file for Prob #51

function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);
m=150;
c=0;
k=4000;
F=15;


if t<1
   dx(1)=x(2);
   dx(2)=-c/m*x(2)- k/m*x(1);
   else
       dx(1)=x(2);
   dx(2)=-c/m*x(2) - k/m*x(1)+ F/m;
end
```

3.59*. Solve Example 3.3.2, Figure 3.9 by numerically integrating rather then using analytical expressions, and plot the response.

Solution: Both Mathcad and Matlab solutions follow:
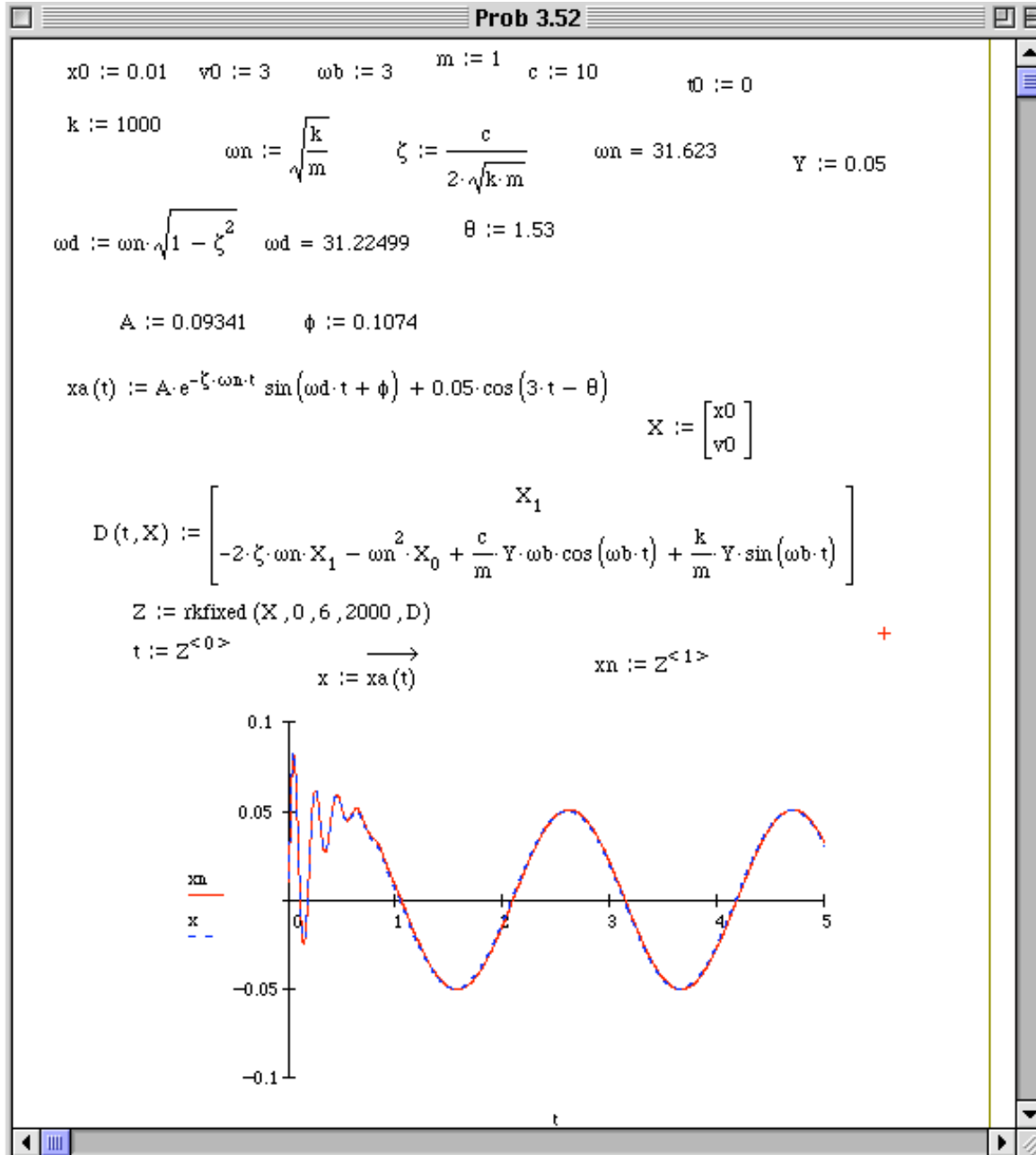
Prob 3.52

$x0 := 0.01$ $v0 := 3$ $\omega b := 3$ $m := 1$ $c := 10$ $t0 := 0$

$k := 1000$

$\omega n := \sqrt{\dfrac{k}{m}}$ $\zeta := \dfrac{c}{2 \cdot \sqrt{k \cdot m}}$ $\omega n = 31.623$ $Y := 0.05$

$\omega d := \omega n \cdot \sqrt{1 - \zeta^2}$ $\omega d = 31.22499$ $\theta := 1.53$

$A := 0.09341$ $\phi := 0.1074$

$xa(t) := A \cdot e^{-\zeta \cdot \omega n \cdot t} \sin(\omega d \cdot t + \phi) + 0.05 \cdot \cos(3 \cdot t - \theta)$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$

$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \dfrac{c}{m} \cdot Y \cdot \omega b \cdot \cos(\omega b \cdot t) + \dfrac{k}{m} \cdot Y \cdot \sin(\omega b \cdot t) \end{bmatrix}$

$Z := rkfixed(X, 0, 6, 2000, D)$

$t := Z^{<0>}$

$x := \overrightarrow{xa(t)}$ $xn := Z^{<1>}$


```
%Numerical Solutions
%Problem #53
clc
clear
close all
%Numerical Solution
```

```
x0=[0;0];
tspan=[0 10];

[t,x]=ode45('prob53a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #53');
xlabel('Time, sec.');
ylabel('Displacement, mm');
hold on

%Analytical Solution
t1=0.2;
t2=0.6;

%for t<to
t=linspace(0,t1);
x=2.5*t-4.56.*sin(0.548.*t);
plot(t,x,'*');

%for t1<t<t2
t=linspace(t1,t2);
x=0.75 - 1.25.*t + 6.84.*sin(0.548*(t-t1))- 4.56.*sin(0.548.*t);
plot(t,x,'*');

%for t2<t
t=linspace(t2,10);
x=6.84.*sin(0.548.*(t-t1))-2.28.*sin(0.548.*(t-t2))-
4.56.*sin(0.548.*t);
plot(t,x,'*');
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions
%Clay
%Vibrations
%Solutions

%M-file for Prob #52

function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);
m=1;
c=10;
k=1000;
Y=0.05;
wb=3;

a=c*Y*wb;
b=k*Y;
alpha=atan(b/a);
AB=sqrt(a^2+b^2)/m;

dx(1)=x(2);
dx(2)=-c/m*x(2)- k/m*x(1)+ a/m*cos(wb*t) + b/m*sin(wb*t);
```

3.60*. Numerically simulate the response of the system of Problem 3.21 and plot the response.

Solution: The solution in Matlab is

```
%Clay
%Vibrations
%Numerical Solutions
%Problem #53

clc
clear
close all

%Numerical Solution

x0=[0;0];
tspan=[0 10];

[t,x]=ode45('prob53a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #53');
xlabel('Time, sec.');
ylabel('Displacement, mm');
hold on

%Analytical Solution

t1=0.2;
t2=0.6;

%for t<to
t=linspace(0,t1);
x=2.5*t-4.56.*sin(0.548.*t);
plot(t,x,'*');

%for t1<t<t2
t=linspace(t1,t2);
x=0.75 - 1.25.*t + 6.84.*sin(0.548*(t-t1))- 4.56.*sin(0.548.*t);
plot(t,x,'*');

%for t2<t
t=linspace(t2,10);
x=6.84.*sin(0.548.*(t-t1))-2.28.*sin(0.548.*(t-t2))-
4.56.*sin(0.548.*t);
plot(t,x,'*');
legend('Numerical', 'Analytical')

%Clay
%Vibrations
%Solutions
```

```
%M-file for Prob #53

function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=5000;
k=1.5e3;
ymax=0.5;
F=k*ymax;
t1=0.2;
t2=0.6;

if t<t1
    dx(1)=x(2);
    dx(2)= - k/m*x(1)+ F/m*(t/t1);
elseif t<t2 & t>t1
        dx(1)=x(2);
    dx(2)= - k/m*x(1)+ F/(2*t1*m)*(t2-t);
else
    dx(1)=x(2);
    dx(2)= - k/m*x(1);
end
```


Figure No. 1
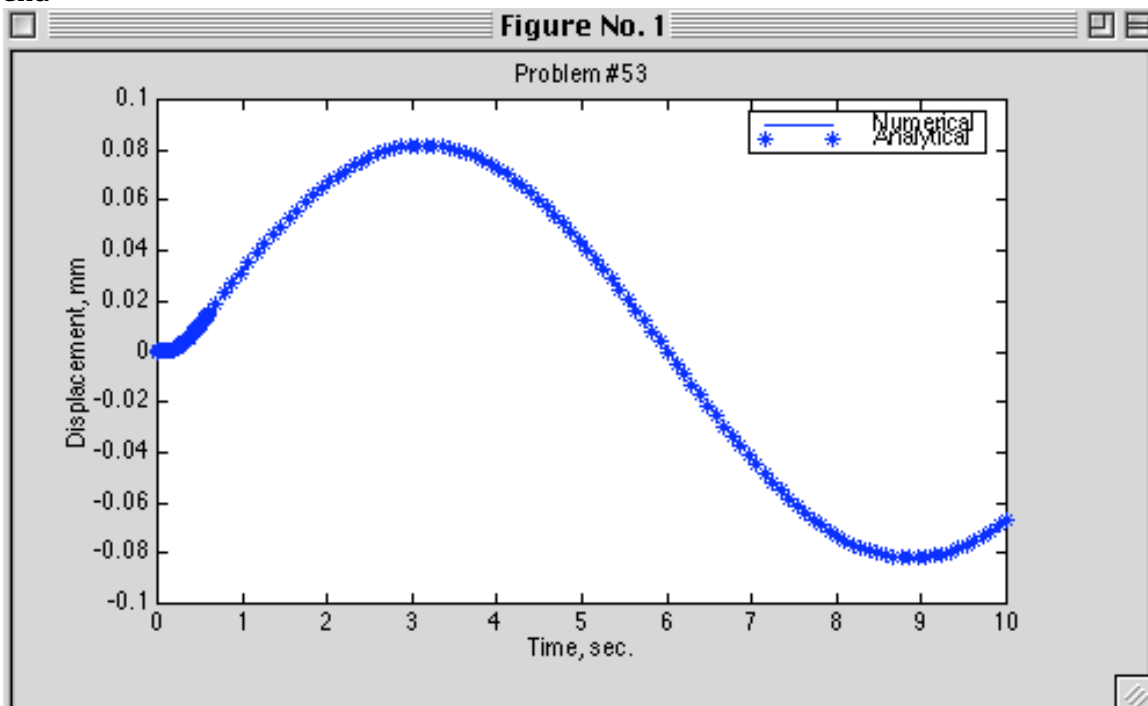
Problem #53

3.61*. Numerically simulate the response of the system of Problem 3.18 and plot the response.

Solution: The solution in Matlab is

```
%Clay
%Vibrations
%Numerical Solutions
%Problem #54

clc
clear
close all

%Numerical Solution

x0=[0;0];
tspan=[0 10];

[t,x]=ode45('prob54a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #54');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

to=4;

%for t<to
t=linspace(0,to);
x=5*(t-sin(t));
plot(t,x,'*');

%for t>=to
t=linspace(to,10);
x=5*(sin(t-to)-sin(t))+20;

plot(t,x,'*');
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions

%M-file for Prob #54


function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=1;
k=1;
F=20;
```

```
to=4;

if t<to
    dx(1)=x(2);
    dx(2)= - k/m*x(1)+ F/m*(t/to);
else
        dx(1)=x(2);
    dx(2)= - k/m*x(1)+ F/m;
end
```
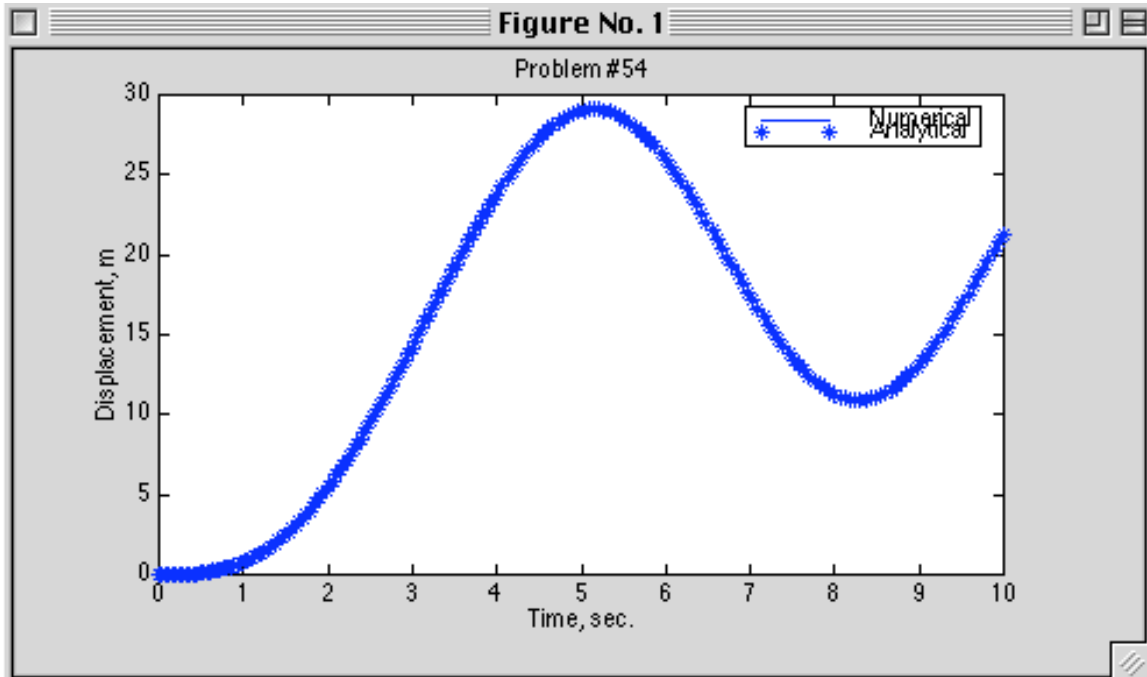

Problem #54

3.62*. Numerically simulate the response of the system of Problem 3.19 for a 2 meter concrete wall with cross section 0.03 m^2 and mass modeled as lumped at the end of 1000 kg. Use $F_0 = 100$ N, and plot the response for the case $t_0 = 0.25$ s.
Solution The solution in Matlab is:

```
%Numerical Solutions
%Problem #3.62

clc
clear
close all

%Numerical Solution

x0=[0;0];
tspan=[0 0.5];

[t,x]=ode45('prob55a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #55');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

m=1000;
E=3.8e9;
A=0.03;
l=2;
k=E*A/l;
F=100;
w=sqrt(k/m);
to=0.25;


%for t<to
t=linspace(0,to);
x=F/k*(1-cos(w*t))+ F/(to*k)*(1/w*sin(w*t)-t);
plot(t,x,'*');

%for t>=to
t=linspace(to,0.5);
x=-F/k*cos(w*t)- F/(w*k*to)*(sin(w*(t-to))-sin(w*t));
plot(t,x,'*');
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions

%M-file for Prob #3.62
```

```
function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=1000;
E=3.8e9;
A=0.03;
l=2;
k=E*A/l;
F=100;
w=sqrt(k/m);
to=0.25;


if t<to
   dx(1)=x(2);
   dx(2)= - k/m*x(1) + F/m*(1-t/to);
   else
       dx(1)=x(2);
   dx(2)= - k/m*x(1);
end
```
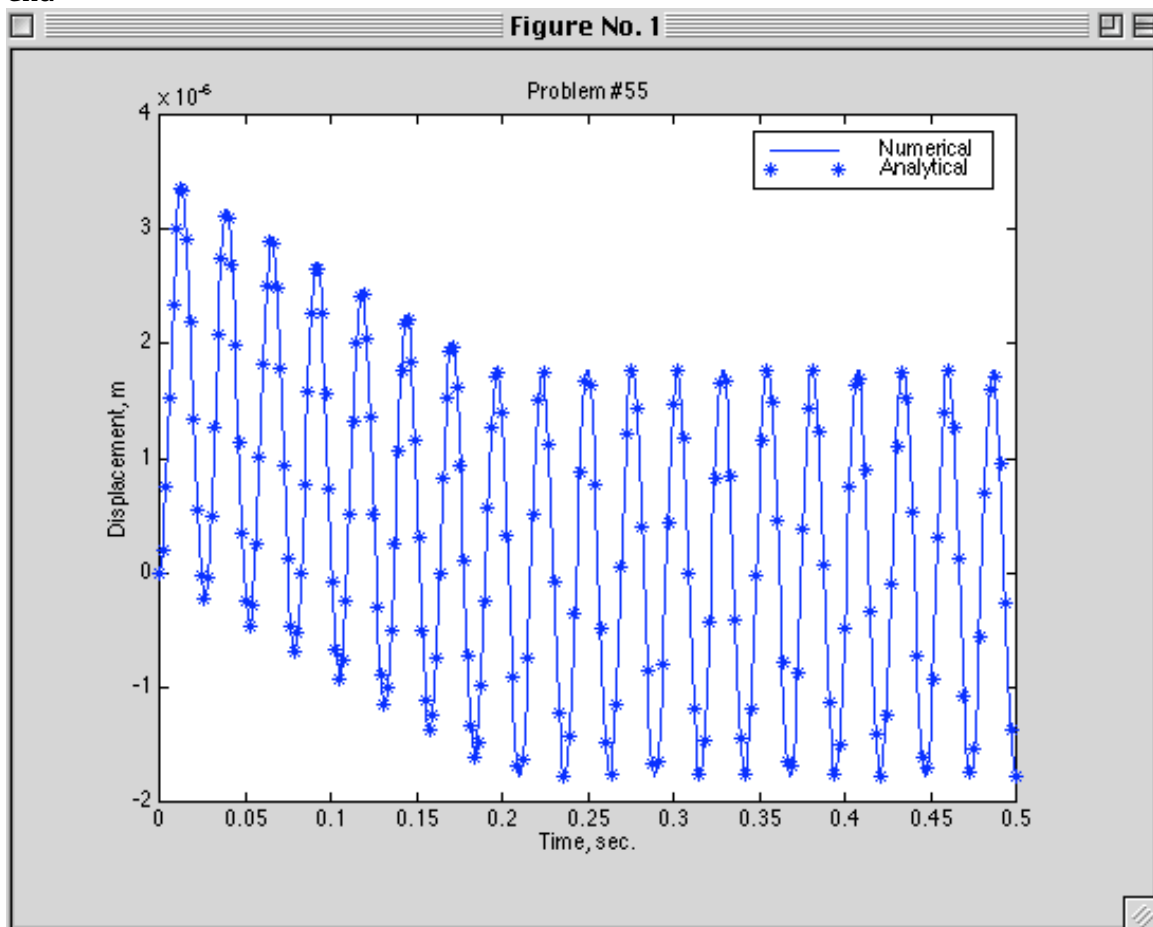

Problem #55

3.63*. Numerically simulate the response of the system of Problem 3.20 and plot the response.

Solution The solution in Matlab is:

```
%Clay
%Vibrations
%Numerical Solutions
%Problem #56

clc
clear
close all

%Numerical Solution

x0=[0;0];
tspan=[0 2];

[t,x]=ode45('prob56a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #56');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

t=linspace(0,2);
x=0.5*t-0.05*sin(10*t);
plot(t,x,'*');
legend('Numerical', 'Analytical')


%Clay
%Vibrations
%Solutions

%M-file for Prob #56


function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=1;
k=100;
F=50;


 dx(1)=x(2);
 dx(2)= - k/m*x(1) + F/m*(t);
```

Figure No. 1

Problem #56
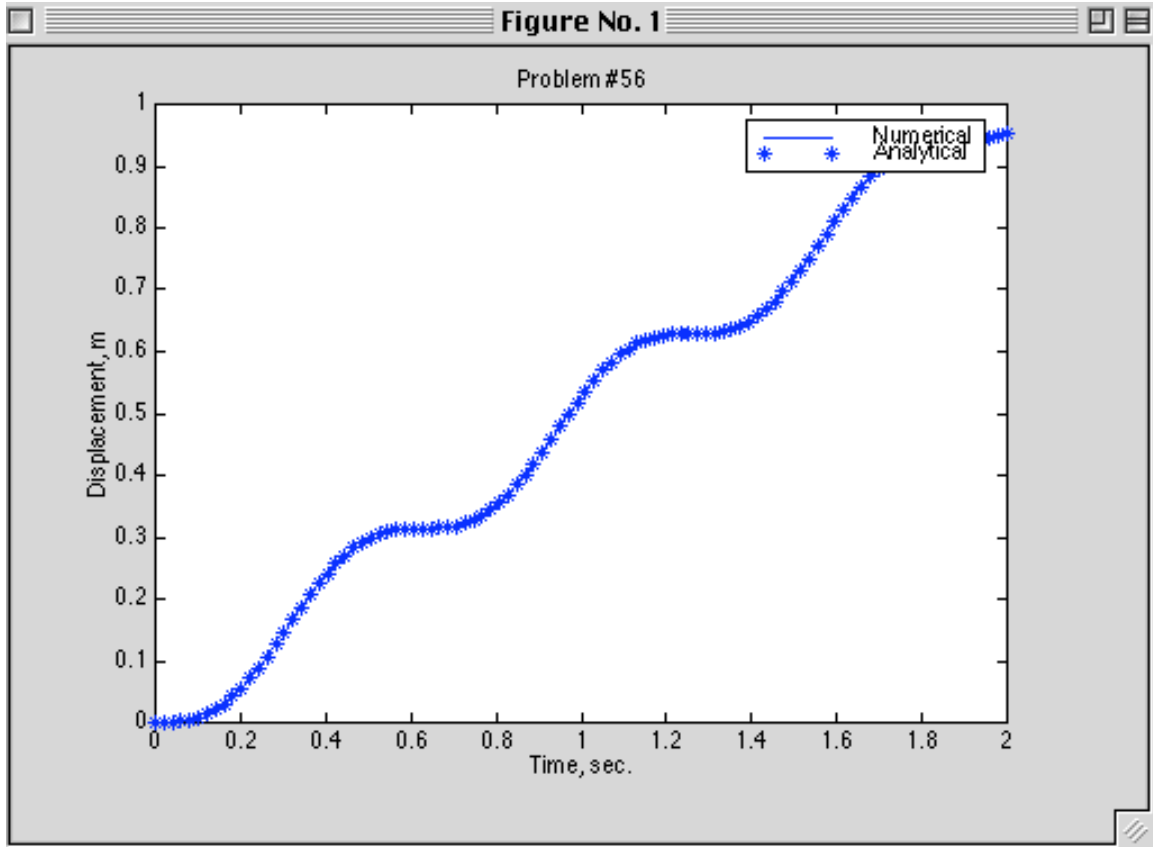
3.64*. Compute and plot the response of the system of following system using numerical integration:

$$10\ddot{x}(t) + 20\dot{x}(t) + 1500x(t) = 20\sin 25t + 10\sin 15t + 20\sin 2t$$

with initial conditions of $x_0 = 0.01$ m and $v_0 = 1.0$ m/s.

Solution The solution in Matlab is:

```
%Clay
%Vibrations
%Numerical Solutions
%Problem #57

clc
clear
close all

%Numerical Solution

x0=[0.01;1];
tspan=[0 5];

[t,x]=ode45('prob57a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #57');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

m=10;
c=20;
k=1500;
w=sqrt(k/m);
d=c/(2*w*m);
wd=w*sqrt(1-d^2);


Y1=0.00419;
ph1=3.04;
wb1=25;

Y2=0.01238;
ph2=2.77;
wb2=15;

Y3=0.01369;
ph3=0.0268;
wb3=2;

A=0.1047;
phi=0.1465;
```

```
x=A.*exp(-d*w.*t).*sin(wd*t+phi)+ Y1.*sin(wb1*t-ph1) + Y2*sin(wb2*t-
ph2) + Y3*sin(wb3*t-ph3);

plot(t,x,'*')
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions


%M-file for Prob #57


function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=10;
c=20;
k=1500;


dx(1)=x(2);
dx(2)= -c/m*x(2) - k/m*x(1) + 20/m*sin(25*t) + 10/m*sin(15*t) +
20/m*sin(2*t) ;
```


Figure No. 1
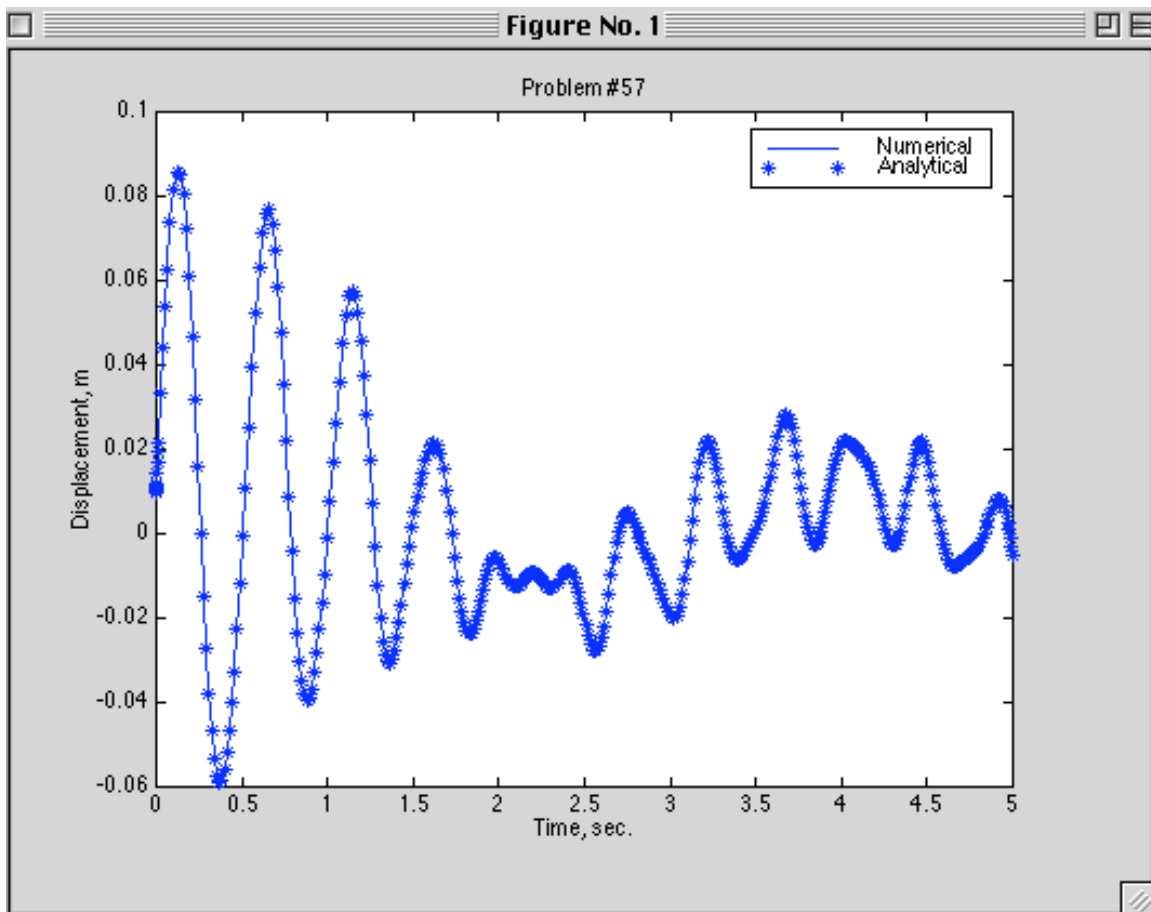
Problems and Solutions Section 3.10 (3.65 through 3.71)

3.65*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form
$$k(x) = kx - k_1 x^3$$
and the system is subject to a excitation of the form ($t_1 = 1.5$ and $t_2 = 1.6$)
$$F(t) = 1500\left[\Phi(t - t_1) - \Phi(t - t_2)\right] \text{ N}$$
and initial conditions of $x_0 = 0.01$ m and $v_0 = 1.0$ m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 300 N/m³. Compute the solution and compare it to the linear solution ($k_1 = 0$). Which system has the largest magnitude? Compare your solution to that of Example 3.10.1.

Solution: The solution in Mathcad is

$x0 := 0.01 \quad v0 := 1 \qquad m := 100 \quad k := 2000 \quad k1 := 300 \qquad c := 30$

$$\omega n := \sqrt{\frac{k}{m}} \qquad\qquad \zeta := \frac{c}{2\sqrt{k \cdot m}} \quad \alpha := \frac{k1}{m} \quad \begin{array}{l} F0 := 1500 \quad t1 := 1.5 \\ f0 := \dfrac{F0}{m} \end{array} \quad \begin{array}{l} t2 := 1.6 \\ \zeta = 0.034 \end{array}$$

$$X := \begin{pmatrix} x0 \\ v0 \end{pmatrix} \quad Y := X \qquad\qquad\qquad\qquad\qquad\qquad \alpha = 3$$

$$f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$$

$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \left[\alpha \cdot (X_0)^3 + f(t)\right] \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 2000, D)$

$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle} \qquad\qquad\qquad W := \text{rkfixed}(Y, 0, 10, 2000, L)$

$xL := W^{\langle 1 \rangle}$

$$t := 0, 0.001 .. 5.5$$

$$F(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$$

Note that for this load the load, which is more like an impulse, the linear and nonlinear responses are similar, whereas in Example 3.10.1 the applied load is a "wider" impulse and the linear and nonlinear responses differ quite a bit.

3.66*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form
$$k(x) = kx - k_1 x^3$$
and the system is subject to a excitation of the form ($t_1 = 1.5$ and $t_2 = 1.6$)
$$F(t) = 1500 \left[\Phi(t - t_1) - \Phi(t - t_2) \right] \text{ N}$$
and initial conditions of $x_0 = 0.01$ m and $v_0 = 1.0$ m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 300 N/m^3. Compute the solution and compare it to the linear solution ($k_1 = 0$). How different are the linear and nonlinear responses? Repeat this for $t_2 = 2$. What can you say regarding the effect of the time length of the pulse?

Solution: The solution in Mathcad for the case $t_2 = 1.6$ is

Note in this case the linear response is almost the same as the nonlinear response.
Next changing the time of the pulse input to $t_2 = 2$ yields the following:

Prob 3.59

$x0 := 0.01$

$v0 := 1$

$m := 100 \qquad k := 2000$

$c := 30$

$\alpha := 3 \qquad F0 := 1500 \qquad t1 := 1.5$

$t2 := 2$

$\omega n := \sqrt{\dfrac{k}{m}}$

$\zeta := \dfrac{c}{2\sqrt{k \cdot m}}$

$f0 := \dfrac{F0}{m} \qquad \zeta = 0.034$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \qquad Y := X$

$f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$

$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \left[\alpha \cdot (X_0)^3 + f(t) \right] \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0 \right) + f(t) \end{bmatrix}$$

$Z := rkfixed(X, 0, 10, 2000, D)$

$t := Z^{<0>} \qquad x := Z^{<1>}$

$W := rkfixed(Y, 0, 10, 2000, L)$

$xL := W^{<1>}$

Note that as the step input last for a longer time, the response of the linear and the nonlinear becomes much different.

3.67*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring stiffness is of the form

$$k(x) = kx - k_1 x^2$$

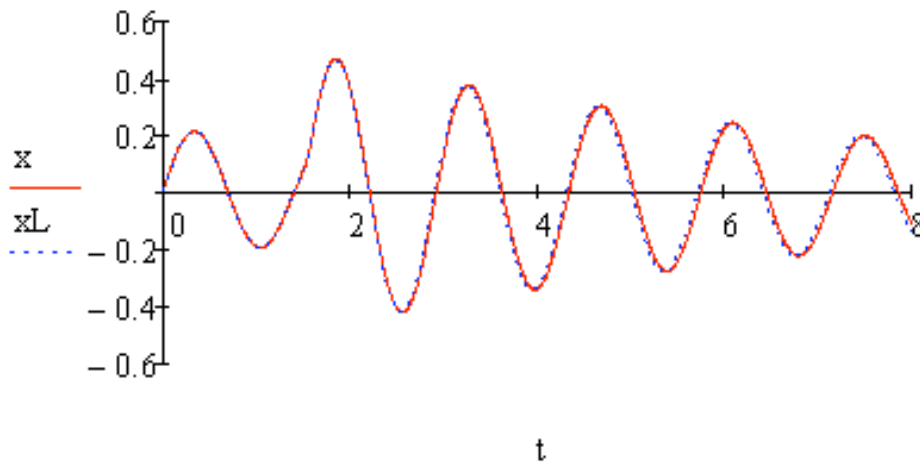and the system is subject to a excitation of the form ($t_1 = 1.5$ and $t_2 = 2.5$)

$$F(t) = 1500\left[\Phi(t - t_1) - \Phi(t - t_2)\right] \text{ N}$$

initial conditions of $x_0 = 0.01$ m and $v_0 = 1$ m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 450 N/m³. Which system has the largest magnitude?

Solution: The solution is computed in Mathcad as follows:

Prob 3.60

$x0 := 0.01$

$v0 := 1$

$m := 100$ $k := 2000$

$c := 30$

$\alpha := 4.5$ $FO := 1500$ $t1 := 1.5$ $t2 := 2.5$

$\omega n := \sqrt{\dfrac{k}{m}}$

$\zeta := \dfrac{c}{2\sqrt{k \cdot m}}$

$f0 := \dfrac{FO}{m}$ $\zeta = 0.034$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$ $Y := X$

$$f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$$

$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \left[\alpha \cdot (X_0)^2 + f(t)\right] \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$$

$Z := rkfixed(X, 0, 10, 2000, D)$

$t := Z^{<0>}$ $x := Z^{<1>}$

$W := rkfixed(Y, 0, 10, 2000, L)$

$xL := W^{<1>}$

Note that the linear response under predicts the actual response

3.68*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring stiffness is of the form
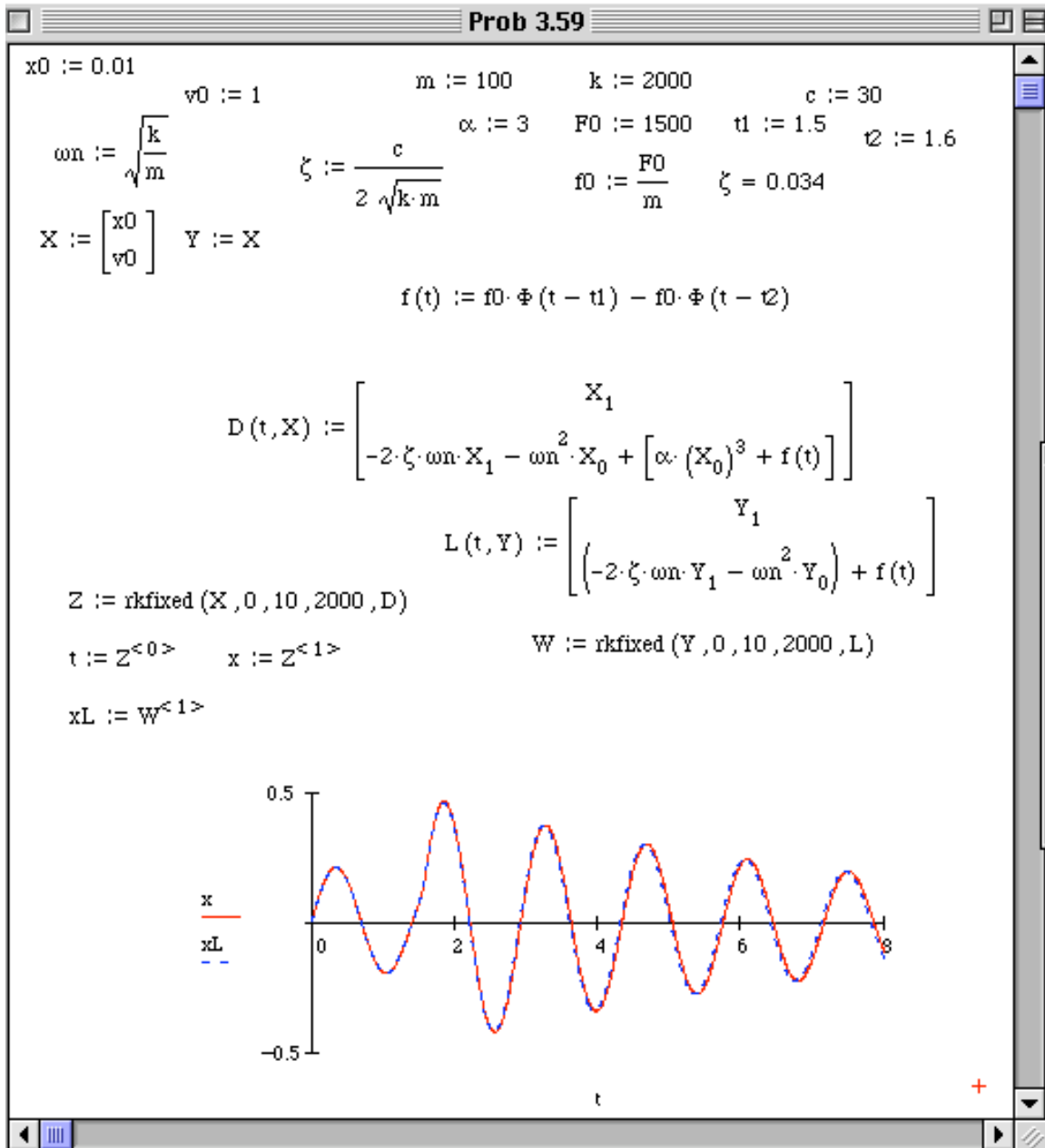
$$k(x) = kx + k_1 x^2$$

and the system is subject to a excitation of the form ($t_1 = 1.5$ and $t_2 = 2.5$)
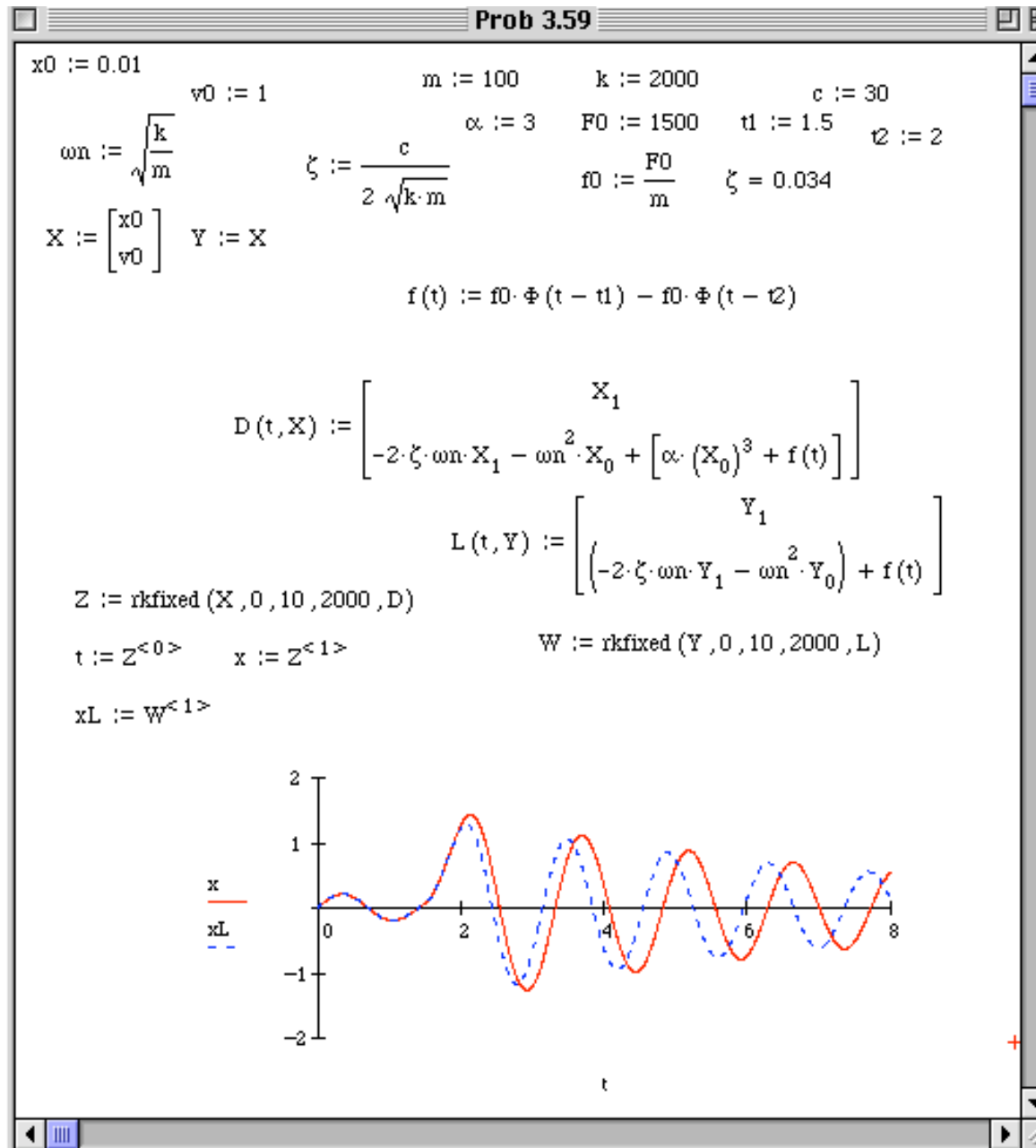
$$F(t) = 1500\left[\Phi(t - t_1) - \Phi(t - t_2)\right] \text{ N}$$

initial conditions of $x_0 = 0.01$ m and $v_0 = 1$ m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 450 N/m³. Which system has the largest magnitude?
Solution: The solution is calculated in Mathcad as follows:

$$x0 := 0.01 \qquad m := 100 \qquad k := 2000$$

$$\omega n := \sqrt{\frac{k}{m}} \qquad v0 := 1 \qquad \alpha := 4.5 \qquad F0 := 1500 \quad t1 := 1.5 \qquad c := 30$$

$$X := \begin{pmatrix} x0 \\ v0 \end{pmatrix} \quad Y := X \qquad \zeta := \frac{c}{2\sqrt{k \cdot m}} \qquad f0 := \frac{F0}{m} \qquad \begin{matrix} t2 := 2.5 \\ \\ \zeta = 0.034 \end{matrix}$$

$$f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$$

$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \left[-\alpha \cdot (X_0)^2 + f(t)\right] \end{bmatrix} \qquad +$$

$$Z := \text{rkfixed}(X, 0, 10, 2000, D) \qquad L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$$

$$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle}$$

$$xL := W^{\langle 1 \rangle} \qquad W := \text{rkfixed}(Y, 0, 10, 2000, L)$$

In this case (compared to the hardening spring of the previous problem, the linear response over predicts the time history.

3.69*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring stiffness is of the form
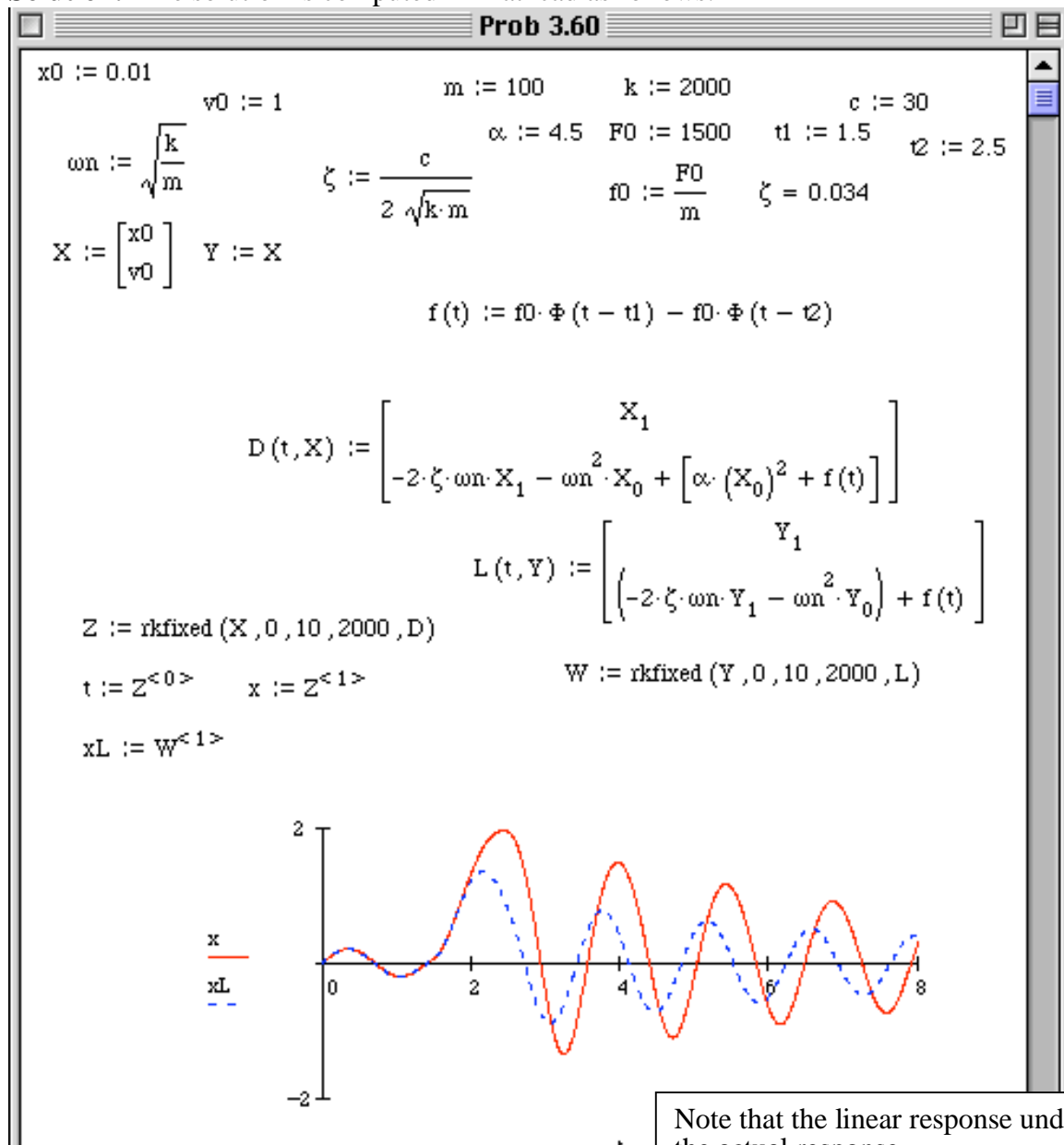
$$k(x) = kx - k_1 x^2$$

and the system is subject to a excitation of the form ($t_1 = 1.5$ and $t_2 = 2.5$)

$$F(t) = 150\big[\Phi(t - t_1) - \Phi(t - t_2)\big] \, \text{N}$$

initial conditions of $x_0 = 0.01$ m and $v_0 = 1$ m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 5500 N/m^3. Which system has the largest magnitude transient? Which has the largest magnitude in steady state?

Solution: The solution in Mathcad is given below. Note that the linear system response is less than that of the nonlinear system, and hence underestimates the actual response.
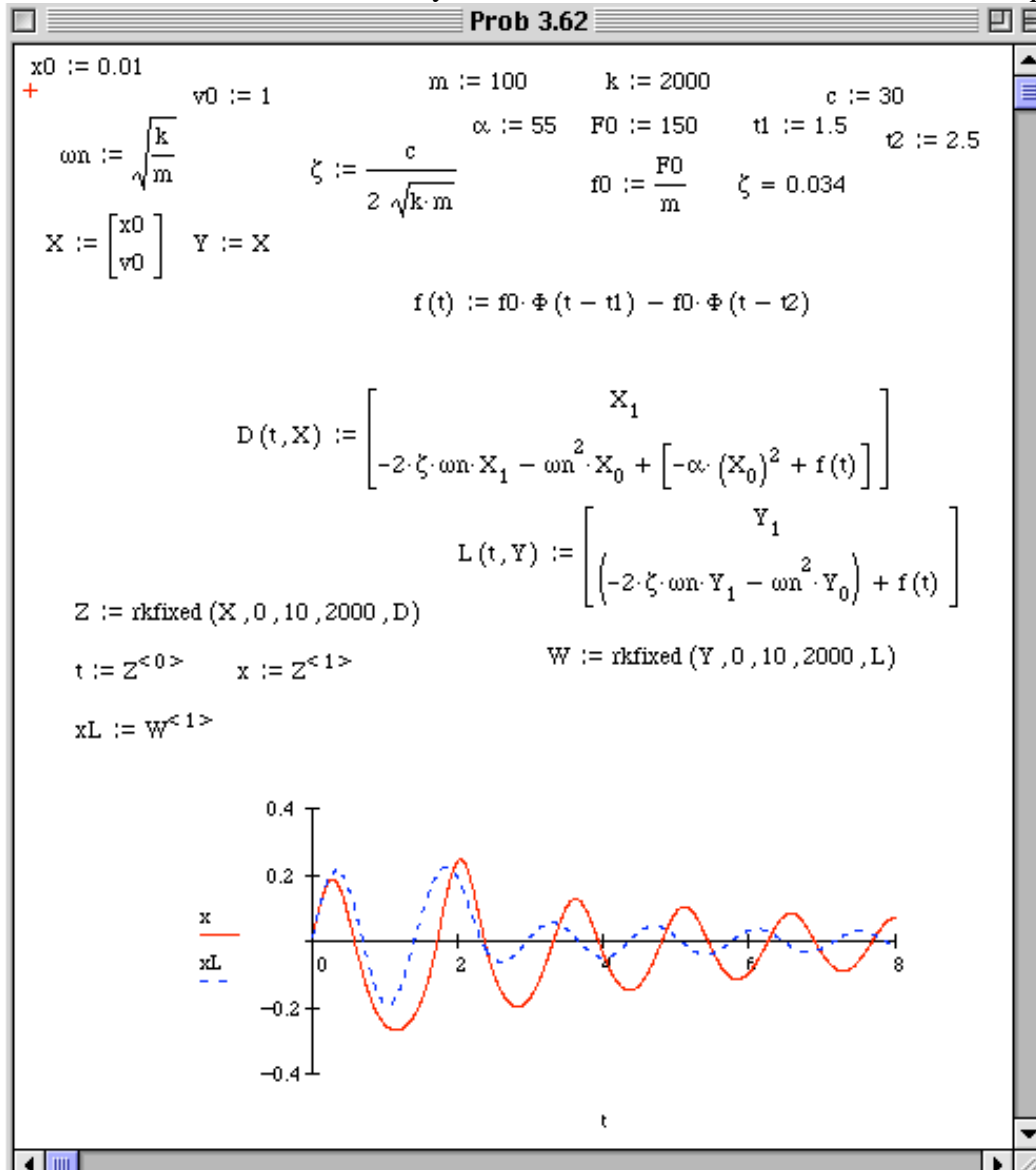
3.70*. Compare the forced response of a system with velocity squared damping as defined in equation (2.129) using numerical simulation of the nonlinear equation to that of the response of the linear system obtained using equivalent viscous damping as
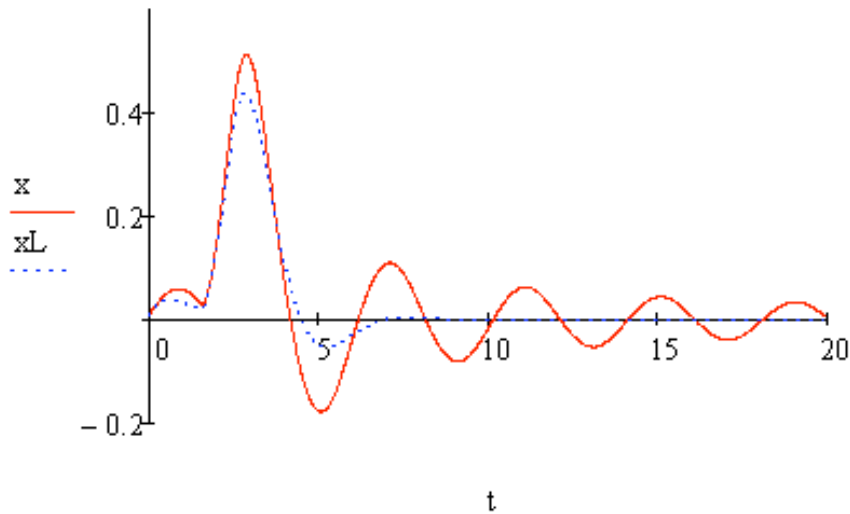
defined by equation (2.131). Use as initial conditions, $x_0 = 0.01$ m and $v_0 = 0.1$ m/s with a mass of 10 kg, stiffness of 25 N/m, applied force of the form ($t_1 = 1.5$ and $t_2 = 2.5$)

$$F(t) = 15[\Phi(t - t_1) - \Phi(t - t_2)] \text{ N}$$

and drag coefficient of $\alpha = 25$.

Solution: The solution calculated in Mathcad is given in the follow:

$x0 := 0.01$ $v0 := 0.1$ $m := 10$ $k := 25$ $\alpha := 25$ $F0 := 15$

$\omega n := \sqrt{\dfrac{k}{m}}$ $f0 := \dfrac{F0}{m}$ $+$

$t1 := 1.5$ $t2 := 2.5$ $ceq := \sqrt{\dfrac{8 \cdot \alpha \cdot m}{3 \cdot \pi}} \cdot f0$

$\zeta := \dfrac{ceq}{2\sqrt{k \cdot m}}$ $f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$

$X := \begin{pmatrix} x0 \\ v0 \end{pmatrix}$

$D(t,X) := \left[\begin{array}{c} X_1 \\ -\omega n^2 \cdot X_0 - \dfrac{\alpha}{m} \cdot (X_1)^2 \cdot \dfrac{X_1}{|X_1|} + f(t) \end{array} \right]$ $Y := X$

$\zeta = 0.564$

$Z := \text{rkfixed}(X, 0, 20, 2000, D)$

$D1(t,Y) := \left[\begin{array}{c} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{array} \right]$

$t := Z^{\langle 0 \rangle}$ $x := Z^{\langle 1 \rangle}$

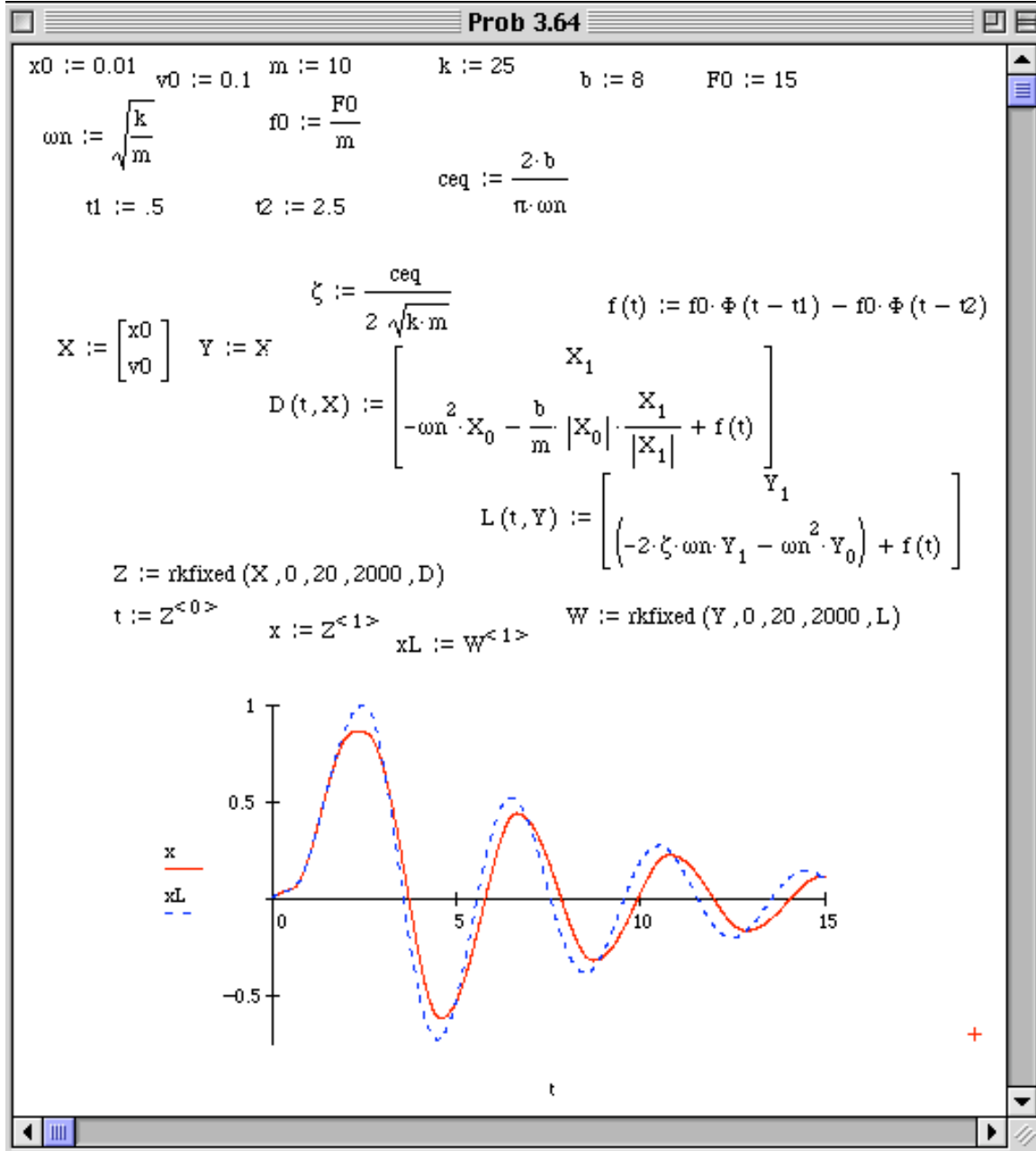$xL := W^{\langle 1 \rangle}$ $W := \text{rkfixed}(Y, 0, 20, 2000, D1)$

Note that the linear solution is very different from the nonlinear solution and dies out more rapidly.

3.71*. Compare the forced response of a system with structural damping (see table 2.2) using numerical simulation of the nonlinear equation to that of the response of the linear system obtained using equivalent viscous damping as defined in Table 2.2. Use the initial conditions, $x_0 = 0.01$ m and $v_0 = 0.1$ m/s with a mass of 10 kg, stiffness of 25 N/m, applied force of the form ($t_1 = 1.5$ and $t_2 = 2.5$)

$$F(t) = 15\left[\Phi(t - t_1) - \Phi(t - t_2)\right] \text{ N}$$

and solid damping coefficient of $b = 8$. Does the equivalent viscous damping linearization, over estimate the response or under estimate it?

Solution: The solution is calculated in Mathcad as follows. Note that the linear solution is an over estimate of the nonlinear response in this case.

```
Prob 3.64
```

$x0 := 0.01$ $v0 := 0.1$ $m := 10$ $k := 25$ $b := 8$ $F0 := 15$

$\omega n := \sqrt{\dfrac{k}{m}}$ $f0 := \dfrac{F0}{m}$

$ceq := \dfrac{2 \cdot b}{\pi \cdot \omega n}$

$t1 := .5$ $t2 := 2.5$

$\zeta := \dfrac{ceq}{2\sqrt{k \cdot m}}$ $f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$ $Y := X$

$D(t, X) := \begin{bmatrix} X_1 \\ -\omega n^2 \cdot X_0 - \dfrac{b}{m} \cdot |X_0| \cdot \dfrac{X_1}{|X_1|} + f(t) \end{bmatrix}$

$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$

$Z := rkfixed(X, 0, 20, 2000, D)$

$t := Z^{<0>}$ $x := Z^{<1>}$ $xL := W^{<1>}$ $W := rkfixed(Y, 0, 20, 2000, L)$

Problems and Solutions for Section 4.1 (4.1 through 4.16)

4.1 Consider the system of Figure P4.1. For $c_1 = c_2 = c_3 = 0$, derive the equation of motion and calculate the mass and stiffness matrices. Note that setting $k_3 = 0$ in your solution should result in the stiffness matrix given by Eq. (4.9).
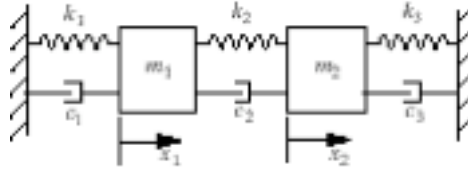
Solution:

For mass 1:

$$m_1\ddot{x}_1 = -k_1 x_1 + k_2\left(x_2 - x_1\right)$$
$$\Rightarrow m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = 0$$

For mass 2:

$$m_2\ddot{x}_2 = -k_3 x_2 - k_2\left(x_2 - x_1\right)$$
$$\Rightarrow m_2\ddot{x}_2 - k_2 x_1 + \left(k_2 + k_3\right)x_2 = 0$$

So, $M\ddot{x} + Kx = 0$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{x} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}x = 0$$

Thus:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

4.2   Calculate the characteristic equation from problem 4.1 for the case

$$m_1 = 9 \text{ kg} \quad m_2 = 1 \text{ kg} \quad k_1 = 24 \text{ N/m} \quad k_2 = 3 \text{ N/m} \quad k_3 = 3 \text{ N/m}$$

and solve for the system's natural frequencies.

Solution: Characteristic equation is found from Eq. (4.9):

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_1 + k_2 + k_3 \end{vmatrix} = \begin{vmatrix} -9\omega^2 + 27 & -3 \\ -3 & -\omega^2 + 6 \end{vmatrix} = 0$$

$$9\omega^4 - 81\omega^2 + 153 = 0$$

Solving for ω:

$$\omega_1 = \mathbf{1.642}$$
$$\text{rad/s}$$
$$\omega_2 = \mathbf{2.511}$$

4.3   Calculate the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ for problem 4.2.

Solution: Calculate $\mathbf{u}_1$:

$$\begin{bmatrix} \left(-2.697\right)\left(9\right) + 27 & -3 \\ -3 & -2.697 + 6 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This yields

$$2.727u_{11} - 3u_{21} = 0$$
$$-3u_{11} + 3.303u_{21} = 0 \qquad \text{or, } u_{21} = 0.909u_{11}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0.909 \end{bmatrix}$$

Calculate $\mathbf{u}_2$:

$$\begin{bmatrix} \left(-6.303\right)\left(9\right) + 27 & -3 \\ -3 & -6.303 + 6 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-29.727u_{12} - 3u_{22} = 0$$

$$-3u_{12} = 0.303u_{22} = 0 \qquad \text{or, } u_{12} = -0.101u_{22}$$

This yields

$$\mathbf{u}_2 = \begin{bmatrix} -0.101 \\ 1 \end{bmatrix}$$

4.4 For initial conditions $\mathbf{x}(0) = [1 \ 0]^T$ and $\dot{\mathbf{x}}(0) = [0 \ 0]^T$ calculate the free response of the system of Problem 4.2. Plot the response x_1 and x_2.

Solution: Given $\mathbf{x}(0) = [1 \ 0]^T$, $\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, The solution is

$$\mathbf{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\mathbf{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\mathbf{u}_2$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) - 0.101A_2 \sin(\omega_2 t + \phi_2) \\ 0.909 A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}$$

Using initial conditions,

$$1 = A_1 \sin\phi_1 - 0.101A_2 \sin\phi_2 \qquad [1]$$
$$0 = 0.909 A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = 1.642A_1 \cos\phi_1 - 0.2536A_2 \cos\phi_2 \quad [3]$$
$$0 = 6.033A_1 \cos\phi_1 + 2.511A_2 \cos\phi_2 \quad [4]$$

From [3] and [4], $\qquad \phi_1 = \phi_2 = \pi/2$

From [1] and [2], $\qquad A_1 = 0.916, \text{ and } A_2 = -0.833$

So,

$$x_1(t) = 0.916\sin(1.642t + \pi/2) + 0.0841\sin(2.511t + \pi/2)$$
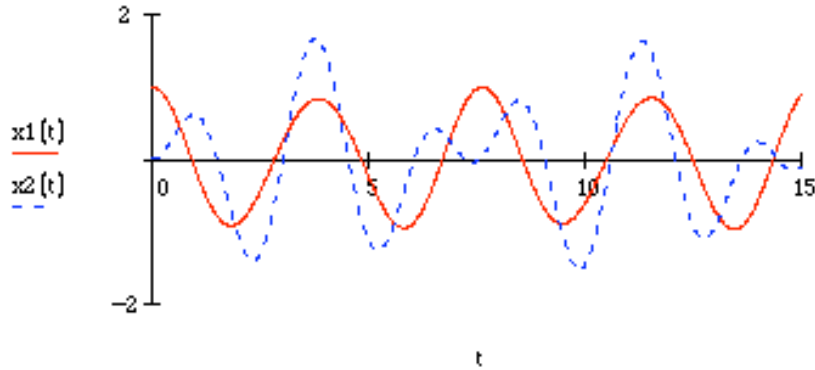$$x_2(t) = 0.833\sin(1.642t + \pi/2) - 0.833\sin(2.511t + \pi/2)$$

$$x_1(t) = 0.916\cos1.642t + 0.0841\cos2.511t$$
$$x_2(t) = 0.833(\cos1.642t - \cos2.511t)$$

$x1 (t) := 0.916 \cdot \cos (1.642 \cdot t) + 0.0841 \cdot \cos (2.511 \cdot t)$

$x2 (t) := 0.833 \cdot (\cos (1.642 \cdot t) - \cos (2.511 \cdot t))$

4.5 Calculate the response of the system of Example 4.1.7 to the initial condition $\mathbf{x}(0) = 0, \dot{\mathbf{x}}$
$(0) = [1 \ 0]^T$, plot the response and compare the result to Figure 4.3.

Solution: Given: $\mathbf{x}(0) = \mathbf{0}, \ \dot{\mathbf{x}}(0) = [1 \ 0]^T$

From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3} A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \dfrac{1}{3} A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

Using initial conditions:

$$\begin{array}{ll} 0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 & [1] \\ 0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 & [2] \\ 3 = \sqrt{2}A_1 \cos\phi_1 - 2A_2 \cos\phi_2 & [3] \\ 0 = \sqrt{2}A_1 \cos\phi_1 + 2A_2 \cos\phi_2 & [4] \end{array}$$

From [1] and [2]:

$$\phi_1 = \phi_2 = 0$$

From [3] and [4]:

$$A_1 = \frac{3\sqrt{2}}{4}, \text{ and } A_2 = -\frac{3}{4}$$
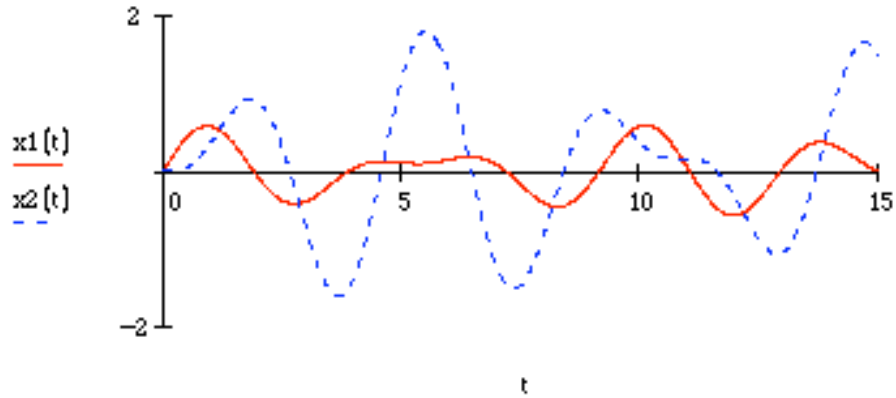
The solution is

$$x_1(t) = 0.25\left(\sqrt{2}\sin\sqrt{2}t + \sin 2t\right)$$

$$x_2(t) = 0.75\left(\sqrt{2}\sin\sqrt{2}t - \sin 2t\right)$$

As in Fig. 4.3, the second mass has a larger displacement than the first mass.

$$x1\ (t) := 0.25 \cdot \left(\sqrt{2} \cdot \sin\left(\sqrt{2} \cdot t\right) + \sin\left(2 \cdot t\right)\right) \quad x2\ (t) := 0.75 \cdot \left(\sqrt{2} \cdot \sin\left(\sqrt{2} \cdot t\right) - \sin\left(2 \cdot t\right)\right)$$

4.6 Repeat Problem 4.1 for the case that $k_1 = k_3 = 0$.

Solution:

The equations of motion are

$$m_1\ddot{x}_1 + k_2 x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

So, $M\ddot{x} + Kx = 0$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{x} + \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}x = 0$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \text{ and } K = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

4.7 Calculate and solve the characteristic equation for Problem 4.6 with $m_1 = 9$, $m_2 = 1$, $k_2 = 10$.

Solution:

The characteristic equation is found from Eq. (4.19):

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -9\omega^2 + 10 & -10 \\ -10 & -\omega^2 + 10 \end{vmatrix} = 9\omega^4 - 100\omega^2 = 0$$

$$\omega_{1,2}^2 = 0, 11.111$$

$$\omega_1 = 0$$

$$\omega_2 = 3.333$$

4.8   Compute the natural frequencies of the following system:

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \ddot{\mathbf{x}}(t) + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x}(t) = 0 .$$

Solution:

$$\det(-\omega^2 M + K) = \det\left(-\omega^2 \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right) = 20\omega^4 - 22\omega^2 + 2 = 0, \ \omega^2 = 0.1, \ 1$$

$$\underline{\omega_{1,2} = 0.316, \ 1 \ \text{rad/s}}$$

4.9   Calculate the solution to the problem of Example 4.1.7, to the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Plot the response and compare it to that of Fig. 4.3.

Solution: Given: $\mathbf{x}(0) = \begin{bmatrix} 1/3 & 1 \end{bmatrix}^T$, $\dot{\mathbf{x}}(0) = 0$

From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3} A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \dfrac{1}{3} A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

Using initial conditions:

$$1 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$1 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = \sqrt{2} A_1 \cos\phi_1 - 2 A_2 \cos\phi_2 \quad [3]$$
$$0 = \sqrt{2} A_1 \cos\phi_1 + 2 A_2 \cos\phi_2 \quad [4]$$

From [3] and [4]: $\phi_1 = \phi_2 = \dfrac{\pi}{2}$

From [1] and [2]: $A_1 = 1$, and $A_2 = 0$

The solution is

$$x_1(t) = \frac{1}{3}\cos\sqrt{2}t$$

$$x_2(t) = \cos\sqrt{2}t$$

In this problem, both masses oscillate at only one frequency.

4.10 Calculate the solution to Example 4.1.7 for the initial condition

$$x(0) = \begin{bmatrix} -\dfrac{1}{3} \\ 1 \end{bmatrix} \quad \dot{x}(0) = 0$$

Solution:

Given: $\mathbf{x}(0) = [-1/3 \quad 1]^T$, $\dot{\mathbf{x}}(0) = 0$

From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3} A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \dfrac{1}{3} A_2 \left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

Using initial conditions:

$$\begin{array}{ll} -1 = A_1 \sin\phi_1 - A_2 \sin\phi_2 & [1] \\ 1 = A_1 \sin\phi_1 + A_2 \sin\phi_2 & [2] \\ 0 = \sqrt{2}A_1 \cos\phi_1 - 2A_2 \cos\phi_2 & [3] \\ 0 = \sqrt{2}A_1 \cos\phi_1 + 2A_2 \cos\phi_2 & [4] \end{array}$$

From [3] and [4]

$$\phi_1 = \phi_2 = \frac{\pi}{2}$$

From [1] and [2]:

$$A_1 = 0$$
$$A_2 = 1$$

The solution is

$$x_1(t) = -\frac{1}{3}\cos 2t$$
$$x_2(t) = \cos 2t$$

In this problem, both masses oscillate at only one frequency (not the same frequency as in Problem 4.9, though.)

4.11 Determine the equation of motion in matrix form, then calculate the natural frequencies and mode shapes of the torsional system of Figure P4.11. Assume that the torsional stiffness values provided by the shaft are equal $(k_1 = k_2)$ and that disk 1 has three times the inertia as that of disk $2(J_1 = 3J_2)$.
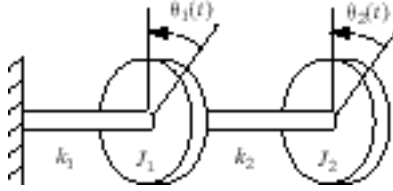
Solution: Let $k = k_1 = k_2$ and $J_1 = 3J_2$. The equations of motion are

$$J_1\ddot{\theta}_1 + 2k\theta_1 - k\theta_2 = 0$$
$$J_2\ddot{\theta}_2 - k\theta_1 + k\theta_2 = 0$$

So,

$$J_2\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\theta} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\theta = 0$$

Calculate the natural frequencies:

$$\det\left(-\omega^2 J + K\right) = \begin{vmatrix} -3\omega^2 J_2 + 2k & -k \\ -k & -\omega^2 J_2 + k \end{vmatrix} = 0$$

$$\omega_1 = 0.482\sqrt{\frac{k}{J_2}}$$

$$\omega_2 = 1.198\sqrt{\frac{k}{J_2}}$$

Calculate the mode shapes: mode shape 1:

$$\begin{bmatrix} -3(0.2324)k + 2k & -k \\ -k & -(0.2324)k + k \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = 0$$

$$u_{11} = 0.7676u_{12}$$

So, $\mathbf{u}_1 = \begin{bmatrix} 0.7676 \\ 1 \end{bmatrix}$

mode shape 2:

$$\begin{bmatrix} -3(1.434)k + 2k & -k \\ -k & -(1.434)k + k \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = 0$$

$$u_{21} = -0.434u_{22}$$

So, $\mathbf{u}_2 = \begin{bmatrix} -0.434 \\ 1 \end{bmatrix}$

4.12 Two subway cars of Fig. P4.12 have 2000 kg mass each and are connected by a coupler. The coupler can be modeled as a spring of stiffness $k = 280,000$ N/m. Write the equation of motion and calculate the natural frequencies and (normalized) mode shapes.
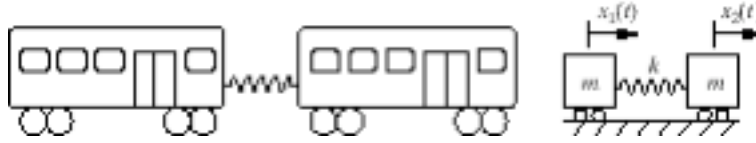
Solution: Given: $m_1 = m_2 = m = 2000$ kg $k = 280,000$ N/m

The equations of motion are:

$$m\ddot{x}_1 + kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 - kx_1 + kx_2 = 0$$

In matrix form this becomes:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \ddot{x} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} x = 0$$

$$\begin{bmatrix} 2000 & 0 \\ 0 & 2000 \end{bmatrix} \ddot{x} + \begin{bmatrix} 280,000 & -280,000 \\ -280,000 & 280,000 \end{bmatrix} x = 0$$

Natural frequencies:

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -2000\omega^2 + 280,000 & -280,000 \\ -280,000 & -2000\omega^2 + 280,000 \end{vmatrix} = 0$$

$$4 \times 10^6 \omega^4 - 1.12 \times 10^9 \omega^2 = 0$$

$$\omega^2 = 0, 280 \Rightarrow \omega_1 = 0 \text{ rad/sec and } \omega_2 = 16.73 \text{ rad/sec}$$

Mode shapes:

Mode 1, $\omega_1^2 = 0$

$$\begin{bmatrix} 280,000 & -280,000 \\ -280,000 & 280,000 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore u_{11} = u_{12}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 280$

$$\begin{bmatrix} -280,000 & -280,000 \\ -280,000 & -280,000 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore u_{21} = u_{22}$$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Normalizing the mode shapes yields

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note that $\mathbf{u}_2 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is also acceptable because a mode shape times a constant (-1 in this case) is still a mode shape.

4.13 Suppose that the subway cars of Problem 4.12 are given the initial position of $x_{10} =$ 0, $x_{20} = 0.1$ m and initial velocities of $v_{10} = v_{20} = 0$. Calculate the response of the cars.

Solution:

Given: $\mathbf{x}(0) = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T$, $\dot{\mathbf{x}}(0) = \mathbf{0}$

From problem 12,

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\omega_1 = 0 \text{ rad/s and } \omega_2 = 16.73 \text{ rad/s}$$

The solution is

$$\mathbf{x}(t) = \left(c_1 + c_2 t\right)\mathbf{u}_1 + A\sin\left(16.73t + \phi\right)\mathbf{u}_2$$

$$\Rightarrow \dot{\mathbf{x}}(0) = c_2\mathbf{u}_1 + 16.73A\cos\left(\phi\right)\mathbf{u}_2 \text{ and } \mathbf{x}(0) = c_1\mathbf{u}_1 + A\sin\left(\phi\right)\mathbf{u}_2$$

Using initial the conditions four equations in four unknowns result:

$$0 = c_1 + A\sin\phi \qquad \begin{bmatrix} 1 \end{bmatrix}$$
$$0.1 = c_1 - A\sin\phi \qquad \begin{bmatrix} 2 \end{bmatrix}$$
$$0 = c_2 + 16.73A\cos\phi \quad \begin{bmatrix} 3 \end{bmatrix}$$
$$0 = c_2 - 16.73A\cos\phi \quad \begin{bmatrix} 4 \end{bmatrix}$$

From [3] and [4]: $c_2 = 0$, and $\phi = \dfrac{\pi}{2}$ rad

From [1] and [2]: $c_1 = 0.05$ m and $A = -0.05$ m

The solution is

$$x_1(t) = 0.05 - 0.05\cos 16.73t$$
$$x_2(t) = 0.05 + 0.05\cos 16.73t$$

Note that if $\mathbf{u}_2 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is chosen as the second mode shape the answer will remain the same. It might be worth presenting both solutions in class, as students are often skeptical that the two choices will yield the same result.

4.14 A slightly more sophisticated model of a vehicle suspension system is given in Figure P4.14. Write the equations of motion in matrix form. Calculate the natural frequencies for $k_1 = 10^3$ N/m, $k_2 = 10^4$ N/m, $m_2 = 50$ kg, and $m_1 = 2000$ kg.
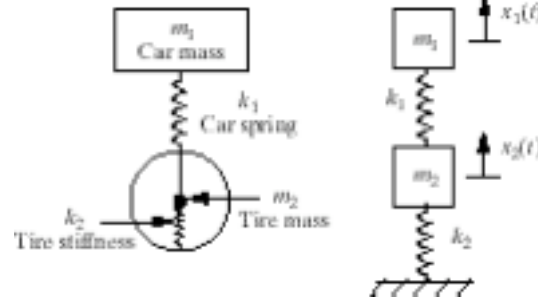
Solution: The equations of motion are

$$2000\ddot{x}_1 + 1000x_1 - 1000x_2 = 0$$

$$50\ddot{x}_2 - 1000x_1 + 11{,}000x_2 = 0$$

In matrix form this becomes:

$$\begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Natural frequencies:

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -2000\omega^2 + 1000 & -1000 \\ -1000 & -50\omega^2 + 11{,}000 \end{vmatrix} = 100{,}000\omega^4 - 2.205 \times 10^7 \omega^2 + 10^7 = 0$$

$$\omega_{1,2}^2 = 0.454, \ \ 220.046 \ \ \Rightarrow \omega_1 = 0.674 \text{ rad/s} \ \ \text{and} \ \ \omega_2 = 14.8 \text{ rad/s}$$

4.15 Examine the effect of the initial condition of the system of Figure 4.1(a) on the responses x_1 and x_2 by repeating the solution of Example 4.1.7, first for $x_{10} = 0, x_{20} = 1$ with $\dot{x}_{10} = \dot{x}_{20} = 0$ and then for $x_{10} = x_{20} = \dot{x}_{10} = 0$ and $\dot{x}_{20} = 1$. Plot the time response in each case and compare your results against Figure 4.3.

Solution: From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3} A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \dfrac{1}{3} A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

(a) $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $\dot{\mathbf{x}}(0) = \mathbf{0}$. Using the initial conditions:

$$0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$1 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = \sqrt{2} A_1 \cos\phi_1 - 2 A_2 \cos\phi_2 \qquad [3]$$
$$0 = \sqrt{2} A_1 \cos\phi_1 + 2 A_2 \cos\phi_2 \qquad [4]$$

From [3] and [4] $\phi_1 = \phi_2 = \dfrac{\pi}{2}$
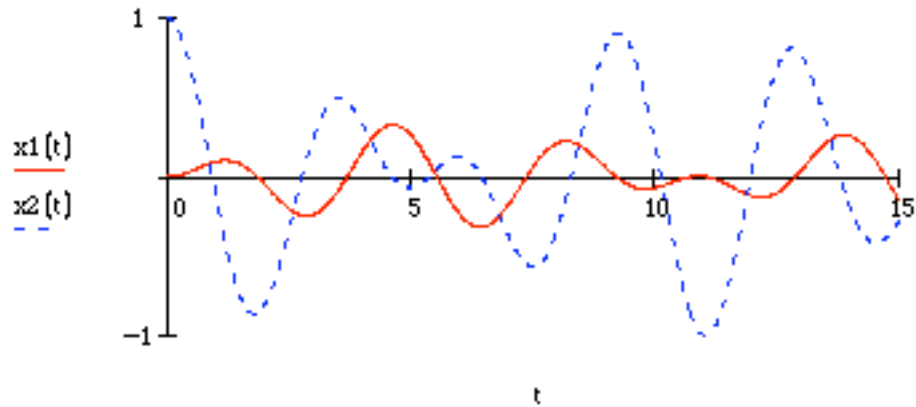
From [1] and [2] $A_1 = A_2 = \dfrac{1}{2}$

The solution is

$$x_1(t) = \frac{1}{6}\cos\sqrt{2}t - \frac{1}{6}\cos 2t$$

$$x_2(t) = \frac{1}{2}\cos\sqrt{2}t + \frac{1}{2}\cos 2t$$

This is similar to the response of Fig. 4.3

$$x1(t) := \frac{1}{6} \cdot \left(\cos\left(\sqrt{2} \cdot t \right) - \cos(2 \cdot t) \right) \qquad x2(t) := \frac{1}{2} \cdot \left(\cos\left(\sqrt{2} \cdot t \right) + \cos(2 \cdot t) \right)$$

(b) $\mathbf{x}(0) = 0$, $\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Using these initial conditions:

$$0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = \sqrt{2} A_1 \cos\phi_1 - 2 A_2 \cos\phi_2 \qquad [3]$$
$$1 = \sqrt{2} A_1 \cos\phi_1 + 2 A_2 \cos\phi_2 \qquad [4]$$

From [1] and [2] $\phi_1 = \phi_2 = 0$

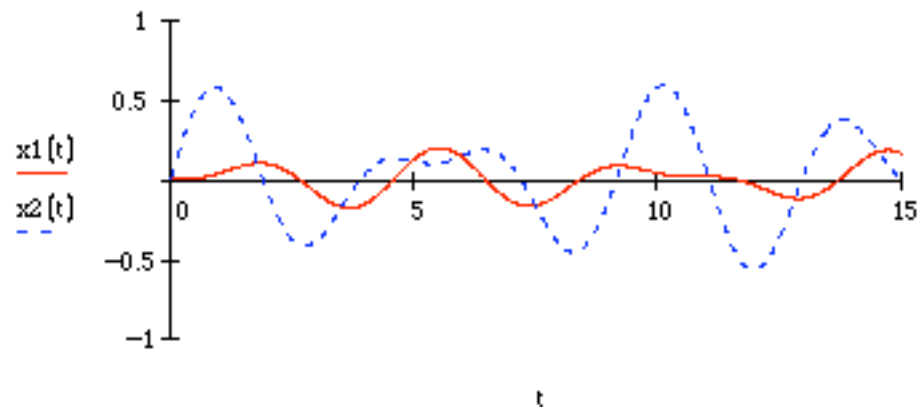From [3] and [4] $A_1 = \dfrac{\sqrt{2}}{4}$, and $A_2 = \dfrac{1}{4}$

The solution is

$$x_1(t) = \frac{\sqrt{2}}{12} \sin \sqrt{2} t - \frac{1}{12} \sin 2t$$

$$x_2(t) = \frac{\sqrt{2}}{4} \sin \sqrt{2} t + \frac{1}{4} \sin 2t$$

This is also similar to the response of Fig. 4.3

$$x1(t) := \frac{\sqrt{2}}{12} \cdot \sin\left(\sqrt{2} \cdot t\right) - \frac{1}{12} \cdot \sin(2 \cdot t) \qquad x2(t) := \frac{\sqrt{2}}{4} \cdot \sin\left(\sqrt{2} \cdot t\right) + \frac{1}{4} \cdot \sin(2 \cdot t)$$

4.16 Refer to the system of Figure 4.1(a). Using the initial conditions of Example 4.1.7, resolve and plot $x_1(t)$ for the cases that k_2 takes on the values 0.3, 30, and 300. In each case compare the plots of x_1 and x_2 to those obtained in Figure 4.3. What can you conclude?

Solution: Let $k_2 = 0.3$, 30, 300 for the example(s) in Section 4.1. Given

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \text{ mm}, \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

$$m_1 = 9, m_2 = 1, k_1 = 24$$

Equation of motion becomes:

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \ddot{x} + \begin{bmatrix} 24 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} x = 0$$

(a) $k_2 = 0.3$

$$\det\left(-\omega^2 M + K\right) = \begin{vmatrix} -9\omega^2 + 24.3 & -0.3 \\ -0.3 & -\omega^2 + 0.3 \end{vmatrix} = 9\omega^4 - 27\omega^2 + 7.2 = 0$$

$$\omega^2 = 0.2598, 2.7042$$

$$\omega_1 = 0.5439$$

$$\omega_2 = 1.6444$$

Mode shapes:

Mode 1, $\omega_1^2 = 0.2958$

$$\begin{bmatrix} 21.6374 & -0.3 \\ -0.3 & 0.004159 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$21.6374u_{11} - 0.3u_{12} = 0$$

$$u_{11} = 0.01386u_{12}$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.01386 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 2.7042$

$$\begin{bmatrix} -0.03744 & -0.3 \\ -0.3 & 2.4042 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-0.3u_{21} = 2.4042u_{22}$$

$$u_{22} = -0.1248u_{21}$$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ -0.1248 \end{bmatrix}$$

The solution is $\quad x(t) = A_1 \sin\left(\omega_1 t + \phi_1\right)u_1 + A_2 \sin\left(\omega_2 t + \phi_2\right)u_2$

Using initial conditions

$$1 = A_1 \left(0.01386\right)\sin\phi_1 + A_2 \sin\phi_2 \qquad\qquad [1]$$
$$0 = A_1 \sin\phi_1 + A_2 \left(-0.1248\right)\sin\phi_2 \qquad\qquad [2]$$
$$0 = A_1 \left(0.01386\right)\left(0.5439\right)\cos\phi_1 + A_2 \left(1.6444\right)\cos\phi_2 \qquad [3]$$
$$0 = A_1 \left(0.5439\right)\cos\phi_1 + A_2 \left(1.6444\right)\left(-0.1248\right)\cos\phi_2 \qquad [4]$$

From [3] and [4],

$$\phi_1 = \phi_2 = \pi/2$$
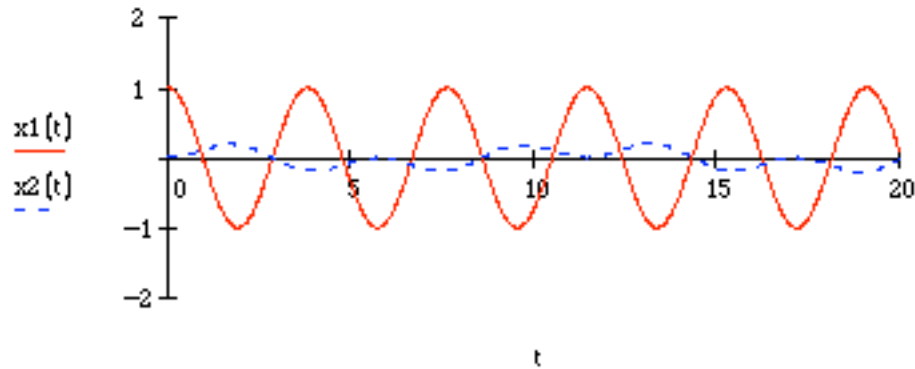
From [1] and [2],

$$A_1 = 0.1246$$
$$A_2 = 0.9983$$

So,

$$x_1(t) = 0.001727\cos(0.5439t) + 0.9983\cos\left(1.6444t\right) \text{ mm}$$
$$x_2(t) = 0.1246\left[\cos\left(0.5439t\right) - \cos\left(1.6444t\right)\right] \text{ mm}$$

$\mathrm{x1}\,(t) := 0.001727 \cdot \cos\,(0.5439 \cdot t) + 0.9983 \cdot \cos\,(1.644 \cdot t)$

$\mathrm{x2}\,(t) := 0.1246 \cdot (\cos\,(0.5439 \cdot t) - \cos\,(1.644 \cdot t))$

(b) $k_2 = 30$

$$\det\left(-\omega^2 M + K\right) = \begin{vmatrix} -9\omega^2 + 54 & -30 \\ -30 & -\omega^2 + 30 \end{vmatrix} = 9\omega^4 - 32\omega^2 + 720 = 0$$

$\omega^2 = 2.3795, 33.6205$

$\omega_1 = 1.5426$

$\omega_2 = 5.7983$

Mode shapes:

Mode 1, $\omega_1^2 = 2.3795$

$$\begin{bmatrix} 32.5845 & -30 \\ -30 & 27.6205 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$30u_{11} = 27.6205u_{12}$

$u_{11} = 0.9207u_{12}$

$$\mathbf{u}_1 = \begin{bmatrix} 0.9207 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 33.6205$

$$\begin{bmatrix} -248.5845 & -30 \\ -30 & -3.6205 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$30u_{21} = -3/6205u_{22}$

$u_{21} = -0.1207u_{22}$

$$\mathbf{u}_2 = \begin{bmatrix} -0.1207 \\ 1 \end{bmatrix}$$

The solution is

$$x(t) = A_1 \sin(\omega_1 t + \phi_1) u_1 + A_2 \sin(\omega_2 t + \phi_2) u_2$$

Using initial conditions,

$1 = A_1(0.9207)\sin\phi_1 + A_2(-0.1207\sin\phi_2$ $[1]$

$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2$ $[2]$

$0 = A_1(0.9207)(1.5426)\cos\phi_1 + A_2(-0.1207)(5.7983)\cos\phi_2$ $[3]$

$0 = A_1(1.5426)\cos\phi_1 + A_2(5.7983)\cos\phi_2$ $[4]$

From [3] and [4]

$$\phi_1 = \phi_2 = \pi / 2$$
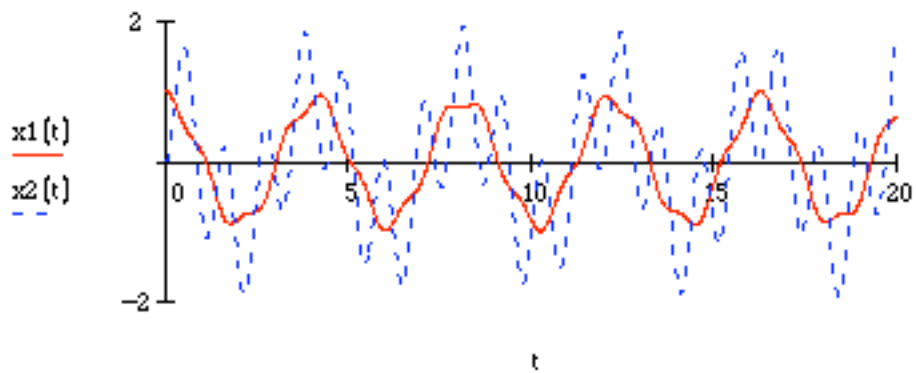
From [1] and [2]

$$A_1 = 0.9602$$
$$A_2 = -0.9602$$

So,

$$x_1(t) = 0.8841\cos(1.5426t) + 0.1159\cos(5.7983t) \text{ mm}$$
$$x_2(t) = 0.9602\left[\cos(1.5426t) - \cos(5.7983t)\right] \text{ mm}$$

x1 (t) := 0.8841·cos (1.5426·t) + 0.1159·cos (5.7983·t)

x2 (t) := 0.9602· (cos (1.5426·t) − cos (5.7983·t))

(c) $k_2 = 300$

$$\det\left(-\omega^2 M + K\right) = \begin{vmatrix} -9\omega^2 + 324 & -300 \\ -300 & -\omega^2 + 300 \end{vmatrix} = 9\omega^4 - 3024\omega^2 + 7200 = 0$$

$$\omega^2 = 2.3981, 333.6019$$
$$\omega_1 = 1.5486$$
$$\omega_2 = 18.2648$$

Mode shapes:

Mode 1, $\omega_1^2 = 2.3981$

$$\begin{bmatrix} 302.4174 & -300 \\ -300 & 297.6019 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$302.4174 u_{11} = 300 u_{12}$$

$$u_{11} = 0.9920 u_{12}$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.9920 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 333.6019$

$$\begin{bmatrix} -2678.4174 & -300 \\ -300 & -33.6019 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$300 u_{21} = 33.6019 u_{22}$$

$$u_{21} = -0.1120 u_{22}$$

$$\mathbf{u}_2 = \begin{bmatrix} -0.1120 \\ 1 \end{bmatrix}$$

The solution is

$$x(t) = A_1 \sin(\omega_1 t + \phi_1) u_1 + A_2 \sin(\omega_2 t + \phi_2) u_2$$

Using initial conditions

$$1 = A_1 (0.9920) \sin\phi_1 + A_2 (-0.1120) \sin\phi_2 \qquad [1]$$
$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = A_1 (0.9920)(1.5486) \cos\phi_1 + A_2 (-0.1120)(18.2648) \quad [3]$$
$$0 = A_1 (1.5486) \cos\phi_1 + A_2 (18.2648) \cos\phi_2 \qquad [4]$$

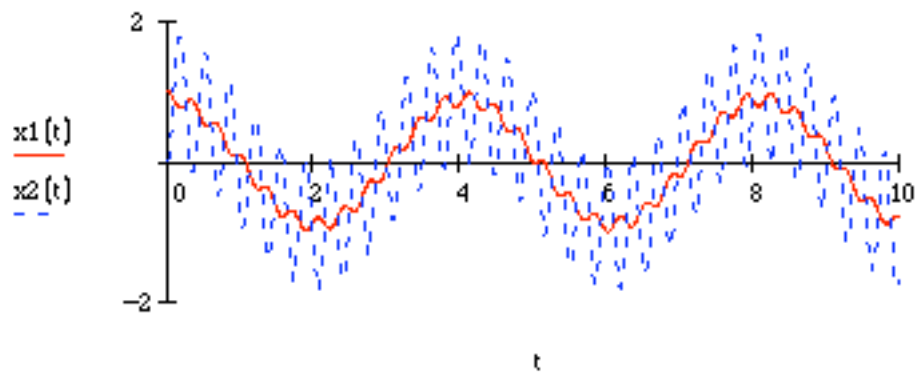From [3] and [4] $\phi_1 = \phi_2 = \pi / 2$
From [1] and [2], $A_1 = 0.9058$ and $A_2 = -0.9058$.
So,

$$x_1(t) = 0.8986 \cos(1.5486t) + 0.1014 \cos(18.2648t) \text{ mm}$$
$$x_2(t) = 0.9058 [\cos(1.5486t) - \cos(18.2648t)] \text{ mm}$$

$$x1(t) := 0.8986 \cdot \cos(1.5486 \cdot t) + 0.1014 \cdot \cos(18.2648 \cdot t)$$

$$x2(t) := 0.9052 \cdot (\cos(1.5486 \cdot t) - \cos(18.2648 \cdot t))$$

As the value of k_2 increases the effect on mass 1 is small, but mass 2 oscillates similar to mass 1 with a superimposed higher frequency oscillation.

4.17 Consider the system of Figure 4.1(a) described in matrix form by Eqs. (4.11), (4.9), and (4.6). Determine the natural frequencies in terms of the parameters m_1, m_2, k_1 and k_2. How do these compare to the two single-degree-of-freedom frequencies $\omega_1 = \sqrt{k_1 / m_1}$ and $\omega_2 = \sqrt{k_2 / m_2}$?

Solution:

The equation of motion is

$$M\ddot{x} + Kx = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{x} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} x = 0$$

The characteristic equation is found from Eq. (4.19):

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{vmatrix}$$

$$m_1 m_2 \omega^4 - \left(k_1 m_2 + k_2\left(m_1 + m_2\right)\right)\omega^2 + k_1 k_2 = 0$$

$$\omega_{1,2}^2 = \frac{k_1 m_2 + k_2\left(m_1 + m_2\right) \pm \sqrt{\left[k_1 m_2 + k_2\left(m_1 + m_2\right)\right]^2 - 4m_1 m_2 k_1 k_2}}{2m_1 m_2}$$

So,

$$\omega_{1,2} = \sqrt{\frac{k_1 m_2 + k_2\left(m_1 + m_2\right) \pm \sqrt{\left[k_1 m_2 + k_2\left(m_1 + m_2\right)\right]^2 - 4m_1 m_2 k_1 k_2}}{2m_1 m_2}}$$

In two-degree-of-freedom systems, each natural frequency depends on all four parameters (m_1, m_2, k_1, k_2), while a single-degree-of-freedom system's natural frequency depends only on one mass and one stiffness.

4.18   Consider the problem of Example 4.1.7 and use a trig identity to show the $x_1(t)$
experiences a beat. Plot the response to show the beat phenomena in the response.

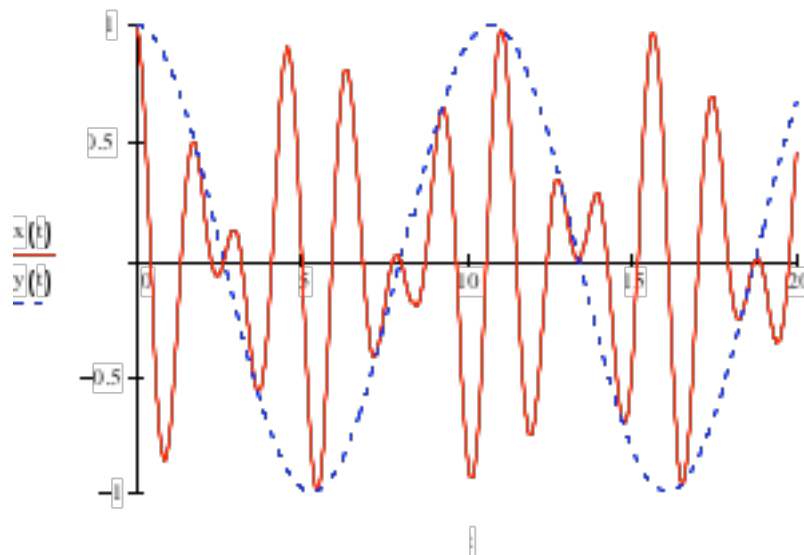**Solution** Applying the trig identity of Example 2.2.2 to x_1 yields

$$x_1(t) = (\cos\sqrt{2}t + \cos 2t) = \cos(\frac{\sqrt{2}-2}{2}t)\cos(\frac{\sqrt{2}+2}{2}t) = \cos 0.586t \cos 3.414t$$

Plotting x_1 and cos(0.586t) yields the clear beat:

$x(t) := \cos(0.586 \cdot t) \cdot \cos(3.414 \cdot t)$

$y(t) := \cos(0.586 \cdot t)$

Problems and Solutions for Section 4.2 (4.19 through 4.33)

4.19 Calculate the square root of the matrix

$$M = \begin{bmatrix} 13 & -10 \\ -10 & 8 \end{bmatrix}$$

$$\left[\text{Hint: Let } M^{1/2} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}; \text{ calculate } \left(M^{1/2} \right)^2 \text{ and compare to } M. \right]$$

Solution: Given:

$$M = \begin{bmatrix} 13 & -10 \\ -10 & 8 \end{bmatrix}$$

If

$$M^{1/2} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}, \text{ then}$$

$$M = M^{1/2} M^{1/2} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}\begin{bmatrix} a & -b \\ -b & c \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & -ab - bc \\ -ab - bc & b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 13 & -10 \\ -10 & 8 \end{bmatrix}$$

This yields the 3 nonlinear algebraic equations:
$$a^2 + b^2 = 13$$
$$ab + bc = 10$$
$$b^2 + c^2 = 8$$

There are several possible solutions but only one that makes $M^{1/2}$ positive definite which is $a = 3$, $b = c = 2$ as determined below in Mathcad. Choosing these values results in

$$M^{1/2} = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$a := 1$ $b := 1$ $c := 1$

given

$a^2 + b^2 = 13$

$a \cdot b + b \cdot c = 10$

$b^2 + c^2 = 8$

$$\text{find}(a, b, c) = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

4.20   Normalize the vectors

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

first with respect to unity (i.e., $x^T x = 1$) and then again with respect to the matrix M (i.e., $x^T M x = 1$), where

$$M = \begin{bmatrix} 3 & -0.1 \\ -0.1 & 2 \end{bmatrix}$$

Solution:

(a) Normalize the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\alpha_1 = \frac{1}{\sqrt{x^T x}} = \frac{1}{\sqrt{5}}$$

Normalized:

$$\mathbf{x}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.4472 \\ -0.8944 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\alpha_2 = \frac{1}{\sqrt{x^T x}} = \frac{1}{5}$$

Normalized:

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$$\alpha_3 = \frac{1}{\sqrt{x^T x}} = \frac{1}{\sqrt{0.02}}$$

Normalized:

$$\mathbf{x}_3 = \sqrt{50} \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

(b) Mass normalize the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\alpha_1 = \frac{1}{\sqrt{\mathbf{x}^T M \mathbf{x}}} = \frac{1}{\sqrt{11.4}}$$

Mass normalized:

$$\mathbf{x}_1 = \frac{1}{\sqrt{11.4}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.2962 \\ -.5923 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\alpha_2 = \frac{1}{\sqrt{\mathbf{x}^T M \mathbf{x}}} = \frac{1}{\sqrt{50}}$$

$$\mathbf{x}_2 = \frac{1}{\sqrt{50}} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7071 \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$$\alpha_3 = \frac{1}{\sqrt{x^T Mx}} = \frac{1}{\sqrt{0.052}}$$

Mass normalized:

$$\mathbf{x}_3 = \frac{1}{\sqrt{0.052}} \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -0.4385 \\ 0.4385 \end{bmatrix}$$

4.21 For the example illustrated in Figure P4.1 with $c_1 = c_2 = c_3 = 0$, calculate the matrix $\tilde{K}$.

Solution:

From Figure 4.1,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{x} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} x = 0$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} m_1^{-1/2} & 0 \\ 0 & m_2^{-1/2} \end{bmatrix} \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} m_1^{-1/2} & 0 \\ 0 & m_2^{-1/2} \end{bmatrix}$$

$$\tilde{K} = \begin{bmatrix} m_1^{-1}\left(k_1 + k_2\right) & -m_1^{-1/2} m_2^{-1/2} k_2 \\ -m_1^{-1/2} m_2^{-1/2} k_2 & m_1^{-1}\left(k_2 + k_3\right) \end{bmatrix}$$

Since $\tilde{K}^T = \tilde{K}, \tilde{K}$ is symmetric.

Using the numbers given in problem 4.2 yields

$$\tilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix}$$

This is obviously symmetric.

4.22 Repeat Example 4.2.5 using eight decimal places. Does $P^T P = 1$, and does
$P^T \tilde{K} P = \Lambda = \text{diag} \begin{bmatrix} \omega_1^2 & \omega_2^2 \end{bmatrix}$ exactly?

Solution: From Example 4.2.5,

$$\tilde{K} = \begin{bmatrix} 12 & -1 \\ -1 & 3 \end{bmatrix} \Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 15\lambda + 35 = 0$$

$$\Rightarrow \lambda_1 = 2.89022777, \text{ and } \lambda_2 = 12.10977223$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 2.89022777$$

$$\begin{bmatrix} 9.10977223 & -1 \\ -1 & 0.10977223 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 9.10977223 v_{11} = v_{12}$$

$$\|v_1\| = \sqrt{v_{11}^2 + v_{12}^2} = \sqrt{v_{11}^2 + \left(9.10977223\right)^2 v_{11}^2} = 1$$

$$\Rightarrow v_{11} = 0.10911677 \text{ and } v_{12} = 0.99402894$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.10911677 & 0.99402894 \end{bmatrix}^T$$

$$\lambda_2 = 12.10977223$$

$$\Rightarrow \begin{bmatrix} -0.10977223 & -1 \\ -1 & -9.10977223 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_{21} = 9.10977223 v_{22}$$

$$\|v_2\| = \sqrt{v_{21}^2 + v_{22}^2} = \sqrt{\left(-9.10977223\right)^2 v_{22}^2 + v_{22}^2} = 1$$

$$v_{21} = -9.10911677, \text{ and } v_{22} = -0.99402894$$

$$\mathbf{v}_2 = \begin{bmatrix} 0.99402894 & 0.10911677 \end{bmatrix}^T$$

Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.10911677 & -0.99402894 \\ 0.99402894 & 0.10911677 \end{bmatrix}$

Check $P^T P = I$

$$P^T P = \begin{bmatrix} 1.00000000 & 0 \\ 0 & 1.00000000 \end{bmatrix} = I \text{ (to 8 decimal places)}$$

Check $P^T \tilde{K} P = \Lambda = \text{diag}\left(\lambda_1, \lambda_2\right)$

$$\Lambda = P^T \tilde{K} P = \begin{bmatrix} 2.89022778 & 0.00000002 \\ 0.00000002 & 12.10977227 \end{bmatrix}$$

$$\mathrm{diag}\left(\lambda_1, \lambda_2\right) = \begin{bmatrix} 2.89022777 & 0 \\ 0 & 12.10977223 \end{bmatrix}$$

This is accurate to 7 decimal places.

4.23 Discuss the relationship or difference between a mode shape of equation (4.54) and an eigenvector of $\tilde{K}$.

Solution:

The relationship between a mode shape, **u**, of $M\ddot{x} + Kx = 0$ and an eigenvector, **v**, of $\tilde{K} = M^{-1/2}KM^{-1/2}$ is given by

$$\mathbf{v}_i = M^{1/2}\mathbf{u}_i \quad \text{or} \quad \mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

If **v** is normalized, then **u** is mass normalized.

This is shown by the relation

$$\mathbf{v}_i^T \mathbf{v}_i = 1 = \mathbf{u}_i^T M\mathbf{u}_i$$

4.24 Calculate the units of the elements of matrix $\tilde{K}$.

Solution:

$\tilde{K} = M^{-1/2}KM^{-1/2}$

$M^{-1/2}$ has units $\text{kg}^{-1/2}$

K has units $\text{N/m} = \text{kg/s}^2$

So, $\tilde{K}$ has units $\left(\text{kg}^{-1/2}\right)\left(\text{kg/s}^2\right)\left(\text{kg}^{-1/2}\right) = s^{-2}$

4.25 Calculate the spectral matrix Λ and the modal matrix P for the vehicle model of Problem 4.14, Figure P4.14.

Solution: From Problem 4.14:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 0.5 & -3.162 \\ -3.162 & 220 \end{bmatrix}$$

$$\begin{vmatrix} 0.5 - \lambda & -3.162 \\ -3.162 & 220 - \lambda \end{vmatrix} = \lambda^2 - 220.5\lambda + 100 = 0$$

$$\lambda_{1,2} = 0.454, 220.05$$

The spectral matrix is

$$\Lambda = \operatorname{diag}\left(\lambda_i\right) = \begin{bmatrix} 0.454 & 0 \\ 0 & 220.05 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0.454$$

$$\begin{bmatrix} 0.0455 & -3.162 \\ -3.162 & 219.55 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \Rightarrow v_{11} = 69.426 v_{12}$$

$$\|v_1\| = \sqrt{v_{11}^2 + v_{22}^2} = \sqrt{(69.426)^2 v_{12}^2 + v_{12}^2} = 69.434 v_{12} = 1$$

$$\Rightarrow v_{12} = 0.0144, \text{ and } v_{11} = 0.9999$$

$$\Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.9999 \\ 0.0144 \end{bmatrix}$$

$$\lambda_2 = 220.05$$

$$\begin{bmatrix} -219.55 & -3.162 \\ -3.162 & -0.0455 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$v_{21} = 0.0144 v_{22}$$

$$\|v_2\| = \sqrt{v_{21}^2 + v_{22}^2} = \sqrt{(-0.0144)^2 v_{22}^2 + v_{22}^2} = 1.0001 v_{22} = 1$$

$$\Rightarrow v_{22} = 0.9999, \text{ and } v_{21} = -0.0144$$

$$\Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.0144 \\ 0.9999 \end{bmatrix}$$

The modal matrix is

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.9999 & -0.0144 \\ 0.0144 & 0.9999 \end{bmatrix}$$

4.26   Calculate the spectral matrix Λ and the modal matrix P for the subway car system of Problem 4.12, Figure P4.12.

Solution: From problem 4.12 and Figure P4.12,

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 2000 & 0 \\ 0 & 2000 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 280,000 & -280,000 \\ -280,000 & 280,000 \end{bmatrix} \mathbf{x} = 0$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix}$$

$$\begin{vmatrix} 140 - \lambda & -140 \\ -140 & 140 - \lambda \end{vmatrix} = \lambda^2 - 280\lambda = 0$$

$$\lambda_{1,2} = 0, 280$$

The spectral matrix is

$$\Lambda = \text{diag}\left(\lambda_i\right) = \begin{bmatrix} 0 & 0 \\ 0 & 280 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0$$

$$\begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$v_{11} = v_{12}$$

$$\|v_1\| = \sqrt{v_{11}^2 + v_{12}^2} = \sqrt{v_{12}^2 + v_{12}^2} = 1.414 v_{12} = 1$$

$$v_{12} = 0.7071$$

$$v_{11} = 0.7071$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

$$\lambda_2 = 280$$

$$\begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \Rightarrow v_{21} = v_{22}$$

$$\Rightarrow \|v_2\| = \sqrt{v_{21}^2 + v_{22}^2} = \sqrt{v_{22}^2 + v_{22}^2} = 14.14 v_{12} = 1 \Rightarrow v_{22} = 0.7071, v_{21} = -0.7071$$

$$\Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

The modal matrix is $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$

4.27   Calculate $\tilde{K}$ for the torsional vibration example of Problem 4.11. What are the units of $\tilde{K}$?

Solution: From Problem 4.11,

$$J\ddot{\theta} + K\theta = J_2 \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\theta} + k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \theta = 0$$

$$\tilde{K} = J^{-1/2} K J^{-1/2}$$

$$J^{-1/2} = J_2^{-1/2} \begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{K} = J_2^{-1/2} \begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix} k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} J_2^{-1/2} \begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{K} = \frac{k}{J_2} \begin{bmatrix} 0.6667 & -0.5774 \\ -0.5774 & 1 \end{bmatrix}$$

The units of $\tilde{K}$ are

$$\left(\frac{kg \cdot m^2}{rad} \right)^{-1/2} \left(\frac{N \cdot m}{rad} \right) \left(\frac{kg \cdot m^2}{rad} \right)^{-1/2} = s^{-2}$$

4.28   Consider the system in the Figure P4.28 for the case where $m_1 = 1$ kg, $m_2 = 4$ kg, $k_1 = 240$ N/m and $k_2 = 300$ N/m. Write the equations of motion in vector form and compute each of the following

a) the natural frequencies
b) the mode shapes
c) the eigenvalues
d) the eigenvectors
e) show that the mode shapes are not orthogonal
f) show that the eigenvectors are orthogonal

g) show that the mode shapes and eigenvectors are related by $M^{-\frac{1}{2}}$
h) write the equations of motion in modal coordinates

Note the purpose of this problem is to help you see the difference between these various quantities.
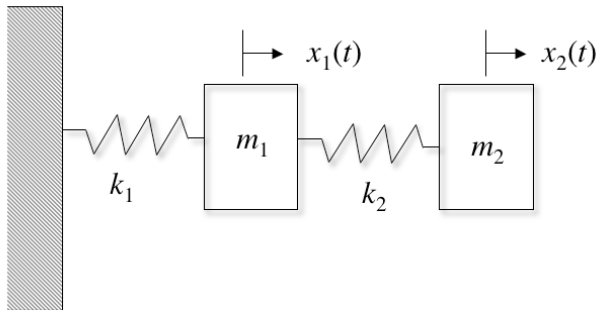

Figure P1.28 A two-degree of freedom system

Solution From a free body diagram, the equations of motion in vector form are

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 540 & -300 \\ -300 & 300 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequencies can be calculated in two ways. The first is using the determinant following example 4.1.5:

a) $\det(-\omega^2 M + K) = 0 \Rightarrow \underline{\omega_1 = 5.5509, \omega_2 = 24.1700 \text{ rad/s}}$

The second approach is to compute the eigenvalues of the matrix $\tilde{K} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$ following example 4.4.4, which yields the same answers. The mode shapes are calculate following the procedures of example 4.1.6 or numerically using eig(K,M) in Matlab

b) $\mathbf{u}_1 = \begin{bmatrix} 0.5076 \\ 0.8616 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0.9893 \\ -0.1457 \end{bmatrix}$

The eigenvectors are vectors that satisfy $\tilde{K}\mathbf{v} = \lambda\mathbf{v}$, where λ are the eigenvalues. These can be computed following example 4.2.2, or using [V,Dv]=eig(Kt) in Matlab. The eigenvalues and eigenvectors are

c) $\lambda_1 = 30.8120, \quad \lambda_2 = 584.1880$,

d) $\mathbf{v}_1 = \begin{bmatrix} 0.2826 \\ 0.9592 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -0.9592 \\ 0.2826 \end{bmatrix}$

To show that the mode shapes are not orthogonal, show that $\mathbf{u}_1^T\mathbf{u}_2 \neq 0$:

e) $\mathbf{u}_1^T\mathbf{u}_2 = (0.5076)(0.9893) + (0.8616)(-0.1457) = 0.3767 \neq 0$

To show that the eigenvectors are orthogonal, compute the inner product to show that $\mathbf{v}_1^T\mathbf{v}_2 = 0$:

f) $\mathbf{v}_1^T\mathbf{v}_2 = (0.2826)(-0.9592) + (0.9592)(0.2826) = 0$

To solve the next part merely compute $M^{-\frac{1}{2}}\mathbf{v}_2$ and show that it is equal to $\mathbf{u}_2$ (see the discussion at the top of page 262.

g) $M^{-\frac{1}{2}}\mathbf{v}_2 = \begin{bmatrix} 0.9592 \\ -0.1413 \end{bmatrix}$, normalize to get $\begin{bmatrix} -0.9893 \\ 0.1457 \end{bmatrix} = -\mathbf{u}_2$

Likewise, $M^{-\frac{1}{2}}\mathbf{v}_1 = \mathbf{u}_1$. Note that if you use Matlab you'll automatically get normalized vectors. But the product $M^{-\frac{1}{2}}\mathbf{v}_2$ will not be normalized, so it must be normalized before comparing it to $\mathbf{u}_2$.

h) We can write down the modal equations, just as soon as we know the eigenvalues (squares of the frequencies). They are:

$$\ddot{r}_1(t) + 30.812r_1(t) = 0$$
$$\ddot{r}_2(t) + 583.189r_2(t) = 0$$

4.29   Consider the following system:

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

where M is in kg and K is in N/m. (a) Calculate the eigenvalues of the system. (b) Calculate the eigenvectors and normalize them.

Solution: Given:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -0.5 \\ -0.5 & 0.25-\lambda \end{vmatrix} = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_{1,2} = 0.162, 3.088$$

The spectral matrix is

$$\Lambda = \text{diag}\left(\lambda_i\right) = \begin{bmatrix} 0.162 & 0 \\ 0 & 3.088 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$\lambda_1 = 0.162$

$$\begin{bmatrix} 2.838 & -0.5 \\ -0.5 & 0.088 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \Rightarrow v_{11} = 1.762 v_{21}$$

$$\|v_1\| = \sqrt{v_{11}^2 + v_{21}^2} = \sqrt{\left(0.1762\right)^2 v_{21}^2 + v_{21}^2} = 1.015 v_{21} = 1$$

$$v_{21} = 0.9848 \text{ and } v_{11} = 0.1735 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.1735 \\ 0.9848 \end{bmatrix}$$

$\lambda_2 = 3.088$

$$\begin{bmatrix} -0.088 & -0.5 \\ -0.5 & -2.838 \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0 \Rightarrow v_{12} = 1.762 v_{22}$$

$$\|v_2\| = \sqrt{v_{12}^2 + v_{22}^2} = \sqrt{\left(-5.676\right)^2 v_{22}^2 + v_{22}^2} = 5.764 v_{22} = 1$$

$$\Rightarrow v_{22} = 0.1735 \text{ and } v_{12} = -0.9848 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.9848 \\ 0.1735 \end{bmatrix}$$

4.30   The torsional vibration of the wing of an airplane is modeled in Figure P4.30. Write the equation of motion in matrix form and calculate the natural frequencies in terms of the rotational inertia and stiffness of the wing (See Figure 1.22).

Airplane wing with engines

Wing modeled as two shafts and two disks for torsional vibration
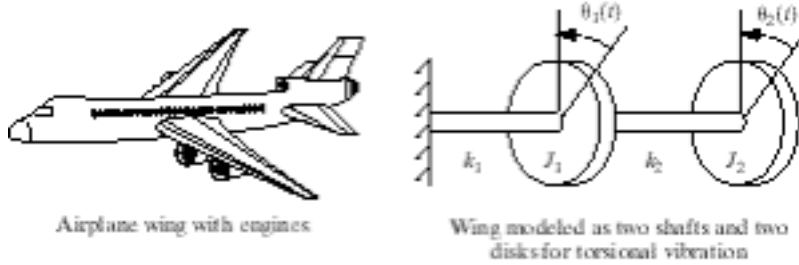
Solution: From Figure 1.22,

$$k_1 = \frac{GJ_p}{l_1} \text{ and } k_2 = \frac{GJ_p}{l_2}$$

Equation of motion:

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \ddot{\theta} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \theta = 0$$

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \ddot{\theta} + \begin{bmatrix} GJ_p\left(\dfrac{1}{l_1} + \dfrac{1}{l_2}\right) & \dfrac{-GJ_p}{l_2} \\ \dfrac{-GJ_p}{l_2} & \dfrac{GJ_p}{l_2} \end{bmatrix} \theta = 0$$

Natural frequencies:

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} \dfrac{GJ_p}{J_1}\left(\dfrac{1}{l_1} + \dfrac{1}{l_2}\right) & \dfrac{-GJ_p}{l_2\sqrt{J_1 J_2}} \\ \dfrac{-GJ_p}{l_2\sqrt{J_1 J_2}} & \dfrac{GJ_p}{J_2 l_2} \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \begin{vmatrix} \dfrac{GJ_p}{J_1}\left(\dfrac{1}{l_1} + \dfrac{1}{l_2}\right) - \lambda & \dfrac{-GJ_p}{l_2\sqrt{J_1 J_2}} \\ \dfrac{-GJ_p}{l_2\sqrt{J_1 J_2}} & \dfrac{GJ_p}{J_2 l_2} - \lambda \end{vmatrix}$$

Solving for λ yields

$$\lambda_{1,2} = \frac{GJ_p}{2}\left[\frac{1}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + \frac{1}{J_2 l_2}\right] \pm \frac{GJ_p}{2}\sqrt{\left[\frac{1}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + \frac{1}{J_2 l_2}\right]^2 - \frac{4}{J_1 J_2 l_1 l_2}}$$

The natural frequencies are

$$\omega_1 = \sqrt{\lambda_1} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2}$$

4.31 Calculate the value of the scalar a such that $\mathbf{x}_1 = [a \quad -1 \quad 1]^T$ and $\mathbf{x}_2 = [1 \quad 0 \quad 1]^T$ are orthogonal.

Solution: To be orthogonal, $\mathbf{x}_1^T\mathbf{x}_2 = 0$

So, $\mathbf{x}_1^T\mathbf{x}_2 = \begin{bmatrix} a & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = a + 1 = 0$. Therefore, $a = -1$.

4.32 Normalize the vectors of Problem 4.31. Are they still orthogonal?

Solution: From Problem 4.31, with $a = -1$,

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left(\alpha x_1\right)^T \left(\alpha x_1\right) = 1$$

Normalize $\mathbf{x}_1$:

$$\alpha^2 \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 3\alpha^2 = 1$$

$$\alpha = 0.5774$$

So, $\mathbf{x}_1 = 0.5774\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Normalize $\mathbf{x}_2$:

$$\left(\alpha x_2\right)^T \left(\alpha x_2\right) = 1$$

$$\alpha^2 \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2\alpha^2 = 1$$

$$\alpha = 0.7071$$

So, $\mathbf{x}_2 = 0.7071\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Check orthogonality:

$$\mathbf{x}_1^T\mathbf{x}_2 = \left(0.5774\right)\left(0.7071\right)\begin{bmatrix} -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \text{ Still orthogonal}$$

4.33 Which of the following vectors are normal? Orthogonal?

$$\mathbf{x}_1 = \begin{bmatrix} \dfrac{1}{\sqrt{2}} \\ 0 \\ \dfrac{1}{\sqrt{2}} \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \end{bmatrix}$$

Solution:

Check vectors to see if they are normal:

$$\|x_1\| = \sqrt{1/2 + 0 + 1/2} = \sqrt{1} = 1 \qquad \text{Normal}$$

$$\|x_2\| = \sqrt{.1^2 + .2^2 + .3^2} = \sqrt{.14} = 0.3742 \quad \text{Not normal}$$

$$\|x_3\| = \sqrt{.3^2 + .4^2 + .3^2} = \sqrt{.34} = 0.5831 \quad \text{Not normal}$$

Check vectors to see if they are orthogonal:

$$\mathbf{x}_1^T\mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} .1 \\ .2 \\ .3 \end{bmatrix} = .2828 \quad \text{Not orthogonal}$$

$$\mathbf{x}_2^T\mathbf{x}_3 = \begin{bmatrix} .1 & .2 & .3 \end{bmatrix} \begin{bmatrix} .3 \\ .4 \\ .3 \end{bmatrix} = 0.2 \qquad \text{Not orthogonal}$$

$$\mathbf{x}_3^T\mathbf{x}_1 = \begin{bmatrix} .3 & .4 & .3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = 0.4243 \qquad \text{Not orthogonal}$$

$\therefore$ Only $\mathbf{x}_1$ is normal, and none are orthogonal.

Problems and Solutions for Section 4.3 (4.34 through 4.43)

4.34 Solve Problem 4.11 by modal analysis for the case where the rods have equal stiffness (i.e., $k_1 = k_2$), $J_1 = 3J_2$, and the initial conditions are $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $\dot{\mathbf{x}}(0) = \mathbf{0}$.

Solution: From Problem 4.11 and Figure P4.11, with $k = k_1 = k_2$ and $J_1 = 3J_2$:

$$J_2 \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\theta} + k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \theta = 0$$

Calculate eigenvalues and eigenvectors:

$$J^{-1/2} = J_2^{-1/2} \begin{bmatrix} \dfrac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{K} = J^{-1/2} K J^{-1/2} = \frac{k}{J_2} \begin{bmatrix} \dfrac{2}{3} & \dfrac{-1}{\sqrt{3}} \\ \dfrac{-1}{\sqrt{3}} & 1 \end{bmatrix} \Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - \frac{5k}{3J_2}\lambda + \frac{k^2}{3J_2^2} = 0$$

$$\lambda_1 = \frac{\left(5 - \sqrt{13}\right)k}{6J_2} \Rightarrow \omega_1 = \sqrt{\lambda_1}, \text{ and } \frac{\left(5 + \sqrt{13}\right)k}{6J_2} \Rightarrow \omega_2 = \sqrt{\lambda_2}$$

$$\lambda_1 = \frac{\left(5 - \sqrt{13}\right)k}{6J_2} \Rightarrow \begin{bmatrix} \dfrac{\left(5 + \sqrt{13}\right)k}{6J_2} & \dfrac{-k}{\sqrt{3J_2}} \\ \dfrac{-k}{\sqrt{3J_2}} & \dfrac{\left(5 + \sqrt{13}\right)k}{6J_2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\Rightarrow v_{11} = 1.3205 v_{12} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.7992 \\ 0.6011 \end{bmatrix}$$

$$\lambda_2 = \frac{\left(5 + \sqrt{13}\right)k}{6J_2} \Rightarrow \begin{bmatrix} \dfrac{\left(-1 - \sqrt{13}\right)k}{6J_2} & \dfrac{-k}{\sqrt{3J_2}} \\[4mm] \dfrac{-k}{\sqrt{3J_2}} & \dfrac{\left(1 - \sqrt{13}\right)k}{6J_2} \end{bmatrix} \begin{bmatrix} v2_{11} \\ v_{22} \end{bmatrix} = 0$$

$$\Rightarrow v_{21} = -0.7522 v_{22} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.6011 \\ 0.7992 \end{bmatrix}$$

Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.7992 & -0.6011 \\ 0.6011 & 0.7992 \end{bmatrix}$

Calculate S and S^{-1}:

$$S = J^{-1/2} P = \frac{1}{\sqrt{J_2}} \begin{bmatrix} 0.4614 & -0.3470 \\ 0.6011 & 0.7992 \end{bmatrix}$$

$$S^{-1} = P^T J^{1/2} = J_2^{1/2} \begin{bmatrix} 1.3842 & 0.6011 \\ -1.0411 & 0.7992 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\boldsymbol{\theta}(0) = S^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = J_2^{1/2} \begin{bmatrix} 0.6011 \\ 0.7992 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\boldsymbol{\theta}}(0) = 0$$

Modal solution:

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + \dot{r}_{10}^2}}{\omega_1} \sin\left[\omega_1 t + \tan^{-1} \frac{\omega_1 r_{10}}{\dot{r}_{10}} \right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + \dot{r}_{20}^2}}{\omega_2} \sin\left[\omega_2 t + \tan^{-1} \frac{\omega_2 r_{10}}{\dot{r}_{20}} \right]$$

$$r_1(t) = 0.6011 J_2^{1/2} \sin\left[\omega_1 t + \frac{\pi}{2} \right] = 0.6011 J_2^{1/2} \cos\omega_1 t$$

$$r_2(t) = 0.7992 J_2^{1/2} \sin\left[\omega_2 t + \frac{\pi}{2} \right] = 0.6011 J_2^{1/2} \cos\omega_2 t$$

$$\mathbf{r}(t) = \begin{bmatrix} 0.6011 J_2^{1/2} \cos\omega_1 t \\ 0.7992 J_2^{1/2} \cos\omega_2 t \end{bmatrix}$$

Convert to physical coordinates:

$$\theta(t) = S\mathbf{r}(t) = J_2^{1/2} \begin{bmatrix} 0.4614 & -0.3470 \\ 0.6011 & 0.7992 \end{bmatrix} \begin{bmatrix} 0.6011 J_2^{1/2} \cos\omega_1 t \\ 0.7992 J_2^{1/2} \cos\omega_2 t \end{bmatrix}$$

$$\theta(t) = \begin{bmatrix} 0.2774\cos\omega_1 t - 0.2774\cos\omega_2 t \\ 0.3613\cos\omega_1 t + 0.6387\cos\omega_2 t \end{bmatrix}$$

where $\omega_1 = 0.4821\sqrt{\dfrac{k}{J_2}}$ and $\omega_2 = 1.1976\sqrt{\dfrac{k}{J_2}}$,

4.35   Consider the system of Example 4.3.1. Calculate a value of $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ such that both masses of the system oscillate with a single frequency of 2 rad/s.

Solution:

From Example 4.3.1,

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1/3 & 1/3 \\ 1 & -1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

From Equations (4.67) and (4.68),

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + r_{10}^2}}{\omega_1} \sin\left[\omega_1 t + \tan^{-1}\frac{\omega_1 r_{10}}{\dot{r}_{10}}\right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + r_{20}^2}}{\omega_2} \sin\left[\omega_2 t + \tan^{-1}\frac{\omega_2 r_{20}}{\dot{r}_{20}}\right]$$

Choose $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ so that $r_1(t) = 0$. This will cause the frequency $\sqrt{2}$ to drop out. For $r_1(t) = 0$, its coefficient must be zero.

$$\frac{\sqrt{\omega_1^2 r_{10}^2 + r_{10}^2}}{\omega_1} = 0 \quad \text{or} \quad \omega_1^2 r_{10}^2 + r_{10}^2 = 0$$

Choose $r_{10} = \dot{r}_{20} = 0$.

Let $r_{20} = 3/\sqrt{2}$ and $\dot{r}_{20} = 0$ as calculated in Example 4.3.1.

So, $\mathbf{r}(0) = \begin{bmatrix} 0 & 3/\sqrt{2} \end{bmatrix}^T$ and $\dot{\mathbf{r}}(0) = \mathbf{0}$.

Solve for $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$:

$$\mathbf{x}(0) = S\mathbf{r}(0) = \frac{1}{\sqrt{12}}\begin{bmatrix} 1/3 & 1/3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 3/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1.5 \end{bmatrix}$$

$$\dot{\mathbf{x}}(0) = S\dot{\mathbf{r}}(0) = \mathbf{0}$$

4.36 Consider the system of Figure P4.36 consisting of two pendulums coupled by a spring. Determine the natural frequency and mode shapes. Plot the mode shapes as well as the solution to an initial condition consisting of the first mode shape for $k = 20$ N/m, $l = 0.5$ m and $m_1 = m_2 = 10$ kg, $a = 0.1$ m along the pendulum.
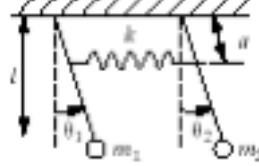
Solution: Given:

$$k = 20 \text{ N/m} \quad m_1 = m_2 = 10 \text{ kg}$$
$$a = 0.1 \text{ m} \qquad l = 0.5 \text{ m}$$

For gravity use $g = 9.81$ m/s^2. For a mass on a pendulum, the inertia is: $I = ml^2$

Calculate mass and stiffness matrices (for small θ). The equations of motion are:

$$
\begin{aligned}
I_1\ddot{\theta}_1 &= ka^2\left(\theta_2 - \theta_1\right) - m_1gl\theta_1 \\
I_2\ddot{\theta}_2 &= -ka^2\left(\theta_2 - \theta_1\right) - m_2gl\theta_2
\end{aligned}
\Rightarrow ml^2\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix} mgl + ka^2 & -ka^2 \\ -ka^2 & mgl + ka^2 \end{bmatrix}\begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}
$$

Substitution of the given values yields:

$$\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} 49.05 & -0.2 \\ -0.2 & 49.05 \end{bmatrix}\theta = 0$$

Natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 19.7 & -0.08 \\ -0.08 & 19.7 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 19.54 \text{ and } \lambda_2 = 19.7 \Rightarrow \omega_1 = 4.42 \text{ rad/s and } \omega_2 = 4.438 \text{ rad/s}$$

Eigenvectors:

$$\lambda_1 = 19.54$$

$$\begin{bmatrix} 0.08 & -0.08 \\ -0.08 & 0.08 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 19.7$$

$$\begin{bmatrix} -0.08 & -0.08 \\ -0.08 & -0.08 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
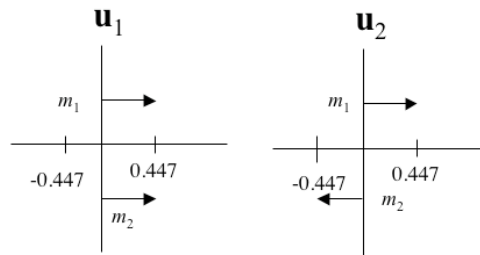
Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Mode shapes:
$$\mathbf{u}_1 = M^{-1/2}\mathbf{v}_1 = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix}$$

$$\mathbf{u}_2 = M^{-1/2}\mathbf{v}_2 = \begin{bmatrix} 0.4472 \\ -0.4472 \end{bmatrix}$$

A plot of the mode shapes is simply

This shows the first mode vibrates in phase and in the second mode the masses vibrate out of phase.

$$\theta(0) = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix} \qquad \dot{\theta}(0) = 0, \ S = M^{-1/2}P = \begin{bmatrix} 0.4472 & 0.4472 \\ 0.4472 & -0.4472 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 1.118 & 1.118 \\ 1.118 & -1.118 \end{bmatrix}, \ \mathbf{r}(0) = S^{-1}\theta(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \dot{\mathbf{r}}(0) = 0$$

From Eq. (4.67) and (4.68): $r_1(t) = \sin\left(4.42t + \dfrac{\pi}{2}\right) = \cos 4.45t, \ r_2(t) = 0$

Convert to physical coordinates: $\theta(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.4472\cos 4.42t \\ 0.4472\cos 4.42t \end{bmatrix}$ rad

$$\theta(t) := 0.4472 \cdot \cos(4.429 \cdot t)$$
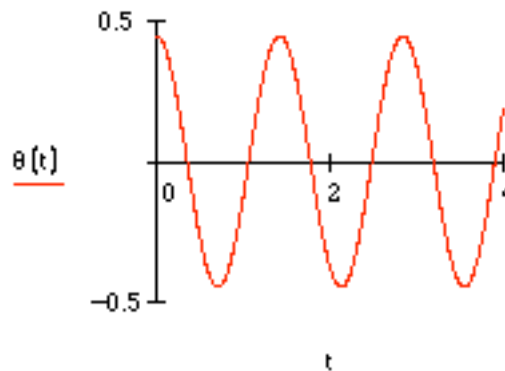
4.37 Resolve Example 4.3.2 with m_2 changed to 10 kg. Plot the response and compare the plots to those of Figure 4.6.

Solution: From examples 4.3.2 and 4.2.5, with $m_2 = 10$ kg,

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 12 & -2 \\ -2 & 12 \end{bmatrix}\mathbf{x} = 0$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sqrt{10}} \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 12 & -0.6325 \\ -0.6325 & 1.2 \end{bmatrix}$$

$$\det(\tilde{K} - \lambda I) = \lambda^2 - 13.2\lambda + 14 = 0$$

$$\lambda_1 = 1.163 \quad \omega_1 = 1.078 \text{ rad/s}$$

$$\lambda_2 = 12.04 \quad \omega_2 = 3.469 \text{ rad/s}$$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.0583 & -0.9983 \\ 0.9983 & 0.0583 \end{bmatrix}$$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \begin{bmatrix} 0.0583 & -0.9983 \\ 0.9983 & 0.0583 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 0.0583 & 3.1569 \\ -0.9983 & 0.1842 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2152 \\ -0.8141 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = 0$$

Modal solution (from Eqs. (4.67) and (4.68)):

$$r_1(t) = 3.2152\sin\left[1.078t + \frac{\pi}{2}\right] = 3.2152\cos1.078t$$
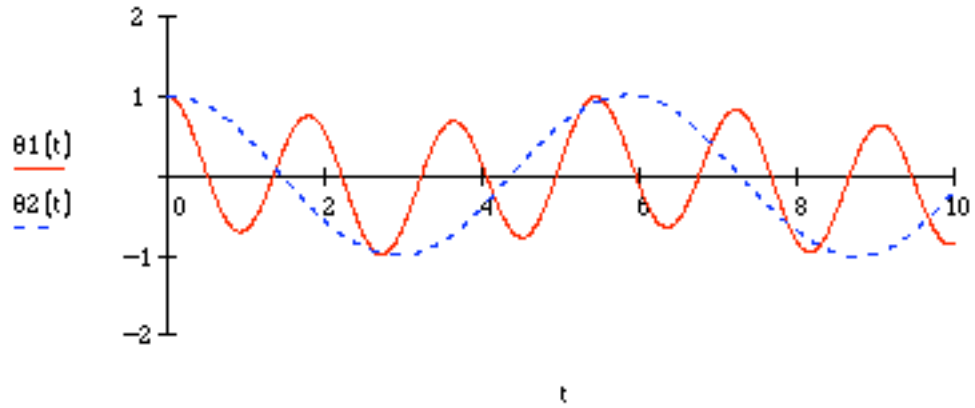
$$r_2(t) = -0.8141\cos3.469t$$

Covert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.0583 & -0.9983 \\ 0.3157 & 0.0184 \end{bmatrix}\begin{bmatrix} 3.2152\cos1.078t \\ -0.8141\cos3.469t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.1873\cos1.078t + 0.8127\cos3.469t \\ 1.015\cos1.078t - 0.0150\cos3.469t \end{bmatrix}$$

$\theta 1\,(t) := 0.1873 \cdot \cos\,(1.078 \cdot t) + 0.8127 \cdot \cos\,(3.469 \cdot t)$

$\theta 2\,(t) := 1.015 \cdot \cos\,(1.078 \cdot t) - 0.0150 \cdot \cos\,(3.469 \cdot t)$

These figures are similar to those of Figure 4.6, except the responses are reversed (θ_2 looks like x_2 in Figure 4.6, and θ_1 looks like x_1 in Figure 4.6)

4.38   Use modal analysis to calculate the solution of Problem 4.29 for the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T (\text{mm}) \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T (\text{mm/s})$$

Solution: From Problem 4.29,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\omega_1 = \sqrt{\lambda_1} = 0.4024 \text{ rad/s}$$

$$\omega_2 = \sqrt{\lambda_2} = 1.7573 \text{ rad/s}$$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.1735 & -0.9848 \\ 0.9848 & 0.1735 \end{bmatrix}$$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \begin{bmatrix} 0.1735 & -0.9848 \\ 0.4924 & 0.0868 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 0.1735 & 1.9697 \\ -0.9848 & 0.3470 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.9697 \\ 0.3470 \end{bmatrix}$$

$$\mathbf{r}(0) = S^{-1}\dot{\mathbf{x}}(0) = 0$$

Modal solution (from Eqs. (4.67) and (4.68)):

$$r_1(t) = 1.9697\cos 0.4024t$$

$$r_2(t) = -.3470\cos 1.7573t$$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.1735 & -0.9848 \\ 0.4924 & 0.0868 \end{bmatrix}\begin{bmatrix} 1.9697\cos 0.4024t \\ 0.3470\cos 1.7573t \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} 0.3417\cos 0.4024t - 0.3417\cos 1.7573t \\ 0.9699\cos 0.4024t + 0.0301\cos 1.7573t \end{bmatrix}\text{mm}$$

4.39 For the matrices

$$M^{-1/2} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 4 \end{bmatrix} \text{ and } P = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

calculate $M^{-1/2}P$, $\left(M^{-1/2}P\right)^T$, and $P^T M^{-1/2}$ and hence verify that the computations in Eq. (4.70) make sense.

Solution:

Given

$$M^{-1/2} = \begin{bmatrix} \dfrac{1}{\sqrt{2}} & 0 \\ 0 & 4 \end{bmatrix} \text{ and } P = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Now

$$M^{-1/2}P = \begin{bmatrix} 0.5 & 0.5 \\ -2\sqrt{2} & -2\sqrt{2} \end{bmatrix}$$

So

$$\left(M^{-1/2}P\right)^T = \begin{bmatrix} 0.5 & -2\sqrt{2} \\ 0.5 & -2\sqrt{2} \end{bmatrix}$$

$$P^T M^{-1/2} = \begin{bmatrix} 0.5 & -2\sqrt{2} \\ 0.5 & -2\sqrt{2} \end{bmatrix}$$

Thus, $\left(M^{-1/2}P\right)^T = P^T M^{-1/2}$ [Equation (4.71)]

4.40 Consider the 2-degree-of-freedom system defined by:

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}.$$

Calculate the response of the system to the initial condition

$$\mathbf{x}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \dot{\mathbf{x}}_0 = \mathbf{0}$$

What is unique about your solution compared to the solution of Example 4.3.1.

Solution: Following the calculations made for this system in Example 4.3.1,

$$\omega_1 = \sqrt{\lambda_1} = 1.414 \text{ rad/s}, \quad \omega_2 = \sqrt{\lambda_2} = 2 \text{ rad/s}$$

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow S = M^{-1/2}P = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \text{ and } S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

Next compute the modal initial conditions

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and } \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solution (from Eqs. (4.67) and (4.68)):

$$\mathbf{r}(t) = \begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

Note that the second coordinate modal coordinate has zero initial conditions and is hence not vibrating. Convert this solution back into physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} 0.236\cos 1.414t \\ 0.707\cos 1.414t \end{bmatrix}$$

The unique feature about the solution is that both masses are vibrating at only one frequency. That is the frequency of the first mode shape. This is because the system is excited with a position vector equal to the first mode of vibration.

4.41   Consider the 2-degree-of-freedom system defined by:

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}.$$

Calculate the response of the system to the initial condition

$$\mathbf{x}_0 = \mathbf{0}, \quad \text{and } \dot{\mathbf{x}}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

What is unique about your solution compared to the solution of Example 4.3.1 and to Problem 4.40, if you also worked that?

Solution: From example 4.3.1,

$$\omega_1 = \sqrt{\lambda_1} = 1.414 \text{ rad/s}, \ \omega_2 = \sqrt{\lambda_2} = 2 \text{ rad/s}, \ P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow S = M^{-1/2} P = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix}, \text{ and } S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \mathbf{0}, \text{ and } \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Modal solution (from Eqs. (4.67) and (4.68)):

$$\mathbf{r}(t) = \begin{bmatrix} 0 \\ \dfrac{1}{\omega_2}\cos 2t \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5\cos 2t \end{bmatrix}$$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5\cos 2t \end{bmatrix} = \begin{bmatrix} 0.118\cos 2t \\ -0.354\cos 2t \end{bmatrix}$$

Compared to Example 4.3.1, only the second mode is excited, because the initial velocity is proportional to the second mode shape, and the displacement is zero. Compared to the previous problem, here it is the second mode rather then the first mode that is excited.

4.42 Consider the system of Problem 4.1. Let $k_1 = 10,000$ N/m, $k_2 = 15,000$ N/m, and $k_3 = 10,000$ N/m. Assume that both masses are 100 kg. Solve for the free response of this system using modal analysis and the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Solution: Given:

$$k_1 = 10,000 \text{ N/m} \quad m_1 = m_2 = 100 \text{ kg}$$
$$k_2 = 15,000 \text{ N/m} \quad \mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$
$$k_3 = 10,000 \text{ N/m} \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Equation of motion:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 25,000 & -15,000 \\ -15,000 & 25,000 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$
$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 250 & -150 \\ -150 & 250 \end{bmatrix}$$
$$\det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 500\lambda + 40,000 = 0$$

$$\lambda_1 = 100 \quad \omega_1 = 10 \text{ rad/s}$$
$$\lambda_2 = 400 \quad \omega_2 = 20 \text{ rad/s}$$

$$\lambda_1 = 100$$
$$\begin{bmatrix} 150 & -150 \\ -150 & 150 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 400$

$$\begin{bmatrix} -150 & -150 \\ -150 & -150 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \frac{1}{\sqrt{2}}\begin{bmatrix} 0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 10 & 10 \\ 10 & -10 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solutions:

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + \dot{r}_{10}^2}}{\omega_1}\sin\left[\omega_1 t + \tan^{-1}\frac{\omega_1 r_{10}}{\dot{r}_{10}}\right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + \dot{r}_{20}^2}}{\omega_2}\sin\left[\omega_2 t + \tan^{-1}\frac{\omega_2 r_{20}}{\dot{r}_{20}}\right]$$

So

$$r_1(t) = 7.071\sin\left(10t + \pi/2\right) = 7.071\cos 10t$$

$$r_2(t) = 7.071\sin\left(20t + \pi/2\right) = 7.071\cos 20t$$

$$\mathbf{r}(t) = \begin{bmatrix} 7.071\cos 10t \\ 7.071\cos 20t \end{bmatrix}$$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 7.071\cos 10t \\ 7.7071\cos 20t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.5\big(\cos 10t + \cos 20t\big) \\ 0.5\big(\cos 10t - \cos 20t\big) \end{bmatrix}$$

4.43 Consider the model of a vehicle given in Problem 4.14 and illustrated in Figure P4.14. Suppose that the tire hits a bump which corresponds to an initial condition of

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Use modal analysis to calculate the response of the car $x_1(t)$. Plot the response for three cycles.

Solution: From Problem 4.14,

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Calculate the eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.0224 & 0 \\ 0 & 0.1414 \end{bmatrix}, \quad \tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 0.5 & -3.1623 \\ -3.1623 & 0.1414 \end{bmatrix}$$

$$\Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 220.05\lambda + 100 = 0 \Rightarrow \begin{matrix} \lambda_1 = 0.4545 & \omega_1 = 0.6741 \text{ rad/s} \\ \lambda_2 = 220.05 & \omega_2 = 14.834 \text{ rad/s} \end{matrix}$$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.9999 & -0.0144 \\ 0.0144 & 0.9999 \end{bmatrix}$$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \begin{bmatrix} 0.0224 & -0.003 \\ 0.0020 & 0.1414 \end{bmatrix}, \quad S^{-1} = P^T M^{1/2} = \begin{bmatrix} 44.7167 & 0.1018 \\ -0.6441 & 7.0703 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 0 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.001018 \\ 0.07070 \end{bmatrix}, \quad \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solution (from equations (4.67) and (4.68)): $\mathbf{r}(t) = \begin{bmatrix} 0.001018\cos 0.6741t \\ 0.07070\cos 14.834t \end{bmatrix}$

Convert to physical coordinates:

$$\mathbf{x}(t) = Sr(t) = \begin{bmatrix} 0.0224 & -0.0003 \\ 0.0020 & 0.1414 \end{bmatrix} \begin{bmatrix} 0.001018 \cos 0.6741t \\ 0.07070 \cos 14.834t \end{bmatrix} = \begin{bmatrix} 2.277 \times 10^{-5} \cos 0.6741t - 2.277 \times 10^{-5} \cos 14.834t \\ 2.074 \times 10^{-6} \cos 0.6741t + 9.998 \times 10^{-3} \cos 14.834t \end{bmatrix}$$

$$x1(t) := 2.277 \cdot 10^{-5} \cdot \cos(0.674 \cdot t) - 2.277 \cdot 10^{-5} \cdot \cos(14.834 \cdot t)$$

+
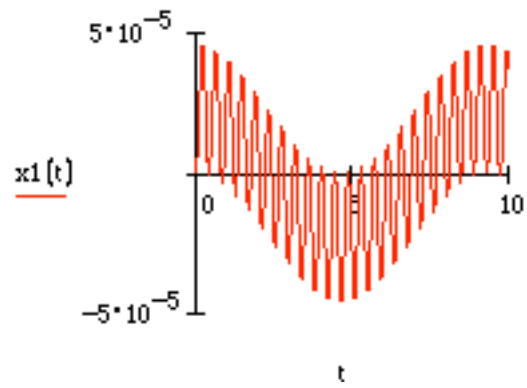
Problems and Solutions for Section 4.4 (4.44 through 4.55)

4.44 A vibration model of the drive train of a vehicle is illustrated as the three-degree-of-freedom system of Figure P4.44. Calculate the undamped free response [i.e. $M(t) = F(t) = 0$, $c_1 = c_2 = 0$] for the initial condition $\mathbf{x}(0) = \mathbf{0}$, $\dot{\mathbf{x}}(0) = [0 \quad 0 \quad 1]^T$. Assume that the hub stiffness is 10,000 N/m and that the axle/suspension is 20,000 N/m. Assume the rotational element J is modeled as a translational mass of 75 kg.
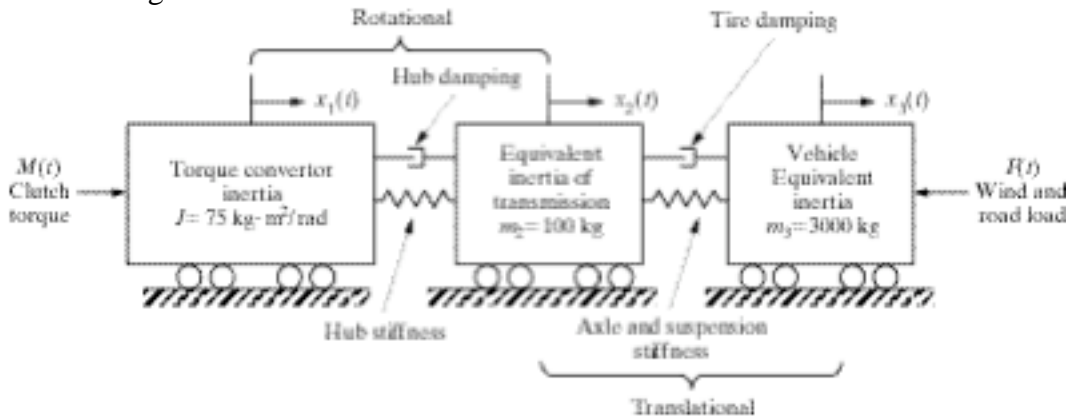
Solution: Let k_1 = hub stiffness and k_2 = axle and suspension stiffness.
The equation of motion is

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \ddot{\mathbf{x}} + 10{,}000 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{0} \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m/s}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.1155 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.0183 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 133.33 & -115.47 & 0 \\ -115.47 & 300 & -36.515 \\ 0 & -36.515 & 6.6667 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 440\lambda^2 + 28{,}222\lambda = 0$$

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$
$$\lambda_2 = 77.951 \quad \omega_2 = 8.8290 \text{ rad/s}$$
$$\lambda_3 = 362.05 \quad \omega_3 = 19.028 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.1537 \\ 0.1775 \\ 0.9721 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -0.8803 \\ -0.4222 \\ 0.2163 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0.4488 \\ -0.8890 \\ 0.0913 \end{bmatrix}$$

Use the mode summation method to find the solution.
Transform the initial conditions:

$$\mathbf{q}(0) = M^{-1/2}\mathbf{x}(0) = \mathbf{0}, \quad \dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 54.7723 \end{bmatrix}^T$$

The solution is given by:

$$\mathbf{q}(t) = \left(c_1 + c_4 t\right)\mathbf{v}_1 + c_2 \sin\left(\omega_2 t + \phi_2\right)\mathbf{v}_2 + c_3 \sin\left(\omega_3 t + \phi_3\right)\mathbf{v}_3$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \quad i = 2,3$$

$$c_i = \frac{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}{\omega_i \cos\phi} \quad i = 2,3$$

Thus,

$$\phi_2 = \phi_3 = 0, c_2 - 1.3417, \text{ and } c_3 = 0.2629$$

So,

$$\mathbf{q}(0) = c_1 \mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_4 \mathbf{v}_1 + \sum_{i=2}^{3} \omega_i c_i \cos\phi_i \mathbf{v}_i$$

Premultiply by v_1^T;

$$\mathbf{v}_1^T \mathbf{q}(0) = 0 = c_1$$

$$\mathbf{v}_1^T \dot{\mathbf{q}}(0) = 53.2414 = c_4$$

So,

$$\mathbf{q}(t) = 53.2414t\mathbf{v}_1 + 1.3417\sin\left(8.8290t\right)\mathbf{v}_2 + 0.2629\sin\left(19.028t\right)\mathbf{v}_3$$

Change to $\mathbf{q}(t)$:

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$$

$$\mathbf{x}(t) = 0.9449t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.1364 \\ -0.05665 \\ 0.005298 \end{bmatrix}\sin 8.8290t + \begin{bmatrix} 0.01363 \\ -0.02337 \\ 0.0004385 \end{bmatrix}\sin 19.028t \text{ m}$$

4.45   Calculate the natural frequencies and normalized mode shapes of

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Solution: Given the indicated mass and stiffness matrix, calculate eigenvalues:

$$M^{-1/2} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 1 & -0.3536 & 0 \\ -0.3536 & 1 & -0.7071 \\ 0 & -0.7071 & 1 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 3\lambda^2 + 2.375\lambda - 0.375 = 0$$

$$\lambda_1 = 0.2094, \quad \lambda_2 = 1, \quad \lambda_3 = 1.7906$$

The natural frequencies are:

$$\omega_1 = 0.4576 \text{ rad/s}$$
$$\omega_2 = 1 \text{ rad/s}$$
$$\omega_3 = 1.3381 \text{ rad/s}$$

The corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{bmatrix} -0.3162 \\ -0.7071 \\ -0.6325 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.8944 \\ 0 \\ -0.4472 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.3162 \\ -0.7071 \\ 0.6325 \end{bmatrix}$$

The relationship between eigenvectors and mode shapes is

$$\mathbf{u} = M^{-1/2}\mathbf{v}$$

The mode shapes are:

$$\mathbf{u}_1 = \begin{bmatrix} -0.1581 \\ -0.5 \\ -0.6325 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0.4472 \\ 0 \\ -0.4472 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0.1581 \\ -0.5 \\ 0.6325 \end{bmatrix}$$

The normalized mode shapes are

$$\hat{\mathbf{u}}_1 = \frac{\mathbf{u}_1}{\sqrt{\mathbf{u}_1^T\mathbf{u}_1}} = \begin{bmatrix} 0.192 \\ 0.609 \\ 0.77 \end{bmatrix}, \quad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0.707 \\ 0 \\ -0.707 \end{bmatrix}, \quad \hat{\mathbf{u}}_3 = \begin{bmatrix} 0.192 \\ -0.609 \\ 0.77 \end{bmatrix}.$$

4.46 The vibration is the vertical direction of an airplane and its wings can be modeled as a three-degree-of-freedom system with one mass corresponding to the right wing, one mass for the left wing, and one mass for the fuselage. The stiffness connecting the three masses corresponds to that of the wing and is a function of the modulus E of the wing. The equation of motion is

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The model is given in Figure P4.46. Calculate the natural frequencies and mode shapes. Plot the mode shapes and interpret them according to the airplane's deflection.
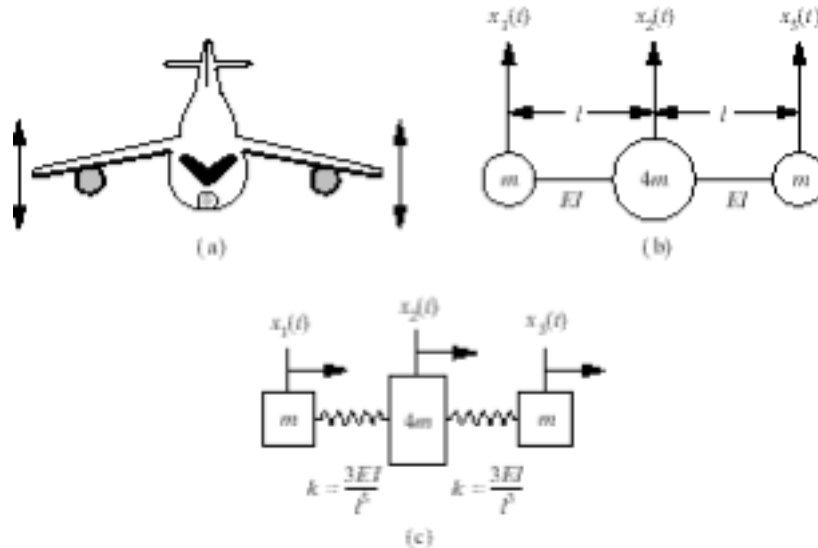
Solution: Given the equation of motion indicated above, the mass-normalized stiffness matrix is calculated to be

$$M^{-\frac{1}{2}} = \frac{1}{\sqrt{m}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{K} = M^{-\frac{1}{2}}KM^{-\frac{1}{2}} = \frac{EI}{ml^3}\begin{bmatrix} 3 & -1.5 & 0 \\ -1.5 & 1.5 & -1.5 \\ 0 & -1.5 & 3 \end{bmatrix}$$

Computing the matrix eigenvalue by factoring out the constant $\dfrac{EI}{ml^3}$ yields

$$\det(\tilde{K} - \lambda I) = 0 \Rightarrow \lambda_1 = 0, \quad \lambda_2 = 3\frac{EI}{ml^3}, \quad \lambda_3 = 4.5\frac{EI}{ml^3}$$

and eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

The natural frequencies are $\omega_1 = 0$, $\omega_2 = 1.7321\sqrt{\dfrac{EI}{m\ell^3}}$ rad/s, and $\omega_3 =$

$2.1213\sqrt{\dfrac{EI}{m\ell^3}}$ rad/s.

The relationship between the mode shapes and eigenvectors **u** is just $\mathbf{u} = M^{-1/2}\mathbf{v}$. The fist mode shape is the rigid body mode. The second mode shape corresponds to one wing up and one down the third mode shape corresponds to the wings moving up and down together with the body moving opposite. Normalizing the mode shapes yields (calculations in Mathcad):

$$\text{Mh} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad K := \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \qquad \text{Kh} := \text{Mh}^{-1} \cdot K \cdot \text{Mh}^{-1}$$

$$\text{Kh} = \begin{bmatrix} 3 & -1.5 & 0 \\ -1.5 & 1.5 & -1.5 \\ 0 & -1.5 & 3 \end{bmatrix}$$

$$\lambda a := \text{eigenvals}(\text{Kh}) \qquad \lambda a = \begin{bmatrix} 4.5 \\ 3 \\ 0 \end{bmatrix}$$

$$\text{v1} := \text{eigenvec}(\text{Kh}, \lambda a_2) \qquad \text{v1} = \begin{bmatrix} 0.408 \\ 0.816 \\ 0.408 \end{bmatrix} \qquad \text{v2} := \text{eigenvec}(\text{Kh}, \lambda a_1) \qquad \text{v2} = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$\text{v3} := \text{eigenvec}(\text{Kh}, \lambda a_0) \qquad \text{v3} = \begin{bmatrix} 0.577 \\ -0.577 \\ 0.577 \end{bmatrix}$$

$$\omega := \begin{bmatrix} \sqrt{\lambda a_2} \\ \sqrt{\lambda a_1} \\ \sqrt{\lambda a_0} \end{bmatrix} \qquad \omega = \begin{bmatrix} 2.967 \cdot 10^{-8} \\ 1.732 \\ 2.121 \end{bmatrix} \qquad \text{u1} := \text{Mh}^{-1} \cdot \text{v1} \qquad \text{u1} = \begin{bmatrix} 0.408 \\ 0.408 \\ 0.408 \end{bmatrix}$$

$$\text{u1n} := \frac{\text{u1}}{|\text{u1}|} \qquad \text{u1n} = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix} \qquad \text{u2} := \text{Mh}^{-1} \cdot \text{v2} \qquad \text{u2} = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$\text{u3} := \text{Mh}^{-1} \cdot \text{v3} \qquad \text{u3} = \begin{bmatrix} 0.577 \\ -0.289 \\ 0.577 \end{bmatrix} \qquad \text{u2n} := \frac{\text{u2}}{|\text{u2}|} \qquad \text{u2n} = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$\text{u3n} := \frac{\text{u3}}{|\text{u3}|} \qquad \text{u3n} = \begin{bmatrix} 0.667 \\ -0.333 \\ 0.667 \end{bmatrix}$$

These are plotted:
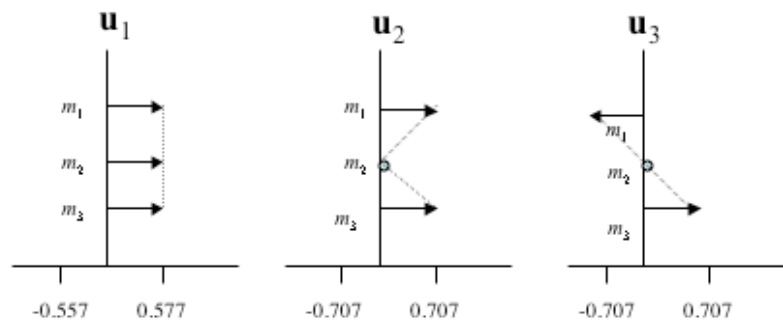
4.47 Solve for the free response of the system of Problem 4.46. Where $E = 6.9 \times 10^9$ N/m^2, $l = 2$ m, $m = 3000$ kg, and $I = 5.2 \times 10^{-6}$ m^4. Let the initial displacement correspond to a gust of wind that causes an initial condition of $\dot{\mathbf{x}}(0) = \mathbf{0}$, $\mathbf{x}(0) = [0.2 \ 0 \ 0]^T$ m. Discuss your solution.

Solution: From problem 4.43 and the given data

$$
\begin{bmatrix} 3000 & 0 & 0 \\ 0 & 12{,}000 & 0 \\ 0 & 0 & 3{,}000 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 1.346 & -1.346 & 0 \\ -1.346 & 2.691 & -1.346 \\ 0 & -1.346 & 1.346 \end{bmatrix} \times 10^4 \mathbf{x} = \mathbf{0}
$$

$$\mathbf{x}(0) = \begin{bmatrix} 0.2 & 0 & 0 \end{bmatrix}^T m$$

$$\dot{\mathbf{x}}(0) = \mathbf{0}$$

Convert to **q**:

$$
I\ddot{\mathbf{q}} + \begin{bmatrix} 4.485 & -2.242 & 0 \\ -2.242 & 2.242 & -2.242 \\ 0 & -2.242 & 4.485 \end{bmatrix} \mathbf{q} = \mathbf{0}
$$

Calculate eigenvalues and eigenvectors:

$$\det\left(\tilde{K} - \lambda I \right) = 0 \Rightarrow$$

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$
$$\lambda_2 = 4.485 \quad \omega_2 = 2.118 \text{ rad/s}$$
$$\lambda_3 = 6.727 \quad \omega_3 = 2.594 \text{ rad/s}$$

$$
\mathbf{v}_1 = \begin{bmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}
$$

The solution is given by

$$\mathbf{q}(t) = \left(c_1 + c_4 t \right)\mathbf{v}_1 + c_2 \sin\left(\omega_2 t + \phi_2\right)\mathbf{v}_2 + c_3 \sin\left(\omega_3 t + \phi_3\right)\mathbf{v}_3$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \qquad i = 2,3$$

$$c_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \qquad i = 2,3$$

Thus, $\phi_2 = \phi_3 = \dfrac{\pi}{2}, c_2 = -7.7459,$ and $c_3 = 6.3251$

So,

$$\mathbf{q}(0) = c_1 \mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_4 \mathbf{v}_i + \sum_{i=2}^{3} \omega_i c_i \cos\phi_i \mathbf{v}_i$$

Premultiply by $\mathbf{v}_i^T$:

$$\mathbf{v}_i^T \mathbf{q}(0) = 4.4716 = c_1$$

$$\mathbf{v}_i^T \dot{\mathbf{q}}(0) = 0 = c_4$$

So, $\mathbf{q}(t) = 4.4716\mathbf{v}_1 - 7.7459\cos(2.118t)\mathbf{v}_2 + 6.3251\cos(2.594t)\mathbf{v}_3$

Convert to physical coordinates:

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) \Rightarrow$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.0333 \\ 0.0333 \\ 0.0333 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \\ -0.1 \end{bmatrix} \cos 2.118t + \begin{bmatrix} 0.0667 \\ -0.0333 \\ 0.0667 \end{bmatrix} \cos 2.594t \text{ m}$$

The first term is a rigid body mode, which represents (in this case) a fixed displacement around which the three masses oscillate. Mode two has the highest amplitude (0.1 m).

4.48 Consider the two-mass system of Figure P4.48. This system is free to move in the $x_1 - x_2$ plane. Hence each mass has two degrees of freedom. Derive the linear equations of motion, write them in matrix form, and calculate the eigenvalues and eigenvectors for $m = 10$ kg and $k = 100$ N/m.
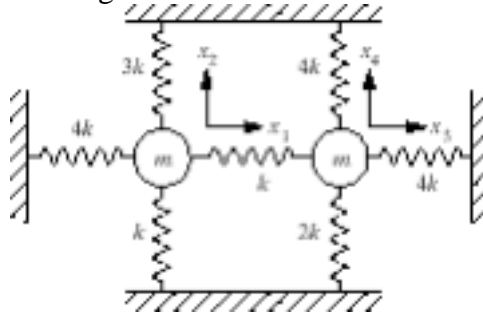
Solution: Given: $m = 10$ kg, $k = 100$ N/m

Mass 1

$$x_1 - \text{direction:} \quad m\ddot{x}_1 = -4kx_1 + k\left(x_3 - x_1\right) = -5kx_1 + kx_3$$

$$x_2 - \text{direction:} \quad m\ddot{x}_2 = -3kx_2 - kx_2 = -4kx_2$$

Mass 2

$$x_3 - \text{direction:} \quad m\ddot{x}_3 = -4kx_3 - k\left(x_3 - x_1\right) = -kx_1 - 5kx_3$$

$$x_4 - \text{direction:} \quad m\ddot{x}_4 = -4kx_4 - 2kx_4 = -6kx_4$$

In matrix form with the values given:

$$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 500 & 0 & -100 & 0 \\ 0 & 400 & 0 & 0 \\ -100 & 0 & 500 & 0 \\ 0 & 0 & 0 & 600 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 50 & 0 & -10 & 0 \\ 0 & 40 & 0 & 0 \\ -10 & 0 & 50 & 0 \\ 0 & 0 & 0 & 60 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^4 - 200\lambda^3 + 14{,}800\lambda^2 - 480{,}000\lambda + 5{,}760{,}000 = 0$$

$$\Rightarrow \lambda_1 = 40, \quad \lambda_2 = 40, \quad \lambda_3 = 60, \quad \lambda_4 = 60$$

The corresponding eigenvectors are found from solving $(\tilde{K} - \lambda_i)\mathbf{v}_i = 0$ for each value of the index and normalizing:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

These are not unique.

4.49 Consider again the system discussed in Problem 4.48. Use modal analysis to calculate the solution if the mass on the left is raised along the x_2 direction exactly 0.01 m and let go.

Solution: From Problem 4.48:

$$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 500 & 0 & -100 & 0 \\ 0 & 400 & 0 & 0 \\ -100 & 0 & 500 & 0 \\ 0 & 0 & 0 & 600 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$M^{-1/2} = 0.3162 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 50 & 0 & -10 & 0 \\ 0 & 40 & 0 & 0 \\ -10 & 0 & 50 & 0 \\ 0 & 0 & 0 & 60 \end{bmatrix}$$

$\lambda_1 = 40 \quad \omega_1 = 6.3246$ rad/s

$\lambda_2 = 40 \quad \omega_2 = 6.3246$ rad/s

$\lambda_3 = 60 \quad \omega_3 = 7.7460$ rad/s

$\lambda_4 = 60 \quad \omega_4 = 7.7460$ rad/s

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Also, $\mathbf{x}(0) = [0 \quad 0.01 \quad 0 \quad 0]^T$ m and $\dot{\mathbf{x}}(0) = \mathbf{0}$

Use the mode summation method to find the solution.
Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 0 & 0.003162 & 0 & 0 \end{bmatrix}^T$$
$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \mathbf{0}$$

The solution is given by Eq. (4.103),

$$\mathbf{x}(t) = \sum_{i=1}^{4} d_i \sin(\omega_i t + \phi_i)\mathbf{u}_i$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \quad i = 1,2,3,4 \quad (\text{Eq.}(4.97))$$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \quad i = 1,2,3,4 \quad \left(\text{Eq. }(4.98)\right)$$

$$\mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

Substituting known values yields

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = \frac{\pi}{2} \text{ rad}$$

$$d_1 = 0.003162$$
$$d_2 = d_3 = d_4 = 0$$

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0.3162 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.2236 \\ 0 \\ 0.2236 \\ 0 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.2236 \\ 0 \\ -0.2236 \\ 0 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.3162 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} 0 \\ 0.001\cos 6.3246t \\ 0 \\ 0 \end{bmatrix}$$

4.50   The vibration of a floor in a building containing heavy machine parts is modeled in Figure P4.50. Each mass is assumed to be evenly spaced and significantly larger than the mass of the floor.   The equation of motion then becomes $\left(m_1 = m_2 = m_3 = m\right)$.

$$m I \ddot{\mathbf{x}} + \frac{EI}{l^3} \begin{bmatrix} \dfrac{9}{64} & \dfrac{1}{6} & \dfrac{13}{192} \\ \dfrac{1}{6} & \dfrac{1}{3} & \dfrac{1}{6} \\ \dfrac{13}{192} & \dfrac{1}{6} & \dfrac{9}{64} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

Calculate the natural frequencies and mode shapes. Assume that in placing box m_2 on the floor (slowly) the resulting vibration is calculated by assuming that the initial displacement at m_2 is 0.05 m. If $l = 2$ m, $m = 200$ kg, $E = 0.6 \times 10^9$ N/m^2, $I = 4.17 \times 10^{-5}$ m^4. Calculate the response and plot your results.
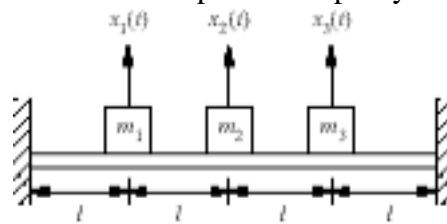
Solution:

The equations of motion can be written as

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \frac{EI}{l^3} \begin{bmatrix} \dfrac{9}{64} & \dfrac{1}{6} & \dfrac{13}{192} \\ \dfrac{1}{6} & \dfrac{1}{3} & \dfrac{1}{6} \\ \dfrac{13}{192} & \dfrac{1}{6} & \dfrac{9}{64} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = 0$$

or $m I \ddot{x} + K x = 0$ where I is the 3x3 identity matrix.

The natural frequencies of the system are obtained using the characteristic equation

$$\left| K - \omega^2 M \right| = 0$$

Using the given mass and stiffness matrices yields the following characteristic equation

$$m^3\omega^6 - \frac{59\,EI\,m^3}{96l^3}\omega^4 + \frac{41\left(EI\right)^2 m}{768l^6}\omega^2 - \frac{7\left(EI\right)^3}{6912l^9} = 0$$

Substituting for E, I, m, and l yields the following answers for the natural frequency

$$\omega_1 = \pm\sqrt{\frac{\left(13-\sqrt{137}\right)EI}{ml^3}} \;,\; \omega_2 = \pm\sqrt{\frac{7\,EI}{96ml^3}} \;,\; \omega_3 = \pm\sqrt{\frac{\left(13+\sqrt{137}\right)EI}{48ml^3}}$$

The plus minus sign shown above will cause the exponential terms to change to trigonometric terms using Euler's formula. Hence, the natural frequencies of the system are 0.65 rad/sec, 1.068 rad/sec and 2.837 rad/sec.

Let the mode shapes of the system be $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. The mode shapes should satisfy the following equation

$$\left[K - \omega_1^2 M\right]\begin{Bmatrix} u_{i1} \\ u_{i2} \\ u_{i3} \end{Bmatrix} = 0, i = 1,2,3$$

Notice that the system above does not have a unique solution for $\mathbf{u}_1$ since $\left[K - \omega_1^2 M\right]$ had to be singular in order to solve for the natural frequency ω. Solving the above equation yields the following relations

$$\frac{u_{i2}}{u_{i3}} = \frac{1}{3}\frac{96m\omega_i^2 l^3 - 7\,EI}{13m\omega_i^2 l^3 + EI}, i = 1,2,3$$

and $u_{i1} = u_{i3}, i = 1,3$ but for the second mode shape this is different $u_{21} = u_{23}$

Substituting the values given yields

$$\frac{u_{12}}{u_{13}} = \frac{1}{3}\frac{96m\omega_1^2 l^3 - 7\,EI}{13m\omega_1^2 l^3 + EI} = -1.088$$

$$\frac{u_{22}}{u_{23}} = \frac{1}{3}\frac{96m\omega_2^2 l^3 - 7\,EI}{13m\omega_2^2 l^3 + EI} = 0$$

$$\frac{u_{32}}{u_{33}} = \frac{1}{3}\frac{96m\omega_3^2 l^3 - 7\,EI}{13m\omega_3^2 l^3 + EI} = 1.838$$

If we let $u_{i3} = 1, i = 1,2,3$, then

$$u_1 = \begin{Bmatrix} 1 \\ -1.088 \\ 1 \end{Bmatrix}, u_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}, u_3 = \begin{Bmatrix} 1 \\ 1.838 \\ 1 \end{Bmatrix}$$

These mode shapes can be normalized to yield

$$u_1 = \begin{Bmatrix} 0.5604 \\ -0.6098 \\ 0.5604 \end{Bmatrix}, u_2 = \begin{Bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{Bmatrix}, u_3 = \begin{Bmatrix} 0.4312 \\ 0.7926 \\ 0.4312 \end{Bmatrix}$$

This solution is the same if obtained using MATLAB

$$u_1 = \begin{Bmatrix} -0.5604 \\ 0.6098 \\ -0.5604 \end{Bmatrix}, u_2 = \begin{Bmatrix} -0.7071 \\ 0.0000 \\ 0.7071 \end{Bmatrix}, u_3 = \begin{Bmatrix} 0.4312 \\ 0.7926 \\ 0.4312 \end{Bmatrix}$$

The second box, m_2, is placed slowly on the floor; hence, the initial velocity can be safely assumed zero. The initial displacement at m_2 is given to be 0.05 m.

Hence, the initial conditions in vector form are given as

$$x(0) = \begin{Bmatrix} 0 \\ -0.05 \\ 0 \end{Bmatrix} \text{ and } \dot{x}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The equations of motion given by $M\ddot{x}(t) + Kx(t) = 0$ can be transformed into the modal coordinates by applying the following transformation

$\mathbf{x}(t) = S\mathbf{r}(t) = M^{-\frac{1}{2}}P\mathbf{r}(t)$ where P is the basis formed by the mode shapes of the system, given by

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$$

Hence, the transformation S is given by

$$S = \begin{bmatrix} -0.04 & -0.05 & 0.03 \\ 0.043 & 0 & 0.056 \\ -0.04 & 0.05 & 0.03 \end{bmatrix}$$

The initial conditions will be also transformed

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \begin{Bmatrix} -0.431 \\ 0 \\ -0.56 \end{Bmatrix}$$

Hence, the modal equations are

with the above initial conditions.

The solution will then be

$$\mathbf{r}(t) = \begin{Bmatrix} 0.431\cos(0.65t) \\ 0 \\ 0.56\cos(2.837t) \end{Bmatrix}$$

The solution can then be determined by

$$\mathbf{x}(t) = \begin{Bmatrix} 0.0172\cos(0.65t) - 0.0168\cos(2.837t) \\ -0.0185\cos(0.65t) - 0.0313\cos(2.837t) \\ 0.0172\cos(0.65t) - 0.0168\cos(2.837t) \end{Bmatrix}$$

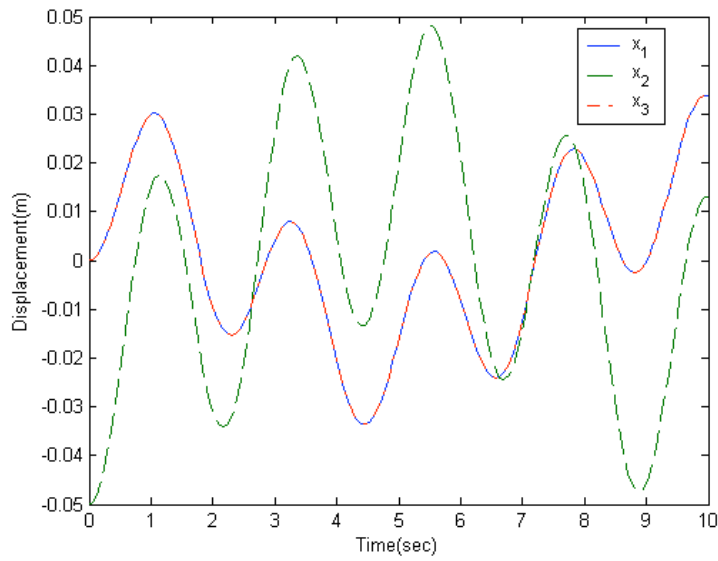The equations of motion can be also be solved using MATLAB to yield the following response.

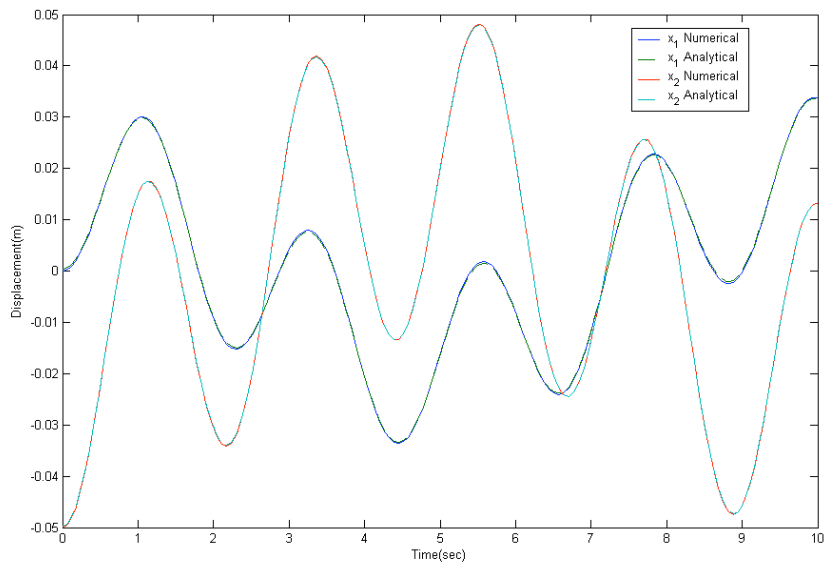**Figure 1 Numerical response due to initial deflection at m_2**

Figure 2 Numerical vs. Analytical Response (shown for x_1 and x_2 only)

The MATLAB code is attached below

```
% Set the values of the physical parameters
%
**********************************************************************
*

% Declare global variables to be used in the differential equation file
global M K
```

```matlab
% Define the mass of the each box
m=200;

% Define the distance l
l=2;

% Define the area moment of inertia
I=4.17*10^-5;

% Define the modulus of elasticity
E=0.6*10^9;

% Define the flexural rigidity
EI=E*I;

% Define the system matrices
%
******************************************************************************
*

% Define the mass matrix
M=m*eye(3,3);

% Define the stiffness matrix
K=EI/l^3*[9/64 1/6 13/192;1/6 1/3 1/6;13/192 1/6 9/64];

% Solve the eigen value problem
[u,lambda]=eig(M\K);

% Simulate the response of the system to the given initial conditions
% The states are arranges as: [x1;x2;x3;x1_dot;x2_dot;x3_dot]
[t,xn]=ode45('sys4p47',[0 10],[0 ; -0.05 ; 0 ; 0 ; 0 ; 0]);

% Plot the results
plot(t,xn(:,1),t,xn(:,2),'--',t,xn(:,3),'-.');
set(gcf,'Color','White');
xlabel('Time(sec)');
ylabel('Displacement(m)');
legend('x_1','x_2','x_3');

% Analytical solution

for i=1:length(t)
    xa(:,i)=[0.0172*cos(0.65*t(i))-0.0168*cos(2.837*t(i));
          -0.0185*cos(0.65*t(i))-0.0313*cos(2.837*t(i)) ;
```

```
        0.0172*cos(0.65*t(i))-0.0168*cos(2.837*t(i)))];
end;

% Camparison
figure;
plot(t,xn(:,1),t,xa(1,:),'--',t,xn(:,2),t,xa(2,:),'--');
set(gcf,'Color','White');
xlabel('Time(sec)');
ylabel('Displacement(m)');
legend('x_1 Numerical','x_1 Analytical','x_2 Numerical','x_2 Analytical');
```

4.51   Recalculate the solution to Problem 4.50 for the case that m_2 is increased in mass to 2000 kg. Compare your results to those of Problem 4.50. Do you think it makes a difference where the heavy mass is placed?

Solution: Given the data indicated the equation of motion becomes:

$$\begin{bmatrix} 200 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 200 \end{bmatrix} \ddot{\mathbf{x}} + 3.197 \times 10^{-4} \begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = [0 \quad 0.05 \quad 0]^T, \dot{\mathbf{x}}(0) = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.07071 & 0 & 0 \\ 0 & 0.02246 & 0 \\ 0 & 0 & 0.07071 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 2.2482 & 0.8246 & 1.0825 \\ 0.8246 & 0.5329 & 0.8246 \\ 1.0825 & 0.8246 & 2.2482 \end{bmatrix} \times 10^{-7}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 9.8255 \times 10^{-7}\lambda^2 + 1.3645 \times 10^{-14}\lambda - 4.1382 \times 10^{-22} = 0$$

$$\lambda_1 = 4.3142 \times 10^{-9} \quad \omega_1 = 2.0771 \times 10^{-5} \text{ rad/s}$$

$$\lambda_2 = 1.1657 \times 10^{-7} \quad \omega_2 = 3.4143 \times 10^{-4} \text{ rad/s}$$

$$\lambda_3 = 8.2283 \times 10^{-7} \quad \omega_3 = 9.0710 \times 10^{-4} \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.2443 \\ -0.9384 \\ 0.2443 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.6636 \\ 0.3455 \\ 0.6636 \end{bmatrix}$$

Use the mode summation method to find the solution. Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 0 & 2.2361 & 0 \end{bmatrix}^T$$
$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \mathbf{0}$$

The solution is given by Eq. (4.103),

$$\mathbf{x}(t) = \sum_{i=1}^{4} d_i \sin(\omega_i t + \phi_i)\mathbf{u}_i$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \quad i = 1,2,3 \quad \left(\text{Eq. } (4.97)\right)$$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \quad i = 1,2,3 \quad \left(\text{Eq. } (4.98)\right)$$

$$\mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

Substituting known values yields

$$\phi_1 = \phi_2 = \phi_3 = \frac{\pi}{2} \text{ rad}$$
$$d_1 = -2.0984$$
$$d_2 = 0$$
$$d_3 = 0.7726$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.0178 \\ -0.02098 \\ 0.01728 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.05 \\ 0 \\ -0.05 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.04692 \\ 0.007728 \\ 0.04692 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} -0.03625 \\ 0.04403 \\ -0.03625 \end{bmatrix} \cos\left(9.7044 \times 10^{-5} t\right) + \begin{bmatrix} 0.03625 \\ 0.005969 \\ 0.0325 \end{bmatrix} \cos\left(6.1395 \times 10^{-4} t\right) \text{ m}$$

The results are very similar to Problem 50. The responses of mass 1 and 3 are the same for both problems, except the amplitudes and frequencies are changed due to the increase in mass 2. There would have been a greater change if the heavy mass was placed at mass 1 or 3.

4.52 Repeat Problem 4.46 for the case that the airplane body is 10 m instead of 4 m as indicated in the figure. What effect does this have on the response, and which design (4m or 10 m) do you think is better as to vibration?

Solution: Given:

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \frac{EI}{l^3}\begin{bmatrix} 3 & -3 & - \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = m^{-1/2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.3612 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \frac{EI}{ml^3}\begin{bmatrix} 3 & -0.9487 & 0 \\ -0.9487 & 0.6 & -0.9487 \\ 0 & -0.9487 & 3 \end{bmatrix}$$

Again choose the parameters so that the coefficient is 1 and compute the eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 6.6\lambda^2 + 10.8\lambda = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = 3$$

$$\lambda_3 = 3.6$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.2887 \\ -0.9129 \\ -0.2887 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.6455 \\ -0.4082 \\ 0.6455 \end{bmatrix}$$

The natural frequencies are

$$\omega_1 = 0 \text{ rad/s}$$

$$\omega_2 = 1.7321 \text{ rad/s}$$

$$\omega_3 = 1.8974 \text{ rad/s}$$

The relationship between eigenvectors and mode shapes is

$$\mathbf{u} = M^{-1/2}\mathbf{v}$$

$$\mathbf{u}_1 = m^{-1/2}\begin{bmatrix} -0.2887 \\ -0.2887 \\ -0.2887 \end{bmatrix} \quad \mathbf{u}_2 = m^{-1/2}\begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.6455 \\ -0.1291 \\ 0.6455 \end{bmatrix}$$

It appears that the mode shapes contain less "amplitude" for the wing masses. This seems to be a better design from a vibration standpoint.

4.53 Often in the design of a car, certain parts cannot be reduced in mass. For example, consider the drive train model illustrated in Figure P4.44. The mass of the torque converter and transmission are relatively the same from car to car. However, the mass of the car could change as much as 1000 kg (e.g., a two-seater sports car versus a family sedan). With this in mind, resolve Problem 4.44 for the case that the vehicle inertia is reduced to 2000 kg. Which case has the smallest amplitude of vibration?

Solution: Let k_1 = hub stiffness and k_2 = axle and suspension stiffness. From Problem 4.44, the equation of motion becomes

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 2000 \end{bmatrix} \ddot{\mathbf{x}} + 10,000 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{0} \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m/s.}$$

Calculate eigenvalues and eigenvectors.

$$M^{-1/2} = \begin{bmatrix} 0.1155 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.0224 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 133.33 & -115.47 & 0 \\ -115.47 & 300 & -44.721 \\ 0 & -44.721 & 10 \end{bmatrix}$$

$$\det(\tilde{K} - \lambda I) = \lambda^3 - 443.33\lambda^2 + 29,000\lambda = 0$$

$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$

$\lambda_2 = 70.765 \quad \omega_2 = 8.9311 \text{ rad/s}$

$\lambda_3 = 363.57 \quad \omega_3 = 19.067 \text{ rad/s}$

$$\mathbf{v}_1 = \begin{bmatrix} -0.1857 \\ -0.2144 \\ -0.9589 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.8758 \\ 0.4063 \\ -0.2065 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4455 \\ -0.8882 \\ 0.1123 \end{bmatrix}$$

Use the mode summation method to find the solution. Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = 0$$

$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 44.7214 \end{bmatrix}^T$$

The solution is given by

$$\mathbf{q}(t) = \left(c_1 + c_4 t\right)\mathbf{v}_1 + c_2 \sin\left(\omega_2 t + \phi_2\right)\mathbf{v}_2 + c_3 \sin\left(\omega_3 t + \phi_3\right)\mathbf{v}_3$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right), \quad i = 2,3$$

$$c_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\omega_i \cos\phi_i}, \quad i = 2,3$$

Thus $\phi_2 = \phi_3 = 0$, $c_2 = -1.3042$ and $c_3 = 0.2635$. Next apply the initial conditions:

$$\mathbf{q}(0) = c_1\mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i\mathbf{v}_i \quad \text{and} \quad \dot{\mathbf{q}}(0) = c_4\mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i\mathbf{v}_i$$

Pre multiply each of these by $\mathbf{v}_1^T$ to get:

$$c_1 = 0 = \mathbf{v}_1^T \mathbf{q}(0) \quad \text{and} \quad c_4 = -42.8845 = \mathbf{v}_1^T \dot{\mathbf{q}}(0)$$

So

$$\mathbf{q}(t) = -42.8845t\mathbf{v}_1 - 1.3042\sin(8.9311t)\mathbf{v}_2 + 0.2635\sin(19.067t)\mathbf{v}_3$$

Next convert back to the physical coordinates by

$$\mathbf{x}(t) = M^{-\frac{1}{2}}\mathbf{q}(t)$$

$$= 0.9195t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.1319 \\ -0.05299 \\ 0.007596 \end{bmatrix}\sin 8.9311t + \begin{bmatrix} 0.01355 \\ -0.02340 \\ 0.0006620 \end{bmatrix}\sin 19.067t \text{ m}$$

Comparing this solution to problem 4.44, the car will vibrate at a slightly higher amplitude when the mass is reduced to 2000 kg.

4.54 Use *mode summation method* to compute the analytical solution for the response of the 2-degree-of-freedom system of Figure P4.28 with the values where $m_1 = 1$ kg, $m_2 = 4$ kg, $k_1 = 240$ N/m and $k_2 = 300$ N/m, to the initial conditions of

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \quad \dot{\mathbf{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution: Following the development of equations (4.97) through (4.103) for the mode summation for the free response and using the values of computed in problem 1, compute the initial conditions for the "**q**" coordinate system:

$$M^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \mathbf{q}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}, \mathbf{q}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From equation (4.97):

$$\phi_1 = \tan^{-1}\left(\frac{x}{0}\right) = \phi_2 = \tan^{-1}\left(\frac{x}{0}\right) = \frac{\pi}{2}$$

From equation (4.98):

$$d_1 = \frac{\mathbf{v}_1^T \mathbf{q}(0)}{\sin(\pi/2)} = \mathbf{v}_1^T \mathbf{q}(0), d_2 = \frac{\mathbf{v}_2^T \mathbf{q}(0)}{\sin(\pi/2)} = \mathbf{v}_2^T \mathbf{q}(0)$$

Next compute $\mathbf{q}(t)$ from (4.92) and multiply by $M^{1/2}$ to get $\mathbf{x}(t)$ or use (4.103) directly to get

$$\mathbf{q}(t) = d_1 \cos(\omega_1 t)\mathbf{v}_1 + d_2 \cos(\omega_2 t)\mathbf{v}_2 = \cos(\omega_1 t)\mathbf{v}_1^T \mathbf{q}(0)\mathbf{v}_1 + \cos(\omega_2 t)\mathbf{v}_1^T \mathbf{q}(0)\mathbf{v}_1$$

$$= \cos(5.551t)\begin{bmatrix} 0.0054 \\ 0.0184 \end{bmatrix} + \cos(24.170t)\begin{bmatrix} -0.0054 \\ 0.0016 \end{bmatrix}$$

Note that as a check, substitute $t = 0$ in this last line to recover the correct initial condition $\mathbf{q}(0)$. Next transform the solution back to the physical coordinates

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = \cos(5.551t)\begin{bmatrix} 0.0054 \\ 0.0092 \end{bmatrix} + \cos(24.170t)\begin{bmatrix} -0.0054 \\ 0.0008 \end{bmatrix} \text{ m}$$

4.55 For a zero value of an eigenvalue and hence frequency, what is the corresponding time response? Or asked another way, the form of the modal solution for a non-zero frequency is $A\sin(\omega_n t + \phi)$, what is the form of the modal solution that corresponds to a zero frequency? Evaluate the constants of integration if the modal initial conditions are: $r_1(0) = 0.1$, and $\dot{r}_1(0) = 0.01$.

Solution: A zero eigenvalue corresponds to the modal equation:
$$\ddot{r}_1(t) = 0 \Rightarrow r_1(t) = a + bt$$
Applying the given initial conditions:
$$r_1(0) = a + b(0) = 0.1 \Rightarrow a = 0.1$$
$$\dot{r}_1(0) = b = 0.01$$
$$\Rightarrow r_1(t) = 0.1 + 0.01t$$

Problems and Solutions for Section 4.5 (4.56 through 4.66)

4.56   Consider the example of the automobile drive train system discussed in Problem 4.44. Add 10% modal damping to each coordinate, calculate and plot the system response.

Solution: Let k_1 = hub stiffness and k_2 = axle and suspension stiffness. From Problem 4.44, the equation of motion with damping is

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \ddot{\mathbf{x}} + 10{,}000 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{0} \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m/s}$$

Other calculations from Problem 4.44 yield:

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$
$$\lambda_2 = 77.951 \quad \omega_2 = 8.8290 \text{ rad/s}$$
$$\lambda_3 = 362.05 \quad \omega_3 = 19.028 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.1537 \\ 0.1775 \\ 0.9721 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.8803 \\ -0.4222 \\ 0.2163 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4488 \\ -0.8890 \\ 0.0913 \end{bmatrix}$$

Use the summation method to find the solution. Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \mathbf{0}$$
$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 54.7723 \end{bmatrix}^T$$

Also, $\zeta_1 = \zeta_2 = \zeta_3 = 0.1$.

$$\omega_{d2} = 8.7848 \text{ rad/s}$$
$$\omega_{d3} = 18.932 \text{ rad/s}$$

The solution is given by

$$\mathbf{q}(t) = \left(c_1 + c_2 t \right) \mathbf{v}_1 + \sum_{2}^{3} d_i e^{-\zeta_i \omega_i t} \sin\left(\omega_{di} t + \phi_i \right) \mathbf{v}_i$$

where $\phi_i = \tan^{-1}\left(\dfrac{\omega_{di} \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0) + \zeta_i \omega_i \mathbf{v}_i^T \mathbf{q}(0)} \right)$ $i = 2,3$ Eq. (4.114)

$$d_i = \frac{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}{\omega_{di} \cos\phi_i - \zeta_i \omega_i \sin\phi_i} \quad i = 2,3$$

Thus,

$$\phi_2 = \phi_3 = 0$$
$$d_2 = 1.3485$$
$$d_3 = 0.2642$$

Now,

$$\mathbf{q}(0) = c_1 \mathbf{v}_1 + \sum_{i=2}^{3} d_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_2 \mathbf{v}_1 + \sum_{i=2}^{3} \left[-\zeta_i \omega_i d_i \sin\phi_i + \omega_{di} d_i \cos\phi_i \right] \mathbf{v}_i$$

Pre-multiply by $\mathbf{v}_1^T$:

$$\mathbf{v}_1^T \mathbf{q}(0) = 0 = c_1$$

$$\mathbf{v}_1^T \dot{\mathbf{q}}(0) = 53.2414 = c_2$$

So,

$$\mathbf{q}(t) = 53.2414 \mathbf{v}_1 - 1.3485 e^{-0.8829t} \sin(8.7848t) \mathbf{v}_2 + 0.2648 t e^{-1.9028t} \sin(18.932t) \mathbf{v}_3$$

The solution is given by

$$\mathbf{x}(t) = M^{-1/2} \mathbf{q}(t)$$

$$\mathbf{x}(t) = 0.9449t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -0.1371 \\ -0.05693 \\ 0.005325 \end{bmatrix} e^{-0.8829t} \sin(8.7848t) + \begin{bmatrix} 0.01369 \\ -0.002349 \\ 0.0004407 \end{bmatrix} e^{-1.9028t} \sin(18.932t) \text{ m}$$
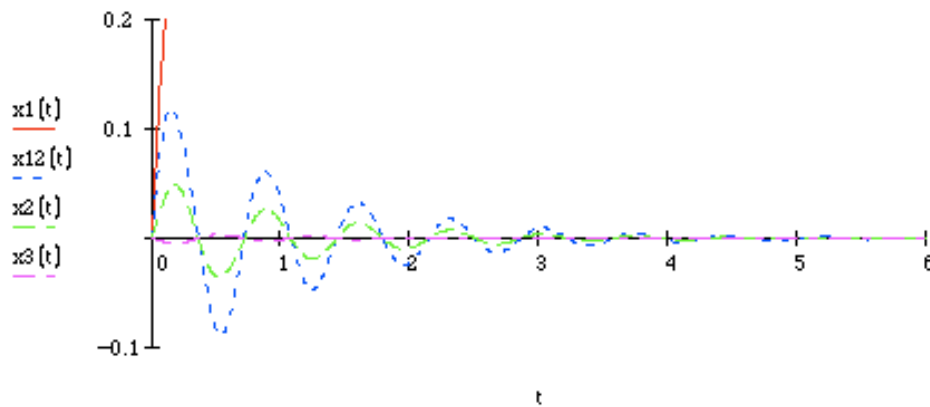
The following Mathcad session illustrates the solution without the rigid body mode (except for x_1 which shows both with and without the rigid mode)

$$x1(t) := 0.9449 \cdot t + 0.1371 \cdot e^{-.8829 \cdot t} \cdot \sin(8.7848 \cdot t) + 0.01369 \cdot e^{-1.9028 \cdot t} \cdot \sin(18.932 \cdot t)$$

$$x12(t) := \left(0.1371 \cdot e^{-.8829 \cdot t} \cdot \sin(8.7848 \cdot t) + 0.01369 \cdot e^{-1.9028 \cdot t} \cdot \sin(18.932 \cdot t) \right)$$

$$x2(t) := 0.05693 \cdot e^{-.8829 \cdot t} \cdot \sin(8.7848 \cdot t) - 0.002349 \cdot e^{-1.9028 \cdot t} \cdot \sin(18.932 \cdot t)$$

$$x3(t) := -0.005325 \cdot e^{-.8829 \cdot t} \cdot \sin(8.7848 \cdot t) + 0.000447 \cdot e^{-1.9028 \cdot t} \cdot \sin(18.932 \cdot t)$$

The read solid line is the first mode with the rigid body mode included.

4.57   Consider the model of an airplane discussed in problem 4.47, Figure P4.46. (a) Resolve the problem assuming that the damping provided by the wing rotation is $\zeta_i = 0.01$ in each mode and recalculate the response. (b) If the aircraft is in flight, the damping forces may increase dramatically to $\zeta_i = 0.1$. Recalculate the response and compare it to the more lightly damped case of part (a).

Solution:

From Problem 4.47, with damping

$$\begin{bmatrix} 3000 & 0 & 0 \\ 0 & 12,000 & 0 \\ 0 & 0 & 3,000 \end{bmatrix} \ddot{\mathbf{x}} + C\dot{\mathbf{x}} + \begin{bmatrix} 13455 & -13455 & 0 \\ -13,455 & 26910 & -13,455 \\ 0 & -13,455 & 13,455 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0.02 & 0 & 0 \end{bmatrix}^T \text{ m}$$

$$\dot{\mathbf{x}}(0) = \mathbf{0}$$

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$

$$\lambda_2 = 4.485 \quad \omega_2 = 2.118 \text{ rad/s}$$

$$\lambda_3 = 6.727 \quad \omega_3 = 2.594 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.4082 \\ -0.8165 \\ -0.4082 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

The solution is given by

$$\mathbf{q}(t) = \left(c_1 + c_2 t \right) \mathbf{v}_1 + \sum_{i=2}^{3} d_i e^{-\zeta_i \omega_i t} \sin\left(\omega_{di} t + \phi_i \right) \mathbf{v}_i$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_{di} \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0) + \zeta_i \omega_i \mathbf{v}_i^T \mathbf{q}(0)} \right) \quad i = 2,3$$

(Eq. (4.114))

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin \phi_i} \quad i = 2,3$$

Now,

$$\mathbf{q}(0) = c_1 \mathbf{v}_1 + \sum_{i=2}^{3} d_i \sin \phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_2 \mathbf{v}_1 + \sum_{i=2}^{3} \left[-\zeta_i \omega_i d_i \sin \phi_i + \omega_{di} d_i \cos \phi_i \right] \mathbf{v}_i$$

Premultiply by $\mathbf{v}_1^T$:

$$\mathbf{v}_1^T \mathbf{q}(0) = 4.4721 = c_1$$
$$\mathbf{v}_1^T \dot{\mathbf{q}}(0) = 0 = c_2$$

(a) $\zeta_1 = \zeta_2 = \zeta_3 = 0.01$

$$\omega_{d2} = 2.1177 \text{ rad/s}, \quad \omega_{d3} = 2.593 \text{ rad/s}$$
$$\phi_2 = -1.5808 \text{ rad}, \quad \phi_3 = 1.5608 \text{ rad}$$
$$d_2 = 7.7464, \quad d_3 = 6.3249$$

Mode shapes:

$$\mathbf{u}_i = M^{-1/2} \mathbf{v}_i$$

$$\mathbf{u}_1 = \begin{bmatrix} -0.007454 \\ -0.007454 \\ -0.007454 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.01291 \\ 0 \\ -0.01291 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.01054 \\ -0.005270 \\ 0.01054 \end{bmatrix}$$

The solution is given by

$$\mathbf{x}(t) = \left(c_1 + c_2 t \right) \mathbf{u}_1 + \sum_{i=2}^{3} d_i e^{-\zeta_i \omega_i t} \sin\left(\omega_{di} t + \phi_i \right) \mathbf{u}_i$$

$$x(t) = 0.0333 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.100 \\ 0 \\ 0.100 \end{bmatrix} e^{-0.0212t} \sin\left(2.1178t - 1.5808 \right)$$

$$+ \begin{bmatrix} 0.0667 \\ -0.0333 \\ 0.0677 \end{bmatrix} e^{-0.0259t} \sin\left(2.5937t + 1.5608 \right)$$

b) $\zeta_1 = \zeta_2 = \zeta_3 = 0.1$

Same thing as part (a), but now the following values are obtained

$$\omega_{d2} = 2.1072 \text{ rad/sec} \quad \omega_{d3} = 2.5807 \text{ rad/sec}$$
$$\phi_2 = -1.6710 \text{ rad} \quad \phi_3 = 1.4706 \text{ rad}$$
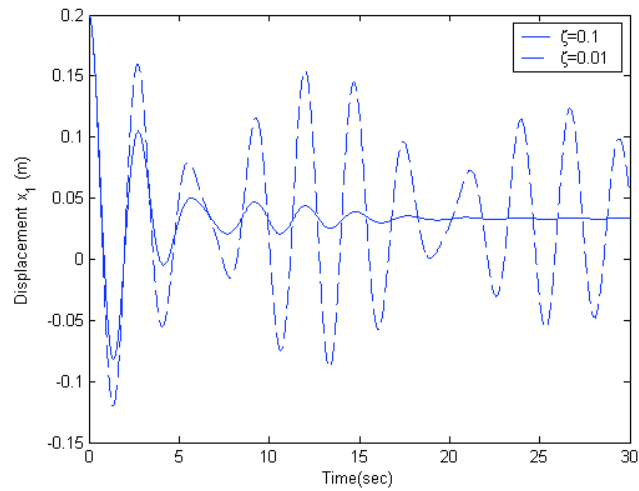$$d_2 = 7.7850 \quad d_3 = 6.3564$$

Notice that the rigid mode is not effected by changing the damping ratio, and hence

$$c = 4.4721$$

Consequently, the solution becomes

$$x(t) = 0.0333 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.1005 \\ 0 \\ 0.1005 \end{bmatrix} e^{-0.2118t} \sin\left(2.1072t - 1.6710\right)$$

$$+ \begin{bmatrix} 0.0670 \\ -0.0335 \\ 0.0670 \end{bmatrix} e^{-0.2594t} \sin\left(2.5807t + 1.4706\right)$$

Below is the plot of the displacement of the left wing

4.58 Repeat the floor vibration problem of Problem 4.50 using modal damping ratios of

$$\zeta_1 = 0.01 \quad \zeta_2 = 0.1 \quad \zeta_3 = 0.2$$

Solution: The equation of motion will be of the form:

$$200\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + 3.197 \times 10^{-4} \begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 & 0.05 & 0 \end{bmatrix}^T \text{ m and } \dot{\mathbf{x}}(0) = \mathbf{0}.$$

$$M^{-1/2} = 0.7071$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 2.2482 & 2.6645 & 1.0825 \\ 2.6645 & 5.3291 & 2.6645 \\ 1.0825 & 2.6645 & 2.2482 \end{bmatrix} \times 10^{-7}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 9.8255 \times 10^{-7}\lambda^2 + 1.3645 \times 10^{-13}\lambda - 4.1382 \times 10^{-21} = 0$$

$$\lambda_1 = 4.3142 \times 10^{-8} \quad \omega_1 = 2.0771 \times 10^{-4} \text{ rad/s}$$

$$\lambda_2 = 1.1657 \times 10^{-7} \quad \omega_2 = 3.34143 \times 10^{-4} \text{ rad/s}$$

$$\lambda_3 = 8.2283 \times 10^{-7} \quad \omega_3 = 9.0710 \times 10^{-4} \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.5604 \\ -0.6098 \\ 0.5604 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4312 \\ 0.7926 \\ 0.4312 \end{bmatrix}$$

Use the mode summation method to find the solution. First transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 0 & 0.7071 & 0 \end{bmatrix}^T$$

$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \mathbf{0}$$

The solution is given by Eq. (4.115):

$$\mathbf{x}(t) = \sum_{i=1}^{3} d_i e^{-\zeta_i \omega_i t} \sin\left(\omega_{di} t + \phi_i\right) \mathbf{u}_i$$

where $\qquad \phi_i = \tan^{-1}\left(\dfrac{\omega_{di} \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0) + \zeta_i \omega_i \mathbf{v}_i^T \mathbf{q}(0)} \right) \qquad i = 1,2,3$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}'(0)}{\sin\phi_i} \qquad i = 1,2,3, \quad \mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

$$\zeta_1 = 0.01, \quad \zeta_2 = 0.1, \qquad \zeta_3 = 0.2$$

Substituting

$\omega_{d1} = 2.0770 \times 10^{-4}$ rad/s, $\omega_{d2} = 3.3972 \times 10^{-4}$ rad/s $\omega_{d3} = 8.8877 \times 10^{-4}$ rad/s

$\phi_1 = 1.5808$ rad, $\quad \phi_2 = 1.6710$ rad, $\quad \phi_3 = 1.3694$ rad

$d_1 = 0.4312, \quad d_2 = 0, \qquad d_3 = 0.5720$

The mode shapes are

$$\mathbf{u}_1 = \begin{bmatrix} 0.03963 \\ -0.04312 \\ 0.03963 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -0.05 \\ 0 \\ 0.05 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.03049 \\ 0.05604 \\ 0.03049 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} 0.01709 \\ -0.01859 \\ 0.01709 \end{bmatrix} e^{-2.0771 \times 10^{-4} t} \sin\left(2.0770 \times 10^{-4} t - 1.5808\right)$$

$$+ \begin{bmatrix} 0.01744 \\ 0.03206 \\ 0.01744 \end{bmatrix} e^{-2.0771 \times 10^{-4} t} \sin\left(8.8877 \times 10^{-4} t + 1.3694\right) \text{m}$$

4.59 Repeat Problem 4.58 with constant modal damping of ζ_1, ζ_2, $\zeta_3 = 0.1$ and compare this with the solution of Problem 4.58.

Solution: Use the equations of motion and initial conditions from Problem 4.58. The mode shapes, natural frequencies and transformed initial conditions remain the same. However the constants of integration are effected by the damping ratio so the solution

$$\mathbf{x}(t) = \sum_{i=1}^{3} d_i e^{-\zeta_i \omega_i t} \sin\left(\omega_{di} t + \phi_i\right) \mathbf{u}_i$$

has new constants determined by $\phi_i = \tan^{-1}\left(\dfrac{\omega_{di} \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0) + \zeta_i \omega_i \mathbf{v}_i^T \mathbf{q}(0)}\right)$ $i = 1,2,3$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}'(0)}{\sin \phi_i} \qquad i = 1,2,3$$

$$\mathbf{u}_i = M^{-1/2} \mathbf{v}_i$$

$$\zeta_1 = \zeta_2 = \zeta_3 = 0.1$$

Substituting yields

$$\omega_{d1} = 2.0667 \times 10^{-4} \text{ rad/s}, \quad \omega_{d2} = 3.3972 \times 10^{-4} \text{ rad/s}, \quad \omega_{d3} = 9.0255 \times 10^{-4} \text{ rad/s}$$

$$\phi_1 = -1.6710 \text{ rad}, \quad \phi_2 = -1.6710 \text{ rad}, \quad \phi_3 = 1.4706 \text{ rad}$$
$$d_1 = 0.4334, \quad d_2 = 0.0, \quad d_3 = 0.5633$$

Mode shapes:

$$\mathbf{u}_1 = \begin{bmatrix} 0.03963 \\ -0.04312 \\ 0.03963 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -0.05 \\ 0 \\ 0.05 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.03049 \\ 0.05604 \\ 0.03049 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} 0.01717 \\ -0.01869 \\ 0.01717 \end{bmatrix} e^{-2.0771 \times 10^{-4} t} \sin\left(2.0667 \times 10^{-4} t - 1.6710\right)$$

$$+ \begin{bmatrix} 0.01717 \\ 0.03157 \\ 0.01717 \end{bmatrix} e^{-9.0710 \times 10^{-4} t} \sin\left(9.0255 \times 10^{-4} t + 1.4706\right) \text{m}$$

The primary difference between problems 4.58 and 4.59 is the settling time; the responses in Problem 4.59 decay faster than those of Problem 4.58.

4.60   Consider the damped system of Figure P4.1. Determine the damping matrix and use the formula of Eq. (4.119) to determine values of the damping coefficient c_I for which this system would be proportionally damped.

Solution:

From Fig. 4.29,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

From Eq. (4.119)

$$C = \alpha M + \beta K$$

$$\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} \alpha m_1 + \beta\left(k_1 + k_2\right) & -\beta k_2 \\ -\beta k_2 & \alpha m_1 + \beta\left(k_2 + k_3\right) \end{bmatrix}$$

To be proportionally damped,

$$c_2 = \beta k_2$$
$$c_1 = \alpha m_1 + \beta k_1$$
$$c_3 = \alpha m_2 + \beta k_3$$

Alternately, compute $KM^{-1}C$ symbolically and show that the condition for symmetry:

$$\begin{bmatrix} k1 + k2 & -k2 \\ -k2 & k2 + k3 \end{bmatrix}\begin{bmatrix} \dfrac{1}{m1} & 0 \\ 0 & \dfrac{1}{m2} \end{bmatrix}\begin{bmatrix} c1 + c2 & -c2 \\ -c2 & c2 + c3 \end{bmatrix}$$

$$\begin{array}{cc} \dfrac{(m2 \cdot k1 \cdot c1 + m2 \cdot k1 \cdot c2 + m2 \cdot k2 \cdot c1 + m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1)}{(m1 \cdot m2)} & \dfrac{-(m2 \cdot k1 \cdot c2 + m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1 + k2 \cdot m1 \cdot c3)}{(m1 \cdot m2)} \\ \dfrac{-(m2 \cdot k2 \cdot c1 + m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1 + c2 \cdot m1 \cdot k3)}{(m1 \cdot m2)} & \dfrac{(m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1 + k2 \cdot m1 \cdot c3 + c2 \cdot m1 \cdot k3 + m1 \cdot k3 \cdot c3)}{(m1 \cdot m2)} \end{array}$$

Requiring the off diagonal elements to be equal enforces symmetry. This requires

$$m_1 k_2 c_3 = m_2 k_2 c_1 + (m_2 k_1 - m_1 k_3)c_2$$

4.61 Let $k_3 = 0$ in Problem 4.60. Also let $m_1 = 1, m_2 = 4, k_1 = 2, k_2 = 1$ and calculate c_1, c_2 and c_3 such that $\zeta_1 = 0.01$ and $\zeta_2 = 0.1$.

Solution:

From Figure P4.1 the equation of motion is,

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \dot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Calculate natural frequencies:

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_1 = 0.1619 \quad \omega_1 = 0.4024 \text{ rad/s}$$

$$\lambda_2 = 3.0881 \quad \omega_2 = 1.7573 \text{ rad/s}$$

From Eq. (4.124)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega}{2}$$

So, $0.01 = \dfrac{\alpha}{2(0.4024)} + \dfrac{\beta(0.4024)}{2}$

and $0.1 = \dfrac{\alpha}{2(1.7573)} + \dfrac{\beta(1.7573)}{2}$

Solving for α and β yields

$$\alpha = -0.01096$$
$$\beta = 0.1174$$

From Eq. (4.119),

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \alpha M + \beta K = \begin{bmatrix} 0.3411 & -0.1174 \\ -0.1174 & 0.07354 \end{bmatrix}$$

$$c_1 = 0.2238$$

Thus, $\qquad\qquad\qquad\qquad c_2 = 0.1174$

$$c_3 = -0.04382$$

Since negative damping is not usually possible, this design would not work.

4.62 Calculate the constants α and β for the two-degree-of-freedom system of Problem 4.29 such that the system has modal damping of $\zeta_1 = \zeta_2 = 0.3$.

Solution:

From Problem 4.29 with proportional damping added,

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{x}} + \left(\alpha M + \beta K \right) \dot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Calculate natural frequencies:

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I \right) = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_1 = 0.1619 \quad \omega_1 = 0.4024 \text{ rad/s}$$
$$\lambda_2 = 3.0881 \quad \omega_2 = 1.7573 \text{ rad/s}$$

From Eq. (4.124)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta \omega_i}{2}$$

So, $0.3 = \dfrac{\alpha}{2(0.4024)} + \dfrac{\beta(0.4024)}{2}$

and $0.3 = \dfrac{\alpha}{2(1,7573)} + \dfrac{\beta(1.7573)}{2}$

Solving for α and β yields

$$\alpha = 0.1966$$
$$\beta = 0.2778$$

4.63   Equation (4.124) represents n equations in only two unknowns and hence cannot be used to specify all the modal damping ratios for a system with $n > 2$. If the floor vibration system of Problem 4.51 has measured damping of $\zeta_1 = 0.01$ and $\zeta_2 = 0.05$, determine ζ_3.

Solution:

From Problem 4.51

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 9.8255 \times 10^{-7} \lambda^2 + 1.3645 \times 10^{-14} \lambda - 4.1382 \times 10^{-22} = 0$$

$$\lambda_1 = 4.3142 \times 10^{-9} \quad \omega_1 = 2.0771 \times 10^{-5} \text{ rad/s}$$

$$\lambda_2 = 1.1657 \times 10^{-7} \quad \omega_2 = 3.4143 \times 10^{-4} \text{ rad/s}$$

$$\lambda_3 = 8.2283 \times 10^{-7} \quad \omega_3 = 9.0710 \times 10^{-4} \text{ rad/s}$$

Eq. (4.124)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}$$

Since the problem contains three modes only, and since the first and second modal damping ratios are give as $\zeta_1 = 0.01$ and $\zeta_2 = 0.05$ then the following linear system can be set up

$$\frac{\alpha}{2\left(2.0771 \times 10^{-5}\right)} + \frac{\beta\left(2.0771 \times 10^{-5}\right)}{2} = 0.01$$

$$\frac{\alpha}{2\left(3.4143 \times 10^{-4}\right)} + \frac{\beta\left(3.4143 \times 10^{-4}\right)}{2} = 0.05$$

which can be solve to yield $\alpha = 2.9 \times 10^{-7}$ and $\beta = 290.397$. Hence, the modal damping of the third mode can be obtained using 4.124

$$\zeta_3 = \frac{\alpha}{2\omega_3} + \frac{\beta\omega_3}{2} = 0.132$$

4.64 Does the following system decouple? If so, calculate the mode shapes and write the equation in decoupled form.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Solution:

The system will decouple if

$$C = \alpha M + \beta K$$

$$\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} \alpha + 5\beta & -\beta \\ -\beta & \alpha + \beta \end{bmatrix}$$

Clearly the off-diagonal terms require

$$\beta = 3$$

Therefore, the diagonal terms require

$$5 = \alpha + 15$$
$$3 = \alpha + 3$$

These yield different values of α, so the system does not decouple. An easier approach is to compute $CM^{-1}K$ to see if it is symmetric:

$$CM^{-1}K = \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -12 & 6 \end{bmatrix}$$

Since this is not symmetric, the system cannot be decoupled.

4.65 Calculate the damping matrix for the system of Problem 4.63. What are the units of the elements of the damping matrix?

Solution:

From Problem 4.58,

$$\alpha = -8.8925 \times 10^{-7}$$
$$\beta = 3.0052 \times 10^{2}$$

From Problem 4.48

$$M = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 200 \end{bmatrix}$$

$$K = 3.197 \times 10^{-4} \begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix}$$

So,
$$C = \alpha M + \beta K$$

$$C = \begin{bmatrix} 0.01334 & 0.01602 & 0.006506 \\ 0.01602 & 0.03025 & 0.01602 \\ 0.006506 & 0.01602 & 0.01334 \end{bmatrix}$$

The units are kg/s

4.66 Show that if the damping matrix satisfies $C = \alpha M + \beta K$, then the matrix $CM^{-1}K$ is symmetric and hence that $CM^{-1}K = KM^{-1}C$.

Solution: Compute the product $CM^{-1}K$ where C has the form: $C = \alpha M + \beta K$.
$$CM^{-1} = (\alpha M + \beta K)M^{-1} = \alpha I + \beta KM^{-1} \Rightarrow CM^{-1}K = \alpha K + \beta KM^{-1}K$$
$$KM^{-1}C = KM^{-1}(\alpha M + \beta K) = \alpha K + \beta KM^{-1}K$$
$$\Rightarrow KM^{-1}C = CM^{-1}K$$

Problems and Solutions for Section 4.6 (4.67 through 4.76)

4.67 Calculate the response of the system of Figure 4.16 discussed in Example 4.6.1 if $F_1(t) = \delta(t)$ and the initial conditions are set to zero. This might correspond to a two-degree-of-freedom model of a car hitting a bump.

Solution: From example 4.6.1, with $F_1(t) = \delta(t)$, the modal equations are
$$\ddot{r}_1 + 0.2\dot{r}_1 + 2r_1 = 0.7071\delta(t)$$
$$\ddot{r}_2 + 0.4\dot{r}_2 + 4r_2 = 0.7071\delta(t)$$
Also from the example,

$$\omega_{n1} = \sqrt{2} \text{ rad/s} \qquad \zeta_1 = 0.07071 \qquad \omega_{d1} = 1.4106 \text{ rad/s}$$
$$\omega_{n2} = 2 \text{ rad/s} \qquad \zeta_2 = 0.1 \qquad \omega_{d2} = 1.9899 \text{ rad/s}$$

The solution to an impulse is given by equations (3.7) and (3.8):

$$r_i(t) = \frac{\hat{F}}{m_i \omega_{di}} e^{-\zeta_i \omega_{ni} t} \sin \omega_{di} t$$

This yields
$$\mathbf{r}(t) = \begin{bmatrix} 0.5012e^{-0.1t} \sin 1.4106t \\ 0.3553e^{-0.2t} \sin 1.9899t \end{bmatrix}$$

The solution in physical coordinates is
$$\mathbf{x}(t) = M^{-1/2} P \mathbf{r}(t) = \begin{bmatrix} .2357 & -.2357 \\ .7071 & .7071 \end{bmatrix} \begin{bmatrix} 0.167e^{-0.1t} \sin 1.4106t \\ -0.118e^{-.02t} \sin 1.9899t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.0394e^{-0.1t} \sin 1.4106t + 0.0279e^{-0.2t} \sin 1.9899t \\ 0.118e^{-0.1t} \sin 1.4106t - 0.0834e^{-0.2t} \sin 1.9899t \end{bmatrix}$$

4.68 For an undamped two-degree-of-freedom system, show that resonance occurs at one or both of the system's natural frequencies.

Solution:

Undamped two-degree-of-freedom system:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{F}(t)$$

Let $\mathbf{F}(t) = \begin{bmatrix} F_1(t) \\ 0 \end{bmatrix}$

Note: placing F_1 on mass 1 is one way to do this. A second force could be placed on mass 2 with or without F_1.

Proceeding through modal analysis,

$$I\ddot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Or,

$$\ddot{r}_1 + \omega_1^2 r_1 = b_1 F_1(t)$$
$$\ddot{r}_2 + \omega_2^2 r_2 = b_2 F_1(t)$$

where b_1 and b_2 are constants from the matrix $P^T M^{-1/2}$.

If $F_1(t) = a \cos \omega t$ and $\omega = \omega_1$ then the solution for r_1 is (from Section 2.1),

$$r_1(t) = \frac{\dot{r}_{10}}{\omega_1}\sin\omega_1 t + r_{10}\cos\omega_1 t + \frac{b_1 a}{2\omega_1}t\sin\omega_1 t$$

The solution for r_2 is

$$r_2(t) = \frac{\dot{r}_{20}}{\omega_2}\sin\omega_2 t + \left(r_{20} - \frac{b_2 a}{\omega_2^2 - \omega_1^2}\right)\cos\omega_2 t + \frac{b_2 a}{\omega_2^2 - \omega_1^2}t\sin\omega_1 t$$

If the initial conditions are zero,

$$r_1(t) = \frac{b_1 a}{2\omega_1} t \sin \omega_1 t$$

$$r_2(t) = \frac{b_2 a}{\omega_2^2 - \omega_1^2} \left(\cos \omega_1 t - \cos \omega_2 t \right)$$

Converting to physical coordinates $X(t) = M^{-1/2} P r(t)$ yields

$$x_1(t) = c_1 r_1(t) + c_2 r_2(t)$$

$$x_2(t) = c_3 r_1(t) + c_4 r_2(t)$$

where c_i is a constant from $M^{-1/2} P$.

So, if the driving force contains just one natural frequency, both masses will be excited at resonance. The driving force could contain the other natural frequency ($\omega = \omega_{n2}$), which would cause r_1 and r_2 to be

$$r_1(t) = \frac{b_1 a}{\omega_1^2 - \omega_2^2} \left(\cos \omega_2 t - \cos \omega_1 t \right)$$

$$r_2(t) = \frac{b_2 a}{2\omega_2} t \sin \omega_2 t$$

and

$$x_1(t) = c_1 r_1(t) + c_2 r_2(t)$$

$$x_2(t) = c_3 r_1(t) + c_4 r_2(t)$$

so both masses still oscillate at resonance.

Also, if $F_1(t) = a_1 \cos \omega_1 t + a_2 \cos \omega_2 t$ where $\omega_1 = \omega_{n1}$ and $\omega_2 = \omega_{n2}$, then both r_1 and r_2 would be at resonance, so $x_1(t)$ and $x_2(t)$ would also be at resonance.

4.69 Use modal analysis to calculate the response of the drive train system of Problem 4.44 to a unit impulse on the car body (i.e., and location q_3). Use the modal damping of Problem 4.56. Calculate the solution in terms of physical coordinates, and after subtracting the rigid-body modes, compare the responses of each part.

Solution:

Let k_1 = hub stiffness and k_2 = axle and suspension stiffness.

From Problems 41 and 51,

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 3000 \end{bmatrix} \ddot{\mathbf{q}} + 10{,}000 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \mathbf{q} = \mathbf{0}$$

$$M^{-1/2} = \begin{bmatrix} .1155 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & .0183 \end{bmatrix}$$

$$P = \begin{bmatrix} .1537 & -.8803 & .4488 \\ .1775 & -.4222 & -.88910 \\ .9721 & .2163 & .0913 \end{bmatrix}$$

$$\lambda_1 = 0 \qquad\qquad \omega_{n1} = 0 \text{ rad/s}$$
$$\lambda_2 = 77.951 \qquad\qquad \omega_{n2} = 8.8290 \text{ rad/s}$$
$$\lambda_3 = 362.05 \qquad\qquad \omega_{n3} = 19.028 \text{ rad/s}$$

The initial conditions are **0**.

Also

$$\zeta_1 = \zeta_2 = \zeta_3 = .1$$
$$\omega_{d1} = 8.7848 \text{ rad/s}$$
$$\omega_{d2} = 18.932 \text{ rad/s}$$

From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i \omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} .01775 \\ .003949 \\ .001668 \end{bmatrix} \delta(t)$$

The modal equations are

$$\ddot{r}_1 = .01775\delta(t)$$
$$\ddot{r}_2 + 1.7658\dot{r}_2 + 77.951r_2 = .003949\delta(t)$$
$$\ddot{r}_3 + 3.8055\dot{r}_3 + 362.05r_3 = .001668\delta(t)$$

The solution for r_1 is

$$r_1(t) = .01775t$$

The solutions for r_2 and r_3 are given by equations 3.7 and (3.8)

$$r_i(t) = \frac{\hat{F}}{m_i \omega_{di}} e^{-\zeta_i \omega_i t} \sin \omega_{di} t$$

This yields

$$r_2(t) = 4.4949 \times 10^{-4} e^{-.8829t} \sin 8.7848t$$
$$r_3(t) = 8.8083 \times 10^{-5} e^{-1.9028t} \sin 18.932t$$

The solution in physical coordinates is

$$\mathbf{q}(t) = M^{-1/2} P \mathbf{r}(t)$$

$$\mathbf{q}(t) = 3.1496 \times 10^{-4} t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -4.5691 \times 10^{-5} \\ -1.8978 \times 10^{-5} \\ 1.7749 \times 10^{-6} \end{bmatrix} e^{-.8829t} \sin 8.7848t$$

$$+ \begin{bmatrix} 4.5647 \times 10^{-6} \\ -7.8301 \times 10^{-6} \\ 1.4689 \times 10^{-7} \end{bmatrix} e^{-1.9028t} \sin 18.932t \text{ m}$$

The magnitude of the components is much smaller than that in problem 51, but they do oscillate at the same frequencies.

4.70 Consider the machine tool of Figure 4.28. Resolve Ex. 4.8.3 if the floor mass $m =$ 1000 kg, is subject to a force of $10 \sin t$ (in Newtons). Calculate the response. How much does this floor vibration affect the machine's toolhead?

Solution:

From example 4.8.3, with $F_3(t) = 10 \sin t$ N and $m_3 = 1000$ kg.

$$
\left(10^3\right)\begin{bmatrix} .4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \left(10^4\right)\begin{bmatrix} 30 & -30 & 0 \\ -30 & 38 & -8 \\ 0 & -8 & 88 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 10\sin t \end{bmatrix}
$$

Calculating the eigenvalues and eigenvectors yields

$$\lambda_1 = 29.980 \qquad \omega_1 = 5.4761 \text{ rad/s}$$
$$\lambda_2 = 868.2743 \qquad \omega_2 = 29.4665 \text{ rad/s}$$
$$\lambda_3 = 921.7378 \qquad \omega_3 = 30.3601 \text{ rad/s}$$

And

$$
P = \begin{bmatrix} -.4215 & .4989 & .7573 \\ -.9048 & -.1759 & -.3877 \\ -.0602 & -.8486 & .5255 \end{bmatrix}
$$

Modal force vector:

$$
P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} -.01904 \\ -.2684 \\ .1662 \end{bmatrix}\sin t
$$

Undamped modal equations:

$$\ddot{r}_1 + 29.9880 r_1 = -.01904 \sin t$$
$$\ddot{r}_2 + 868.2743 r_2 = -.2684 \sin t$$
$$\ddot{r}_3 + 921.7378 r_3 = .1662 \sin t$$

Inserting the damping terms,

$$\zeta_1 = .1 \qquad 2\zeta_1\omega_1 = 1.0952$$

$$\zeta_2 = .01 \qquad 2\zeta_2\omega_2 = .5893$$

$$\zeta_3 = .05 \qquad 2\zeta_3\omega_3 = 3.0360$$

$$\ddot{r}_1 + 1.0952\dot{r}_1 + 29.9880r_1 = -.01904\sin t$$

$$\ddot{r}_2 + .5893\dot{r}_2 + 868.2734r_2 = -.2684\sin t$$

$$\ddot{r}_3 + 3.0360\dot{r}_3 + 921.7378r_3 = .1662\sin t$$

The damped natural frequencies are

$$\omega_{d1} = \omega_{n1}\sqrt{1-\zeta_1^2} = 5.4487 \text{ rad/s}$$

$$\omega_{d2} = \omega_{n2}\sqrt{1-\zeta_2^2} = 29.4650 \text{ rad/s}$$

$$\omega_{d3} = \omega_{n3}\sqrt{1-\zeta_3^2} = 30.3222 \text{ rad/s}$$

The general solution is

$$r_i(t) = A_i e^{-\zeta_i\omega_{ni}t} \sin(\omega_{di}t - \theta_i) + A_{0i}\sin(\omega t - \phi_i)$$

where

$$A_{0i} = \frac{f_{0i}}{\sqrt{\left(\omega_{ni}^2 - \omega^2\right)^2 + \left(2\zeta_i\omega_{ni}\omega\right)^2}} \quad \text{and} \quad \phi_i = \tan^{-1}\left(\frac{2\zeta_i\omega_{ni}\omega}{\omega_{ni}^2 - \omega}\right)$$

Inserting values,

$$A_{01} = -6.5643 \times 10^{-4} \text{ m} \qquad \phi_1 = 3.7764 \times 10^{-2} \text{ rad}$$

$$A_{02} = -3.0943 \times 10^{-4} \text{ m} \qquad \phi_2 = 6.7952 \times 10^{-4} \text{ rad}$$

$$A_{03} = 1.8049 \times 10^{-4} \text{ m} \qquad \phi_3 = 3.2974 \times 10^{-3} \text{ rad}$$

So,

$$r_1(t) = A_1 e^{-.5476t} \sin(5.4487t - \theta_1) - 6.543 \times 10^{-4}\sin(t - 3.7764 \times 10^{-2})$$

$$r_2(t) = A_2 e^{-.2947t} \sin(29.4650t - \theta_2) - 3.0943 \times 10^{-4}\sin(t - 6.7952 \times 10^{-4})$$

$$r_3(t) = A_3 e^{-1.5180t} \sin(30.3222t - \theta_3) + 1.8049 \times 10^{-4}\sin(t - 3.2974 \times 10^{-3})$$

With zero initial conditions:

$$A_1 = 1.2047 \times 10^{-4} \text{ m} \qquad\qquad \theta_1 = .2072 \text{ rad}$$
$$A_2 = 1.0502 \times 10^{-5} \text{ m} \qquad\qquad \theta_2 = .02002 \text{ rad}$$
$$A_3 = -5.9524 \times 10^{-6} \text{ m} \qquad\qquad \theta_3 = .1002 \text{ rad}$$

Now,

$$r_1(t) = 1.2047 \times 10^{-4} e^{-.5476t} \sin(5.4487t - .2027) - 6.543 \times 10^{-4} \sin(t - 3.7764 \times 10^{-2})$$
$$r_2(t) = 1.0502 \times 10^{-5} e^{-.2947t} \sin(29.4650t - .02002) - 3.0943 \times 10^{-4} \sin(t - 6.7952 \times 10^{-4})$$
$$r_3(t) = -5.9524 \times 10^{-6} e^{-1.5180t} \sin(30.3222t - .1002) + 1.8049 \times 10^{-4} \sin(t - 3.2974 \times 10^{-3})$$

Convert to physical coordinates:

$$\mathbf{x}(t) = M^{-1/2} P\mathbf{r}(t) = \begin{bmatrix} -.02108 & .02494 & .03786 \\ -.02023 & -.003993 & -.008670 \\ -.001904 & -.02684 & .01662 \end{bmatrix} \mathbf{r}(t)$$

Therefore

$$x_1(t) = -.02108r_1 + .02494r_2 + .03786r_3$$
$$x_2(t) = -.02023r_1 - .003933r_2 - .008670r_3$$
$$x_3(t) = -.001904r_1 - .02684r_2 + .01662r_3$$

4.71 Consider the airplane of Figure P4.46 with damping as described in Problem 4.57 with $\zeta_1 = 0.1$. Suppose that the airplane hits a gust of wind, which applies an impulse of $3\delta(t)$ at the end of the left wing and $\delta(t)$ at the end of the right wing. Calculate the resulting vibration of the cabin $[x_2(t)]$.

Solution: From Problems 4.46 and 4.57

$$M^{-1/2} = \begin{bmatrix} .01826 & 0 & 0 \\ 0 & .009129 & 0 \\ 0 & 0 & .01826 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.4082 & -0.7071 & 0.5774 \\ 0.8165 & 0 & -0.5774 \\ 0.4082 & 0.7071 & 0.5774 \end{bmatrix}$$

$\lambda_1 = 0$ \qquad $\omega_{n1} = 0$ rad/s

$\lambda_2 = 4.485$ \qquad $\omega_{n2} = 2.118$ rad/s

$\lambda_3 = 6.727$ \qquad $\omega_{n3} = 2.594$ rad/s

Also:

$\zeta_1 = \zeta_2 = \zeta_3 = 0.1$

$$\mathbf{F}(t) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \delta(t)$$

$\omega_{d1} = 0$ rad/s, $\omega_{d2} = 2.1072$ rad/s, $\omega_{d3} = 2.5807$ rad/s

From equation (4.129):

$$\ddot{\mathbf{r}} + \mathrm{diag}(2\zeta_i \omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} -0.0298 \\ 0.0258 \\ 0.0422 \end{bmatrix} \delta(t)$$

The modal equations are

$$\ddot{r_1} = -0.02981\delta(t)$$
$$\ddot{r_2} + 0.424\dot{r_2} + 4.485r_2 = 0.0258\delta(t)$$
$$\ddot{r_3} + 0.519\dot{r_3} + 6.727r_3 = 0.0422\delta(t)$$

The solution for r_1 is

$$r_1(t) = -0.02981t$$

The solutions for r_2 and r_3 are given by equations (3.7) and (3.8)

$$r_i(t) = \frac{\hat{F}}{m_i \omega_{di}} e^{-\zeta_i \omega_i t} \sin \omega_{di} t$$

This yields

$$r_2(t) = 1.2253 \times 10^{-2} e^{-0.212t} \sin 2.107t$$

$$r_3(t) = 1.6338 \times 10^{-2} e^{-0.259t} \sin 2.581t$$

The solution in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2} P \mathbf{r}(t)$$

For x_2:

$$x_2(t) = 2.221 \times 10^{-4} t + 8.06 \times 10^{-5} e^{-0.259t} \sin 2.581t$$

4.72 Consider again the airplane of Figure P4.46 with the modal damping model of Problem 4.57 ($\zeta_i = 0.1$). Suppose that this is a propeller-driven airplane with an internal combustion engine mounted in the nose. At a cruising speed the engine mounts transmit an applied force to the cabin mass ($4m$ at x_2) which is harmonic of the form $50 \sin 10t$. Calculate the effect of this harmonic disturbance at the nose and on the wind tips after subtracting out the translational or rigid motion.

Solution: From Problems 4.47 and 4.57

$$M^{-1/2} = \begin{bmatrix} .01826 & 0 & 0 \\ 0 & .009129 & 0 \\ 0 & 0 & .01826 \end{bmatrix}, \quad P = \begin{bmatrix} -.4082 & .7071 & .5774 \\ -.8165 & 0 & -.5774 \\ -.4082 & -.7071 & .5774 \end{bmatrix}$$

$$\lambda_1 = 0 \qquad\qquad \omega_{n1} = 0 \text{ rad/s}$$
$$\lambda_2 = 17.94 \qquad\quad \omega_{n2} = 4.2356 \text{ rad/s}$$
$$\lambda_3 = 26.91 \qquad\quad \omega_{n3} = 5.1875 \text{ rad/s}$$

Also,
$$\zeta_1 = \zeta_2 = \zeta_3 = 0.1, \Rightarrow \omega_{d1} = 0 \text{ rad/s}, \quad \omega_{d2} = 4.2143 \text{ rad/s}, \quad \omega_{d3} = 5.1615 \text{ rad/s}$$

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 50\sin 10t \\ 0 \end{bmatrix}$$

The initial conditions are **0**. From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i \omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} -.3727 \\ 0 \\ -.2635 \end{bmatrix}\sin 10t$$

The modal equations are

$$\ddot{r}_1 = -.3727\sin 10t$$
$$\ddot{r}_2 + .8471\dot{r}_2 + 17.94r_2 = 0$$
$$\ddot{r}_3 + 1.0375\dot{r}_3 + 26.91r_3 = -.2635\sin 10t$$

The solutions are

$$r_1(t) = .003727 \sin 10$$

$$r_2(t) = 0$$

$$r_3(t) = -.006915 e^{-.5188t} \sin(5.1615t + .0726) + .003569 \sin(10t + .141)$$

The solutions in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2} P \mathbf{r}(t)$$

The wing tips are x_1 and x_3, so

$$x_1(t) = x_3(t) = 2.7780 \times 10^{-5} \sin 10t - 7.2891 \times 10^{-5} e^{-.5188t} \sin(5.1615t + .0726)$$
$$+ 3.7621 \times 10^{-5} \sin(10t + .141)$$

4.73   Consider the automobile model of Problem 4.14 illustrated in Figure P4.14. Add modal damping to this model of $\zeta_1 = 0.01$ and $\zeta_2 = 0.2$ and calculate the response of the body [$x_2(t)$] to a harmonic input at the second mass of 10 sin3t N.

Solution: From problem 4.14

$$M = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}, \quad K = \begin{bmatrix} 1000 & -1000 \\ -1000 & 11000 \end{bmatrix}, \quad P = \begin{bmatrix} .9999 & -.1044 \\ .1044 & .9999 \end{bmatrix}$$

$\lambda_1 = 0.4545$ $\omega_1 = 0.6741$ rad/s, and $\lambda_2 = 220.05$ $\omega_2 = 14.834$ rad/s

Also,

$$\zeta_1 = .01, \quad \zeta_2 = 0.2, \quad \omega_{d1} = 0.6741 \text{ rad/s}, \quad \omega_{d2} = 14.534 \text{ rad/s}$$

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 10\sin 3t \end{bmatrix}$$

The initial conditions are all **0**. From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i\omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} 0.02036 \\ 1.4141 \end{bmatrix}\sin 3t$$

The modal equations are

$$\ddot{r}_1 + 0.01348\dot{r}_1 + 0.454r_1 = 0.02036\sin 3t$$

$$\ddot{r}_2 + 5.9336\dot{r}_2 + 220.046r_2 = 1.4141\sin 3t$$

The solutions are

$$r_1(t) = -0.1088e^{-0.006741t}\sin(0.6741t + 1.0914\times10^{-4}) + .002445\sin(3t - .004857)$$

$$r_2(t) = -0.07500e^{-2.9668t}\sin(14.534t + 1.3087) + .07586\sin(3t + 1.26947)$$

The solutions in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2}P\mathbf{r}(t)$$

The response of the body is

$$x_1(t) = -.002433e^{-0.006741t}\sin(.6471t - 1.0914\times10^{-4})$$

$$+ 5.4665\times10^{-5}\sin(3t - .004857)$$

$$+ 2.4153\times10^{-5}e^{-2.9668t}\sin(14.534t - 1.3087)$$

$$- 2.4430\times10^{-5}\sin(3t + 1.2694)$$

4.74   Determine the *modal equations* for the following system and comment on whether or not the system will experience resonance.

$$\ddot{\mathbf{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(0.618t)$$

Solution: Here $M = I$ so that the eigenvectors and mode shapes are the same. Computing the natural frequencies from $\det(-\omega^2 I + K) = 0$ yields:

$$\omega_1 = 0.618 \text{ rad/s and } \omega_2 = 1.681 \text{ rad/s}$$

Next solve for the mode shapes and normalize them to get

$$P = \begin{bmatrix} 0.526 & -0.851 \\ 0.851 & 0.526 \end{bmatrix}, \quad \text{so that } P^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.526 \\ -0.851 \end{bmatrix}$$

The modal equations then become:

$$\ddot{r}_1 + (0.618)^2 r_1 = \ddot{r}_1 + 0.3819 r_1 = 0.526 \sin(0.618t)$$

$$\ddot{r}_2 + (1.618)^2 r_2 = \ddot{r}_2 + 2.6179 r_2 = -0.851 \sin(0.618t)$$

The driving frequency is equal to the natural frequency of mode one so the system exhibits resonance.

4.75   Consider the following system and compute the solution using the mode summation method.

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution: From Example 4.2.4

$$M^{\frac{1}{2}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, M^{-\frac{1}{2}} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \text{ and } V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \text{ Also } \omega_1 = \sqrt{2}, \omega_2 = 2 \text{ rad/s}$$

$$\text{Appropriate IC are } \mathbf{q}_0 = M^{\frac{1}{2}} \mathbf{x}_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \dot{\mathbf{q}}_0 = M^{\frac{1}{2}} \mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\phi_i = \tan^{-1} \frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)} = \tan^{-1} \frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{0} \Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \dfrac{\pi}{2} \\ \dfrac{\pi}{2} \end{bmatrix}$$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin \phi_i} \Rightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} \big/ 2 \\ 3\sqrt{2} \big/ 2 \end{bmatrix}$$

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \frac{3\sqrt{2}}{2} \sin\left(\sqrt{2}t + \frac{\pi}{2} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3\sqrt{2}}{2} \sin\left(2t + \frac{\pi}{2} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \frac{3}{2}\cos\left(\sqrt{2}t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2}\cos\left(2t\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = \frac{3}{2}\cos\left(\sqrt{2}t\right)\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2}\cos\left(2t\right)\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = \frac{3}{2}\cos\left(\sqrt{2}t\right)\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} + \frac{3}{2}\cos\left(2t\right)\begin{bmatrix} 1/3 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} \dfrac{1}{2}\cos\left(\sqrt{2}t\right) + \dfrac{1}{2}\cos(2t) \\ \dfrac{3}{2}\cos\left(\sqrt{2}t\right) - \dfrac{3}{2}\cos\left(2t\right) \end{bmatrix}$$

Problems and Solutions for Section 4.7 (4.76 through 4.79)

4.76 Use Lagrange's equation to derive the equations of motion of the lathe of Fig. 4.21 for the undamped case.

Solution: Let the generalized coordinates be θ_1, θ_2 and θ_3.
The kinetic energy is

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}J_3\dot{\theta}_3^2$$

The potential energy is

$$U = \frac{1}{2}k_1\left(\theta_2 - \theta_2\right)^2 + \frac{1}{2}k_2\left(\theta_3 - \theta_2\right)^2$$

There is a nonconservative moment $M(t)$ on inertia 3. The Lagrangian is

$$L = T - U = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}J_3\dot{\theta}_3^2 - \frac{1}{2}k_1\left(\theta_2 - \theta_1\right)^2 - \frac{1}{2}k_2\left(\theta_3 - \theta_2\right)^2$$

Calculate the derivatives from Eq. (4.136):

$$\frac{\partial L}{\partial \dot{\theta}_1} = J_1\dot{\theta}_1 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = J_1\ddot{\theta}_1$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = J_2\dot{\theta}_2 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = J_2\ddot{\theta}_2$$

$$\frac{\partial L}{\partial \dot{\theta}_3} = J_3\dot{\theta}_3 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) = J_3\ddot{\theta}_3$$

$$\frac{\partial L}{\partial \theta_1} = -k_1\theta_1 + k_1\theta_2$$

$$\frac{\partial L}{\partial \theta_2} = -k_1\theta_1 - \left(k_1 + k_2\right)\theta_2 + k_2\theta_3$$

$$\frac{\partial L}{\partial \theta_3} = -k_2\theta_2 - k_2\theta_3$$

Using Eq. (4.136) yields

$$J_1\ddot{\theta}_1 + k_1\theta_1 - k_2\theta_2 = 0$$

$$J_2\ddot{\theta}_2 - k_1\theta_1 + \left(k_1 + k_2\right)\theta_2 - k_2\theta_3 = 0$$

$$J_3\ddot{\theta}_3 - k_2\theta_2 + k_2\theta_3 = M\left(t\right)$$

In matrix form this yields

$$
\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \ddot{\theta} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \theta = \begin{bmatrix} 0 \\ 0 \\ M(t) \end{bmatrix}
$$

4.77   Use Lagrange's equations to rederive the equations of motion for the automobile of Example 4.8.2 illustrated in Figure 4.25 for the case $c_1 = c_2 = 0$.

Solution: Let the generalized coordinates be x and θ.
The kinetic energy is

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J\dot{\theta}^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2} k_1 \left(x - l_1\theta \right)^2 + \frac{1}{2} k_2 \left(x + l_2\theta \right)^2$$

The Lagrangian is

$$L = T - U = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J\dot{\theta}^2 - \frac{1}{2} k_1 \left(x - l_1\theta \right)^2 - \frac{1}{2} k_2 \left(x + l_2\theta \right)^2$$

Calculate the derivatives from Eq. (4.136):

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = m\ddot{x}$$

$$\frac{\partial L}{\partial \dot{\theta}} = J\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) = J\ddot{\theta}$$

$$\frac{\partial L}{\partial x} = -\left(k_1 + k_2 \right) x + \left(k_1 l_1 - k_1 l_2 \right)\theta$$

$$\frac{\partial L}{\partial \theta} = \left(k_1 l_1 - k_2 l_2 \right) x - \left(k_1 l_2^2 \right)\theta$$

Using Eq. (4.136) yields

$$m\ddot{x} + \left(k_1 + k_2 \right) x + \left(k_1 l_1 - k_2 l_2 \right)\theta = 0$$

$$J\ddot{\theta} + \left(k_1 l_2 - k_1 l_1 \right) x - \left(k_1 l_1^2 + k_2 l_2^2 \right)\theta = 0$$

In matrix form this yields

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \mathbf{0}$$

4.78   Use Lagrange's equations to rederive the equations of motion for the building model presented in Fig. 4.9 of Ex. 4.4.3 for the undamped case.

Solution:

Let the generalized coordinates be x_1, x_2, x_3 and x_4.
The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}m_4\dot{x}_4^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left(x_2 - x_1\right)^2 + \frac{1}{2}k_3\left(x_3 - x_2\right)^2 + \frac{1}{2}k_4\left(x_4 - x_3\right)^2$$

The Lagrangian is

$$L = T - U = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 + \frac{1}{2}m\dot{x}_4^2$$
$$- \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2\left(x_2 - x_1\right)^2 \frac{1}{2}k_3\left(x_3 - x_2\right)^2 - \frac{1}{2}k_4\left(x_4 - x_3\right)^2$$

Calculate the derivatives from Eq. (4.136):

$$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_2\dot{x}_2 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_2\ddot{x}_2$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_3\dot{x}_3 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_3\ddot{x}_3$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_4\dot{x}_4 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_4\ddot{x}_4$$

$$\frac{\partial L}{\partial x_1} = -\left(k_1 + k_2\right)x_1 + k_2 x_2$$

$$\frac{\partial L}{\partial x_2} = k_2 x_1 - \left(k_2 + k_3\right)x_2 + k_3 x_3$$

$$\frac{\partial L}{\partial x_3} = k_2 x_2 - \left(k_2 + k_4\right)x_3 - k_4 x_4$$

$$\frac{\partial L}{\partial x_4} = k_4 x_3 - k_4 x_4$$

Using Eq. (4.136) yields

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = 0$$

$$m_2\ddot{x}_2 - k_2 x_1 + \left(k_2 + k_3\right)x_2 - k_3 x_3 = 0$$

$$m_3\ddot{x}_3 - k_3 x_2 + \left(k_3 + k_4\right)x_3 - k_4 x_4 = 0$$

$$m_4\ddot{x}_4 - k_4 x_3 + k_4 x_4 = 0$$

In matrix form this yields

$$
\begin{bmatrix}
m_1 & 0 & 0 & 0 \\
0 & m_2 & 0 & 0 \\
0 & 0 & m_3 & 0 \\
0 & 0 & 0 & m_4
\end{bmatrix}
\ddot{\mathbf{x}} +
\begin{bmatrix}
k_1 + k_2 & -k_2 & 0 & 0 \\
-k_2 & k_2 + k_3 & -k_3 & 0 \\
0 & -k_3 & k_3 + k_4 & -k_4 \\
0 & 0 & -k_4 & k_4
\end{bmatrix}
\mathbf{x} = \mathbf{0}
$$

4.79 Consider again the model of the vibration of an automobile of Fig. 4.25. In this case include the tire dynamics as indicated in Fig. P4.79. Derive the equations of motion using Lagrange formulation for the undamped case. Let m_3 denote the mass of the car acting at c.g.
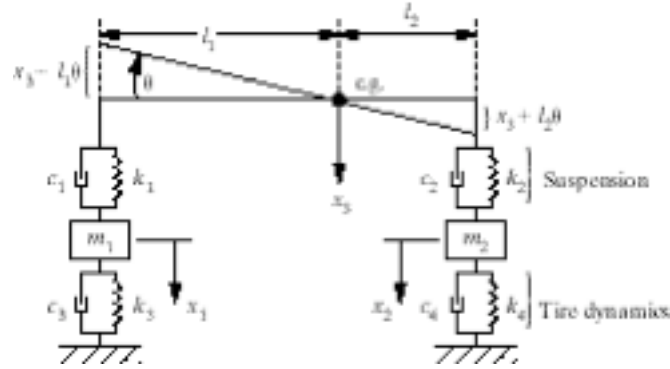
Solution:

Let the generalized coordinates be x_1, x_2, x_3 and θ. The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}J\dot{\theta}^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2}k_1\left(x_3 - \ell_1\theta - x_1\right)^2 + \frac{1}{2}k_2(x_3 - \ell_2\theta - x_2)^2 + \frac{1}{2}k_3x_1^2 + \frac{1}{2}k_4x_2^2$$

The Lagrangian is thus:

$$L = T - U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}k_1\left(x_3 - \ell_1\theta - x_1\right)^2$$

$$- \frac{1}{2}k_2\left(x_3 + \ell_2\theta - x_2\right)^2 - \frac{1}{2}k_3x_1^2 - \frac{1}{2}k_4x_2^2$$

Calculate the derivatives indicated in Eq. (4.146):

$$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$$

$$\frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_1\ddot{x}_2$$

$$\frac{\partial L}{\partial \dot{x}_3} = m_3\dot{x}_3 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = m_3\ddot{x}_3$$

$$\frac{\partial L}{\partial \theta} = J\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{\theta}}\right) = J\ddot{\theta}$$

$$\frac{\partial L}{\partial x_1} = -\left(k_1 + k_3\right)x_1 + k_1 x_3 - k_1 l_1 \theta$$

$$\frac{\partial L}{\partial x_2} = -\left(k_2 + k_4\right)x_2 + k_2 x_3 - k_2 l_2 \theta$$

$$\frac{\partial L}{\partial x_3} = k_1 x_1 + k_2 x_2 - \left(k_1 + k_2\right)x_3 + \left(k_1 l_1 + k_2 l_2\right)\theta$$

$$\frac{\partial L}{\partial \theta} = -k_1 l_1 x_1 - k_2 l_2 x_2 + \left(k_1 l_1 + k_2 l_2\right)x_3 - \left(k_1 l_1^2 + k_2 l_2^2\right)\theta$$

Using Eq. (4.146) yields

$$m_1 \ddot{x}_1 + \left(k_3 + k_1\right)x_1 - k_1 x_3 + k_1 l_1 \theta = 0$$
$$m_2 \ddot{x}_2 + \left(k_4 + k_2\right)x_2 - k_2 x_3 - k_2 l_2 \theta = 0$$
$$m_3 \ddot{x}_3 - k_1 x_1 - k_2 x_2 + \left(k_1 + k_2\right)x_3 - \left(k_1 l_1 - k_2 l_2\right)\theta = 0$$
$$J\ddot{\theta} + k_1 l_1 x_1 - k_2 l_2 x_2 - \left(k_1 l_1 - k_2 l_2\right)x_3 + \left(k_1 l_1^2 + k_2 l_2^2\right) = 0$$

in matrix form

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} \left(k_3 + k_1\right) & 0 & -k_1 & k_1 l_1 \\ 0 & \left(k_4 + k_2\right) & -k_2 & k_2 l_2 \\ -k_1 & -k_2 & \left(k_1 + k_2\right) & -\left(k_2 l_2 + k_1 l_1\right) \\ k_1 l_1 & k_2 l_2 & -\left(k_2 l_2 + k_1 l_1\right) & \left(k_1 l_1^2 + k_2 l_2^2\right) \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \theta \end{Bmatrix} = 0$$

Problems and Solutions for Section 4.9 (4.80 through 4.90)

4.80 Consider the mass matrix

$$M = \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix}$$

and calculate M^{-1}, $M^{-1/2}$, and the Cholesky factor of M. Show that

$$LL^T = M$$

$$M^{-1/2}M^{-1/2} = I$$

$$M^{1/2}M^{1/2} = M$$

Solution: Given

$$M = \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix}$$

The matrix, P, of eigenvectors is

$$P = \begin{bmatrix} -0.1091 & -0.9940 \\ -0.9940 & 0.1091 \end{bmatrix}$$

The eigenvalues of M are

$$\lambda_1 = 0.8902$$

$$\lambda_2 = 10.1098$$

From Equation

$$M^{-1} = P\mathrm{diag}\left[\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right]P^T, \quad M^{-1} = \begin{bmatrix} 0.1111 & 0.1111 \\ 0.1111 & 1.1111 \end{bmatrix}$$

From Equation

$$M^{-1/2} = V diag\left[\lambda_1^{-1/2},\lambda_2^{-1/2}\right]V^T$$

$$M^{-1/2} = \begin{bmatrix} 0.3234 & 0.0808 \\ 0.0808 & 1.0510 \end{bmatrix}$$

The following Mathcad session computes the Cholesky decomposition.

$$M := \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 0.11111 & 0.11111 \\ 0.11111 & 1.11111 \end{bmatrix} \qquad \text{+}$$

$$L := \mathrm{cholesky}(M)$$

$$L = \begin{bmatrix} 3.16228 & 0 \\ -0.31623 & 0.94868 \end{bmatrix} \qquad L \cdot L^T = \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix} \qquad L^{-1} \cdot M \cdot \left(L^T\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.81 Consider the matrix and vector

$$A = \begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \qquad b = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

use a code to solve $Ax = b$ for $\varepsilon = 0.1$, 0.01, 0.001, 10^{-6}, and 1.

Solution:

The equation is

$$\begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \mathbf{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

The following Mathcad session illustrates the effect of ε on the solution, a entire integer difference. Note that no solution exists for the case $\varepsilon = 1$.

$$b := \begin{bmatrix} 10 \\ 10 \end{bmatrix} \qquad A(\varepsilon) := \begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \qquad x(\varepsilon) := A(\varepsilon)^{-1} b$$

$$x(0.1) = \begin{bmatrix} 22.222 \\ 122.222 \end{bmatrix} \qquad x(0.01) = \begin{bmatrix} 20.202 \\ 1.02 \cdot 10^3 \end{bmatrix} \qquad x(0.001) = \begin{bmatrix} 20.02 \\ 1.002 \cdot 10^4 \end{bmatrix}$$

$$x(10^{-6}) = \begin{bmatrix} 20 \\ 1 \cdot 10^7 \end{bmatrix}$$

So the solution to this problem is very sensitive, and ill conditioned, because of the inverse.

4.82 Calculate the natural frequencies and mode shapes of the system of Example 4.8.3. Use the undamped equation and the form given by equation (4.161).

Solution:

The following MATLAB program will calculate the natural frequencies and mode shapes for Example 4.8.3 using Equation (4.161).

```
m=[0.4 0 0;0 2 0;0 0 8]*1e3;
k=[30 -30 0;-30 38 -8;0 -8 88]1e4;
[u, d]=eig(k,m);
w=sqrt (d);
```

The matrix d contains the square of the natural frequencies, and the matrix u contains the corresponding mode shapes.

4.83 Compute the natural frequencies and mode shapes of the undamped version of the system of Example 4.8.3 using the formulation of equation (4.164) and (4.168). Compare your answers.

Solution:

The following MATLAB program will calculate the natural frequencies and mode shapes for Example 4.8.3 using Equation (4.161).

```
m=[0.4 0 0;0 2 0;0 0 8]*1e3;
k=[30 -30 0;-30 38 -8;0 -8 88]1e4;
mi=inv(m);
kt=mi*k;
[u, d]=eig(k,m);
w=sqrt (d);
```

The number of floating point operations needed is 439.
The matrix d contains the square of the natural frequencies, and the matrix u contains the corresponding mode shapes.

The following MATLAB program will calculate the natural frequencies and mode shapes for Example 4.8.3 using Equation (4.168).

```
m=[0.4 0 0;0 2 0;0 0 8]*1e3;
k=[30 -30 0;-30 38 -8;0 -8 88]1e4;
msi=inv(sqrt(m));
kt=msi*k*msi;
[p, d]=eig(kt);
w=sqrt (d);
u=msi*p;
```

The number of floating point operations needed is 461.
The matrix d contains the square of the natural frequencies, and the matrix u contains the corresponding mode shapes.

The method of Equation (4.161) is faster.

4.84 Use a code to solve for the modal information of Example 4.1.5.

Solution: See Toolbox or use the following Mathcad code:

$$\omega := 1$$

Given

$$\left[\omega^4 - \left(6 \cdot \omega^2\right)\right] + 8 = 0$$

$$\text{Find}\,(\omega) = 1.414$$

$$\omega := 2$$

Given

$$\left[\omega^4 - \left(6 \cdot \omega^2\right)\right] + 8 = 0$$

$$\text{Find}\,(\omega) = 2$$

4.85 Write a program to perform the normalization of Example 4.4.2 (i.e., calculate α such that the vector $\alpha\mathbf{v}_1$ is normal).

Solution:

The following MATLAB program will perform the normalization of Example 4.4.2.

```
x=[.4450 .8019 1];
mag=sqrt(sum(x.^2));
xnorm=x/mag;
```

The variable `mag` is the same as α, and `xnorm` is the normalized vector. The original vector `x` can be any length.

4.86 Use a code to calculate the natural frequencies and mode shapes obtained for the system of Example 4.2.5 and Figure 4.4.

Solution: See Toolbox or use the following Mathcad code:

$$M := \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad K := \begin{bmatrix} 12 & -2 \\ -2 & 12 \end{bmatrix} \qquad Mr := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$Kd := Mr^{-1} \cdot K \cdot Mr^{-1}$$

$$Kd = \begin{bmatrix} 12 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\lambda := eigenvals(Kd)$$

$$\lambda = \begin{bmatrix} 12.11 \\ 2.89 \end{bmatrix}$$

$$\omega 1 := \sqrt{\lambda_1} \qquad \omega 2 := \sqrt{\lambda_0} \qquad \omega 1 = 1.7 \qquad \omega 2 = 3.48$$

$$v1 := eigenvec\left(Kd, \lambda_1\right) \qquad v2 := eigenvec\left(Kd, \lambda_0\right)$$

$$v1 = \begin{bmatrix} 0.109 \\ 0.994 \end{bmatrix} \qquad v2 = \begin{bmatrix} -0.994 \\ 0.109 \end{bmatrix} \qquad v1^T \cdot v2 = 0$$

$$v1^T \cdot v1 = 1 \qquad v2^T \cdot v2 = 1$$

$$P := augment(v1, v2)$$

$$P^T \cdot Kd \cdot P = \begin{bmatrix} 2.89 & 0 \\ 0 & 12.11 \end{bmatrix} \qquad P^T \cdot P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.87 Following the modal analysis solution of Window 4.4, write a program to compute the time response of the system of Example 4.3.2.

Solution: The following MATLAB program will compute and plot the time response of the system of Example 4.3.2.

```
t=(0:.1:10)';

m=[1 0;0 4];
k=[12 -2;-2 12];
n=max(size(m));

x0=[1 1]';
xd0=[0 0]';

msi=inv(sqrtm(m));
kt=msi*k*msi;

[p, w]=eig(kt);
for i=1: n-1
  for j=1: n-I
        if w(j,j)>w(j+1,j+1)
              dummy=w(j,j);
              w(j,j)=w(j+1,j+1);
              w(j+1,j+1)=dummy;
              dummy=p(:,j);
              p(:,j)=p(:,j+1);
              p(:,j+1)=dummy;
        end
  end
end
pt=p';
s=msi*p;
si=pt*sqrtm(m);

r0=si*x0;
rd0=si*xd0;
r=[];
for i=1: n,
  wi=sqrt(w(i,i));
  rcol=(swrt((wi*r0(i))^2+rd0(i)^2/wi)*…
        sin(wi*t+atan2(wi*r0(i),rd0(i))));
  r(:,i)=rcol;
end
x=s*r;
plot(t,x);
end
```

4.88 Use a code to solve the damped vibration problem of Example 4.6.1 by calculating the natural frequencies, damping ratios, and mode shapes.

Solution: See Toolbox or use the following Mathcad code (all will do this)

$$M := \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \qquad K := \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \qquad Mr := \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad C := \begin{bmatrix} 2.7 & -.3 \\ -.3 & 0.3 \end{bmatrix}$$

$$Kd := Mr^{-1} \cdot K \cdot Mr^{-1} \qquad\qquad\qquad\qquad Cd := Mr^{-1} \cdot \left(C \cdot Mr^{-1}\right)$$

$$Kd = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$Cd = \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.3 \end{bmatrix}$$

$$\lambda := \text{eigenvals}(Kd)$$

$$\lambda = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\omega 1 := \sqrt{\lambda_1} \qquad \omega 2 := \sqrt{\lambda_0} \qquad \omega 1 = 1.414 \qquad \omega 2 = 2$$

$$v1 := \text{eigenvec}\left(Kd, \lambda_1\right) \qquad v2 := \text{eigenvec}\left(Kd, \lambda_0\right)$$

$$v1 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} \qquad\qquad v2 = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix} \qquad\qquad v1^T \cdot v2 = 0$$

$$v1^T \ v1 = 1 \qquad\qquad v2^T \cdot v2 = 1$$

$$P := \text{augment}(v1, v2)$$

$$P^T \cdot Kd \cdot P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad\qquad P^T \cdot P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \qquad\qquad Cz := P^T \cdot Cd \cdot P$$

$$Cz = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$\zeta 1 := \frac{Cz_{0,0}}{2 \cdot \omega 1} \qquad \zeta 1 = 0.071 \qquad \zeta 2 := \frac{Cz_{1,1}}{2 \cdot \omega 2} \qquad \zeta 2 = 0.1$$

$$\omega d1 := \omega 1 \cdot \sqrt{1 - \zeta 1^2} \qquad \omega d2 := \omega 2 \cdot \sqrt{1 - \zeta 2^2}$$

$$\omega d1 = 1.411 \qquad\qquad\qquad\qquad \omega d2 = 1.99$$

4.89 Consider the vibration of the airplane of Problems 4.46 and 4.47 as given in Figure P4.46. The mass and stiffness matrices are given as

$$M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad K = \frac{EI}{l^3}\begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

where $m = 3000$ kg, $l = 2$ m, $I = 5.2 \times 10^{-6}$ m^4, $E = 6.9 \times 10^9$ N/m^2, and the damping matrix C is taken to be $C = (0.002)K$. Calculate the natural frequencies, normalized mode shapes, and damping ratios.

Solution: Use the Toolbox or use a code directly such as the following Mathcad session:

$E := 6.9 \cdot 10^9$ $\qquad$ $I := 5.2 \cdot 10^{-6}$ $\qquad$ $m := 3000$ $\qquad$ $L := 2$

$$M := m \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad K := \frac{E \cdot I}{L^3}\begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \qquad C := 0.002 \cdot K$$

$$M = \begin{bmatrix} 3 \cdot 10^3 & 0 & 0 \\ 0 & 1.2 \cdot 10^4 & 0 \\ 0 & 0 & 3 \cdot 10^3 \end{bmatrix} \qquad K = \begin{bmatrix} 1.346 \cdot 10^4 & -1.346 \cdot 10^4 & 0 \\ -1.346 \cdot 10^4 & 2.691 \cdot 10^4 & -1.346 \cdot 10^4 \\ 0 & -1.346 \cdot 10^4 & 1.346 \cdot 10^4 \end{bmatrix}$$

$i := 0, 1 .. 2 \qquad j := 0, 1 .. 2$

$\mathrm{Mr}_{i,j} := \sqrt{M_{i,j}}$

$\mathrm{Kh} := \mathrm{Mr}^{-1} \cdot K \cdot \mathrm{Mr}^{-1} \qquad \mathrm{Kh} = \begin{bmatrix} 4.485 & -2.242 & 0 \\ -2.242 & 2.242 & -2.242 \\ 0 & -2.242 & 4.485 \end{bmatrix} \qquad \mathrm{Ch} := \mathrm{Mr}^{-1} \cdot \left(C \cdot \mathrm{Mr}^{-1} \right)$

$\text{eigenvals (Kh)} = \begin{bmatrix} 6.727 \\ 4.485 \\ 0 \end{bmatrix} \qquad \lambda_1 := 0 \qquad \omega_1 := \sqrt{\lambda_1}$

$$\lambda_2 := 4.485 \quad \omega_2 := \sqrt{\lambda_2} \quad \omega_2 = 2.118$$

$$v_1 := \text{eigenvec}\left(Kh, \lambda_1\right) \quad v_1 = \begin{bmatrix} 0.408 \\ 0.816 \\ 0.408 \end{bmatrix}$$

$$\lambda_3 := 6.727 \quad \omega_3 := \sqrt{\lambda_3} \quad \omega_3 = 2.594$$

$$v_2 := \text{eigenvec}\left(Kh, \lambda_2\right) \quad v_2 = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix} \quad v_3 := \text{eigenvec}\left(Kh, \lambda_3\right) \quad v_3 = \begin{bmatrix} 0.577 \\ -0.577 \\ 0.577 \end{bmatrix}$$

$$P1 := \text{augment}\left(v_1, v_2\right) \quad P := \text{augment}\left(P1, v_3\right) \quad Ac := P^T \cdot Ch \cdot P$$

$$Ac = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8.97 \cdot 10^{-3} & 0 \\ 0 & 0 & 0.013 \end{bmatrix} \quad \zeta2 := \frac{Ac_{1,1}}{2 \cdot \omega_2} \quad \zeta2 = 2.118 \cdot 10^{-3}$$

$$\zeta3 := \frac{Ac_{2,2}}{2 \cdot \omega_3} \quad \zeta3 = 2.594 \cdot 10^{-3}$$

The normalized mode shapes are

$$u1 := Mr^{-1} \cdot v_1 \quad u1n := \frac{u1}{|u1|} \quad u1n = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}$$

$$u2 := Mr^{-1} \cdot v_2 \quad u2n := \frac{u2}{|u2|} \quad u2n = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$u3 := Mr^{-1} \cdot v_3 \quad u3n := \frac{u3}{|u3|} \quad u3n = \begin{bmatrix} 0.667 \\ -0.333 \\ 0.667 \end{bmatrix}$$

4.90 Consider the proportionally damped, dynamically coupled system given by

$$M = \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} 49 & -2 \\ -2 & 2 \end{bmatrix}$$

and calculate the mode shapes, natural frequencies, and damping ratios.

Solution: Use the Toolbox or any of the codes. A Mathcad solution is shown:

$$M := \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix} \qquad K := \begin{bmatrix} 49 & -2 \\ -2 & 2 \end{bmatrix} \qquad C := \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$L := cholesky(M)$$

$$Kh := L^{-1} \cdot K \cdot (L^T)^{-1} \qquad Kh = \begin{bmatrix} 5.444 & 1.218 \\ 1.218 & 2.431 \end{bmatrix} \qquad Ch := L^{-1} \cdot \left[C \cdot (L^T)^{-1} \right]$$

$$Ch = \begin{bmatrix} 0.333 & -0.236 \\ -0.236 & 0.917 \end{bmatrix} \qquad eigenvals(Kh) = \begin{bmatrix} 5.875 \\ 2 \end{bmatrix}$$

$$\lambda_1 := 2 \qquad \omega_1 := \sqrt{\lambda_1} \qquad \omega_1 = 1.414 \qquad \lambda_2 := 5.875 \qquad \omega_2 := \sqrt{\lambda_2} \qquad \omega_2 = 2.424$$

$$v_1 := eigenvec(Kh, \lambda_1) \qquad v_1 = \begin{bmatrix} -0.333 \\ 0.943 \end{bmatrix} \qquad v_2 := eigenvec(Kh, \lambda_2) \qquad v_2 = \begin{bmatrix} 0.943 \\ 0.333 \end{bmatrix}$$

$$P := augment(v_1, v_2) \qquad Ac := P^T \cdot Ch \cdot P$$

$$Ac = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$\zeta 1 := \frac{Ac_{0,0}}{2 \cdot \omega_1} \qquad \zeta 1 = 0.354 \qquad \zeta 2 := \frac{Ac_{1,1}}{2 \cdot \omega_2} \qquad \zeta 2 = 0.052$$

+

Computing the mode shapes from the eigenvectors yields:

$$M := \begin{pmatrix} 9 & -1 \\ -1 & 1 \end{pmatrix}$$

$$R := cholesky(M)$$

$$R^{-1} = \begin{pmatrix} 0.333 & 0 \\ 0.118 & 1.061 \end{pmatrix}$$

$$u1 := R^{-1} \cdot \begin{pmatrix} -0.333 \\ 0.943 \end{pmatrix} \quad u1 = \begin{pmatrix} -0.111 \\ 0.961 \end{pmatrix} \quad u2 := R^{-1} \cdot \begin{pmatrix} 0.934 \\ 0.333 \end{pmatrix} \quad u2 = \begin{pmatrix} 0.311 \\ 0.463 \end{pmatrix}$$

Problems and Solutions Section 4.10 (4.91 through 4.98)

4.91* Solve the system of Example 1.7.3 for the vertical suspension system of a car with $m = 1361$ kg, $k = 2.668$ x 10^5 N/m, and $c = 3.81$ x 10^4 kg/s subject to the initial conditions of $x(0) = 0$ and $v(0) = 0.01$ m/s^2.

Solution: Use a Runge Kutta routine such as the one given in Mathcad here or use the toolbox:

$$m := 1361 \qquad k := 2.668 \cdot 10^5 \qquad c := 3.81 \cdot 10^4$$

$$X := \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \qquad A := \begin{bmatrix} 0 & 1 \\ \dfrac{-k}{m} & \dfrac{-c}{m} \end{bmatrix} \qquad D(t,X) := A \cdot X$$

$$Z := \text{rkfixed}(X, 0, 20, 3000, D)$$

$$t := Z^{<0>} \qquad x1 := Z^{<1>}$$
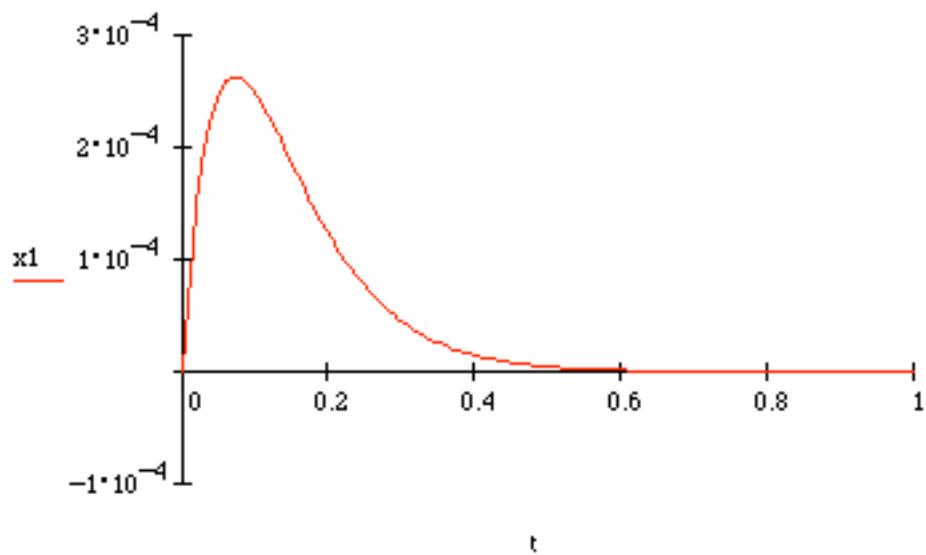
4.92* Solve for the time response of Example 4.4.3 (i.e., the four-story building of Figure 4.9). Compare the solutions obtained with using a modal analysis approach to a solution obtained by numerical integration.

Solution: The following code provides the numerical solution.

$$I := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad O := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M := 4000 \cdot I$$

$$K := \begin{bmatrix} 10000 & -5000 & 0 & 0 \\ -5000 & 10000 & -5000 & 0 \\ 0 & -5000 & 10000 & -5000 \\ 0 & 0 & -5000 & 5000 \end{bmatrix} \qquad C := O \qquad X := \begin{bmatrix} 0.025 \\ 0.02 \\ 0.01 \\ 0.001 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} \, C\right)\right)$$

$$D(t, X) := A \cdot X \qquad Z := \text{rkfixed}(X, 0, 200, 3000, D) \qquad +$$

$$t := Z^{<0>}$$

$$x1 := Z^{<1>} \qquad x2 := Z^{<2>} \qquad x3 := Z^{<3>} \qquad x4 := Z^{<4>}$$
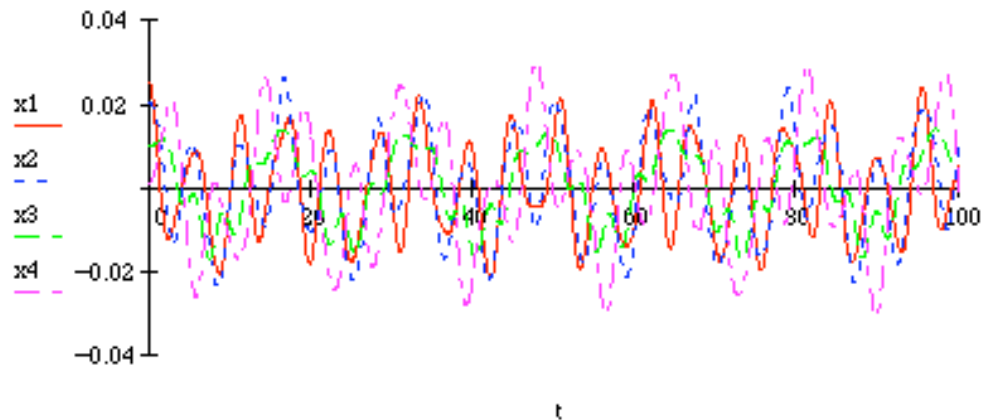

which compares very well with the plots given in Figure 4.11 obtained by plotting the modal equations. One could also plot the modal response and numerical response on the same graph to see a more rigorous comparison.

4.93* Reproduce the plots of Figure 4.13 for the two-degree of freedom system of Example 4.5.1 using a code.

Solution: Use any of the codes. The trick here is to construct the damping matrix from the given modal information by first creating it in modal form and then transforming it back to physical coordinates as indicated in the following Mathcad session:

$$M := \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \qquad P := \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad K := \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \qquad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Mr := \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \omega 1 := \sqrt{2} \qquad \omega 2 := 2 \qquad \zeta 1 := 0.05 \qquad \zeta 2 := 0.1$$

$$Ac := \begin{bmatrix} 2 \cdot \zeta 1 \cdot \omega 1 & 0 \\ 0 & 2 \cdot \zeta 2 \cdot \omega 2 \end{bmatrix} \qquad C := Mr \cdot P \cdot Ac \cdot P^T \cdot Mr$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} \, C\right)\right) \qquad X := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(t, X) := A \cdot X$$

+

$$Z := \text{rkfixed}(X, 0, 80, 4000, D)$$

$$t := Z^{<0>} \qquad x1 := Z^{<1>} \qquad x2 := Z^{<2>}$$
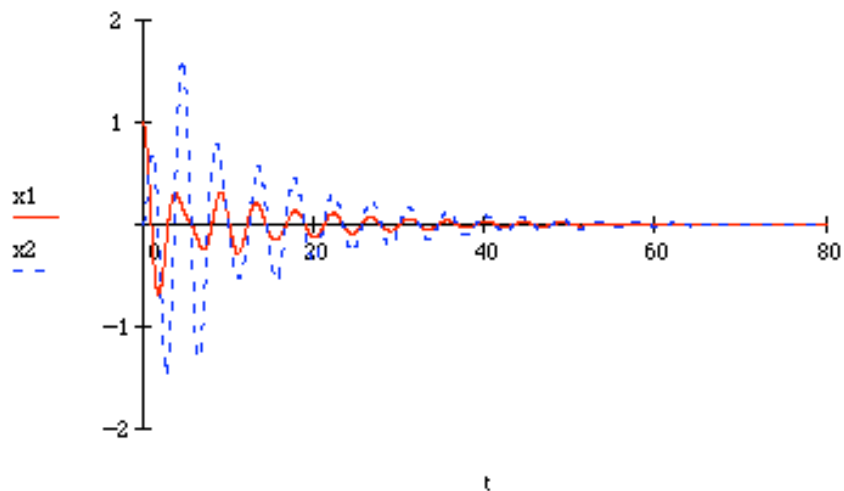
4.94*. Consider example 4.8.3 and a) using the damping ratios given, compute a damping matrix in physical coordinates, b) use numerical integration to compute the response and plot it, and c) use the numerical code to design the system so that all 3 physical coordinates die out within 5 seconds (i.e., change the damping matrix until the desired response results).

Solution: A Mathcad solution is presented. The damping matrix is found, as in the previous problem, by keeping track of the various transformations. Using the notation of the text, the damping matrix is constructed from:

$$C = M^{\frac{1}{2}} P \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 \\ 0 & 2\zeta_2\omega_2 & 0 \\ 0 & 0 & 2\zeta_3\omega_3 \end{bmatrix} P^T M^{\frac{1}{2}} = \begin{bmatrix} 1.062\times10^3 & -679.3 & 187.0 \\ -679.3 & 2.785\times10^3 & 617.8 \\ 187.0 & 617.8 & 2.041\times10^3 \end{bmatrix}$$

as computed using the code that follows. With this form of the matrix the damping ratios are adjusted until the desired criteria are met:

$$I := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad O := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M := \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \cdot 10^3$$

$$\omega1 := 5.3872$$
$$\omega2 := 10.6755$$
$$\omega3 := 30.1166$$

$$K := \begin{bmatrix} 30 & -30 & 0 \\ -30 & 38 & -8 \\ 0 & -8 & 88 \end{bmatrix} \cdot 10^4 \qquad Mr := \begin{bmatrix} \sqrt{0.4 \cdot 10^3} & 0 & 0 \\ 0 & \sqrt{2 \cdot 10^3} & 0 \\ 0 & 0 & \sqrt{8 \cdot 10^3} \end{bmatrix}$$

$$\zeta1 := 0.2$$
$$\zeta2 := 0.05$$
$$\zeta3 := 0.05$$

$$P := \begin{bmatrix} -0.4116 & -0.1021 & 0.9056 \\ -0.8848 & -0.1935 & -0.4239 \\ -0.2185 & 0.9758 & 0.0106 \end{bmatrix} \qquad Ac := \begin{bmatrix} 2 \cdot \omega1 \cdot \zeta1 & 0 & 0 \\ 0 & 2 \cdot \omega2 \cdot \zeta2 & 0 \\ 0 & 0 & 2 \cdot \omega3 \cdot \zeta3 \end{bmatrix}$$

$$C := Mr \cdot P \cdot Ac \cdot P^T \cdot Mr \qquad A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} \, C\right)\right)$$

In changing the damping ratios it is best to start with the rubber component which is the first mode-damping ratio. Doubling it nails the first two coordinates but does not affect the third coordinate enough. Hence the second mode-damping ratio must be changed (doubled here) to attack this mode. This could be accomplished by adding a viscoelastic strip as described in Chapter 5 to the metal. Thus the ratios given in the code above do the trick as the following plots show. Note also how much the damping matrix changes.

$$X := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad B := \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} \qquad f := M^{-1} \cdot B \qquad C = \begin{bmatrix} 1.138 \cdot 10^3 & -313.286 & 208.132 \\ -313.286 & 4.536 \cdot 10^3 & 805.984 \\ 208.132 & 805.984 & 8.958 \cdot 10^3 \end{bmatrix}$$

$$D(t,X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_0 \\ f_1 \\ f_2 \end{bmatrix} \cdot \left(\Phi(t) - \Phi(t - 0.001) \right)$$

$$\text{kfixed}(X, 0, 15, 4000, D)$$

$$t := Z^{<0>} \qquad x1 := Z^{<1>} \qquad x2 := Z^{<2>} \qquad x3 := Z^{<3>}$$
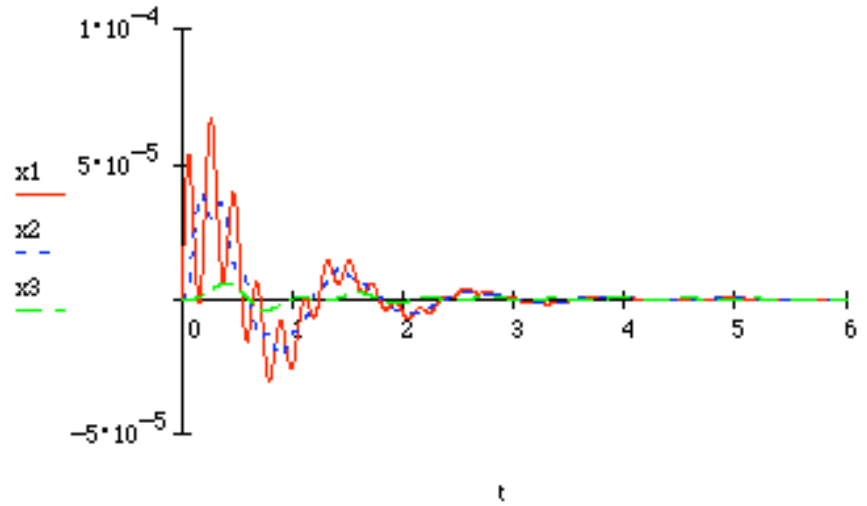
4.95*. Compute and plot the time response of the system (Newtons):

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(4t)$$

subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{m}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{m/s}$$

Solution: The following Mathcad session illustrates the numerical solution of this problem using a Runge Kutta solver.

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \quad K := \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$X := \begin{bmatrix} 0 \\ 0.1 \\ 1 \\ 0 \end{bmatrix}$$

$$C := \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} C\right)\right)$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ 1 \\ \dfrac{1}{5} \\ 1 \end{bmatrix}$$

$$Z := \text{rkfixed}(X, 0, 100, 3000, D)$$

$$t := Z^{<0>} \qquad x1 := Z^{<1>} \qquad x2 := Z^{<2>}$$
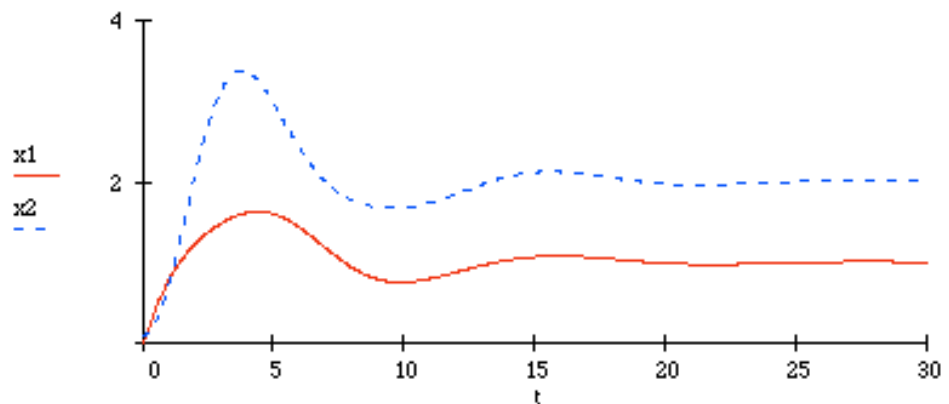
4.96* Consider the following system excited by a pulse of duration 0.1 s (in Newtons):

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0.3 & -0.05 \\ -0.05 & 0.05 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[\Phi(t-1) - \Phi(t-3)]$$

and subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix}\text{ m,}\quad \mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\text{ m/s}$$

Compute and plot the response of the system. Here Φ indicates the Heaviside Step Function introduced in Section 3.2.

Solution: The following Mathcad solution (see example4.10.3 for the other codes) gives the solution:

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad K := \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$X := \begin{bmatrix} 0 \\ -0.1 \\ 0 \\ 0 \end{bmatrix}$$

$$C := \begin{bmatrix} 0.33 & -0.05 \\ -0.05 & 0.05 \end{bmatrix} \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad f := M^{-1} \cdot B$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} C\right)\right)$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ f_0 \\ f_1 \end{bmatrix} \cdot \left(\Phi(t-1) - \Phi(t-3)\right)$$
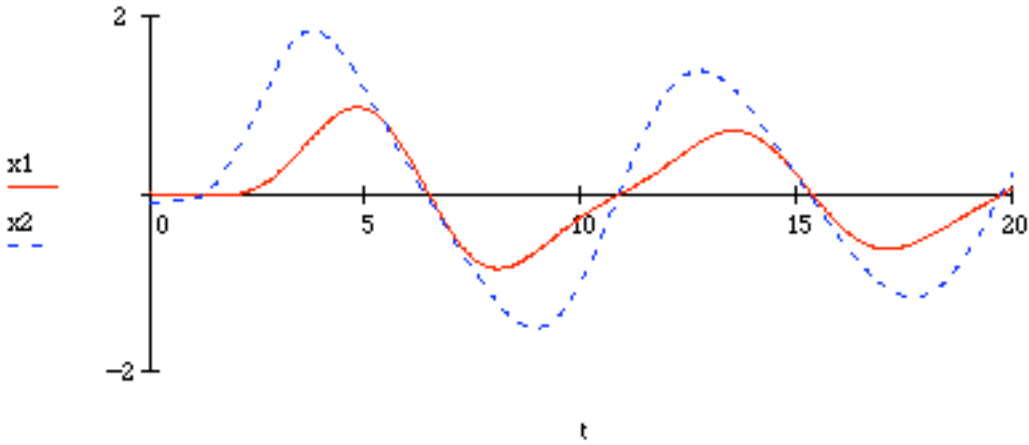
$$Z := \text{rkfixed}(X, 0, 120, 3000, D)$$

$$t := Z^{<0>}$$

$$x1 := Z^{<1>} \qquad x2 := Z^{<2>} \qquad +$$
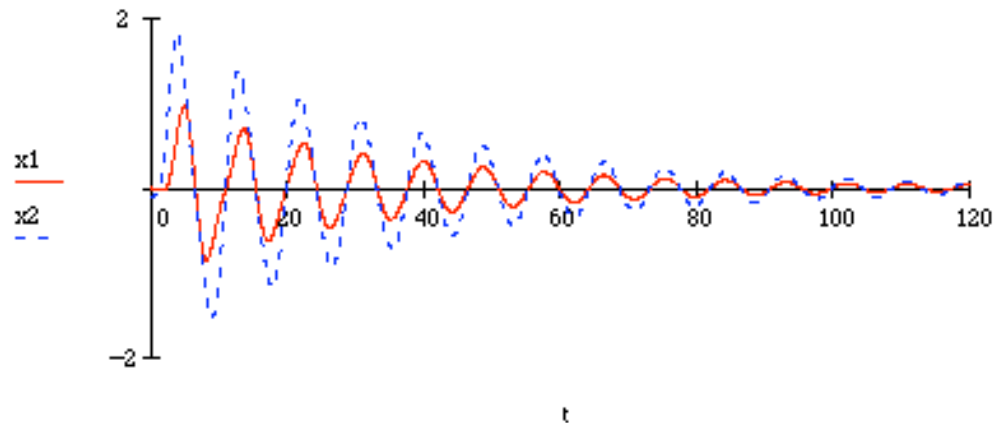
It is also interesting to examine the first 20 seconds more closely to see the effect of the impact:

Note that the impact has much more of an effect on the response than does the initial condition.

4.97.* Compute and plot the time response of the system (Newtons):

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 30 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(4t)$$

subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{ m}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ m/s}$$

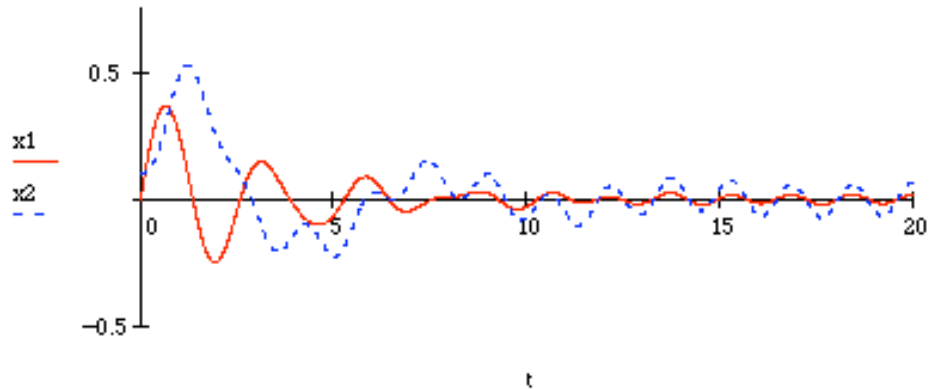**Solution:** Following the codes of Example 4.10.2 yields the solution directly.

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M := \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \qquad K := \begin{bmatrix} 30 & -1 \\ -1 & 1 \end{bmatrix}$$

$$X := \begin{bmatrix} 0 \\ 0.1 \\ 1 \\ 0 \end{bmatrix}$$

$$C := \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \qquad B := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad f := M^{-1} \cdot B$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} C\right)\right)$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ f_0 \\ f_1 \end{bmatrix} \cdot \sin(4 \cdot t)$$

$$Z := \text{rkfixed}(X, 0, 20, 3000, D)$$

$$t := Z^{<0>} \qquad x1 := Z^{<1>} \qquad x2 := Z^{<2>} \qquad +$$

**4.98.*** Compute and plot the time response of the system (Newtons):

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} 500 & -100 & 0 & 0 \\ -100 & 200 & -100 & 0 \\ 0 & -100 & 200 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\sin(4t)$$

subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.01 \end{bmatrix} \text{m}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{m/s}$$

Solution: Again follow Example 4.10.2 for the various codes. Mathcad is given.

$$I := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad O := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M := \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$K := \begin{bmatrix} 500 & -100 & 0 & 0 \\ -100 & 200 & -100 & 0 \\ 0 & -100 & 200 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \qquad C := \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad X := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.01 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad f := M^{-1} \cdot B$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} \, C\right)\right)$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \cdot \sin(4 \cdot t)$$

$$Z := \text{rkfixed}(X, 0, 200, 3000, D)$$

$$x2 := Z^{<2>} \qquad x3 := Z^{<3>} \qquad x4 := Z^{<4>}$$

$$t := Z^{<0>}$$

$$x1 := Z^{<1>}$$
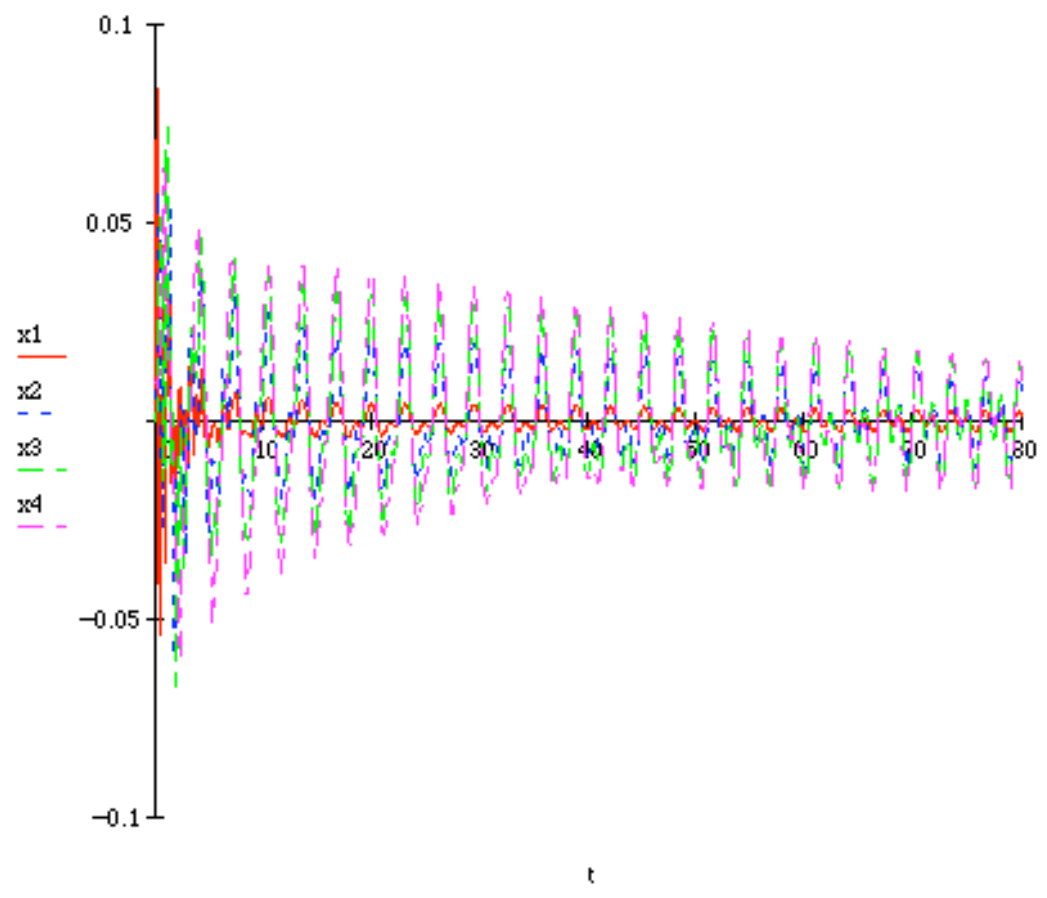
Problems and Solutions Section 5.1 (5.1 through 5.5)

5.1   Using the nomograph of Figure 5.1, determine the frequency range of vibration for which a machine oscillation remains at a satisfactory level under rms acceleration of $1g$.

Solution:

An rms acceleration of $1\ g$ is about 9.81 m/s^2. From Figure 5.1, a satisfactory level would occur at frequencies above 650 Hz.

5.2   Using the nomograph of Figure 5.1, determine the frequency range of vibration for which a structure's rms acceleration will not cause wall damage if vibrating with an rms displacement of 1 mm or less.

Solution:

From Figure 5.1, an rms displacement of 1 mm (1000 μm) would not cause wall damage at frequencies below 3.2 Hz.

5.3   What natural frequency must a hand drill have if its vibration must be limited to a minimum rms displacement of 10 μm and rms acceleration of 0.1 m/s^2? What rms velocity will the drill have?

Solution:

From Figure 5.1, the natural frequency would be about 15.8 Hz or 99.6 rad/s. The rms velocity would be 1 mm/s.

5.4   A machine of mass 500 kg is mounted on a support of stiffness 197,392,000 N/m. Is the vibration of this machine acceptable (Figure 5.1) for an rms amplitude of 10 μm? If not, suggest a way to make it acceptable.

Solution:

The frequency is $\omega_n = \sqrt{\dfrac{k}{m}} = 628.3$ rad/s $= 100$ Hz.

For an rms displacement of 10 μm the vibration is unsatisfactory. To make the vibration satisfactory, the frequency should be reduced to 31.6 Hz. This can be accomplished by reducing the stiffness and/or increasing the mass of the machine.

5.5 Using the expression for the amplitude of the displacement, velocity and acceleration of an undamped single-degree-of-freedom system, calculate the velocity and acceleration amplitude of a system with a maximum displacement of 10 cm and a natural frequency of 10 Hz. If this corresponds to the vibration of the wall of a building under a wind load, is it an acceptable level?

Solution:

The velocity amplitude is

$$\left| v(t) \right| = A\omega_n = \left(0.1 \text{ m} \right)\left(\frac{10}{2\pi} \right) = 0.159 \text{ m/s}$$

The acceleration amplitude is

$$\left| a(t) \right| = A\omega_n^2 = \left(0.1 \text{ m} \right)\left(\frac{10}{2\pi} \right)^2 = 0.253 \text{ m/s}^2$$

The rms displacement is $\dfrac{A}{\sqrt{2}} = \dfrac{0.1}{\sqrt{2}} = 0.0707 \text{ m} = 70{,}700 \text{ μm}$ (from equation (1.21)). At 10 Hz and 70,700 μm , this could be destructive to a building.

Problems and Solutions Section 5.1 (5.6 through 5.26)

5.6   A 100-kg machine is supported on an isolator of stiffness 700×10^3 N/m. The machine causes a vertical disturbance force of 350 N at a revolution of 3000 rpm. The damping ratio of the isolator is $\zeta = 0.2$. Calculate (a) the amplitude of motion caused by the unbalanced force, (b) the transmissibility ratio, and (c) the magnitude of the force transmitted to ground through the isolator.

Solution:

(a) From Window 5.2, the amplitude at steady-state is

$$ X = \frac{F_o / m}{\left[\left(\omega_n^2 - \omega^2 \right)^2 + \left(2\zeta\omega_n\omega \right)^2 \right]^{1/2}} $$

Since $\omega_n = \sqrt{\dfrac{k}{m}} = 83.67$ rad/s and $\omega = 3000\left(\dfrac{2\pi}{60}\right) = 314.2$ rad/s,

(b) From equation (5.7), the transmissibility ratio is

$$ \frac{F_T}{F_0} = \sqrt{\frac{1 + \left(2\zeta r \right)^2}{\left(1 - r^2 \right)^2 + \left(2\zeta r \right)^2}} $$

Since $r = \dfrac{\omega}{\omega_n} = 3.755$, this becomes

$$ \frac{F_T}{F_0} = 0.1368 $$

(c) The magnitude is

$$ F_T = \left(\frac{F_T}{F_0} \right) F_0 = \left(0.1368 \right)\left(350 \right) = \left(47.9 \right) $$
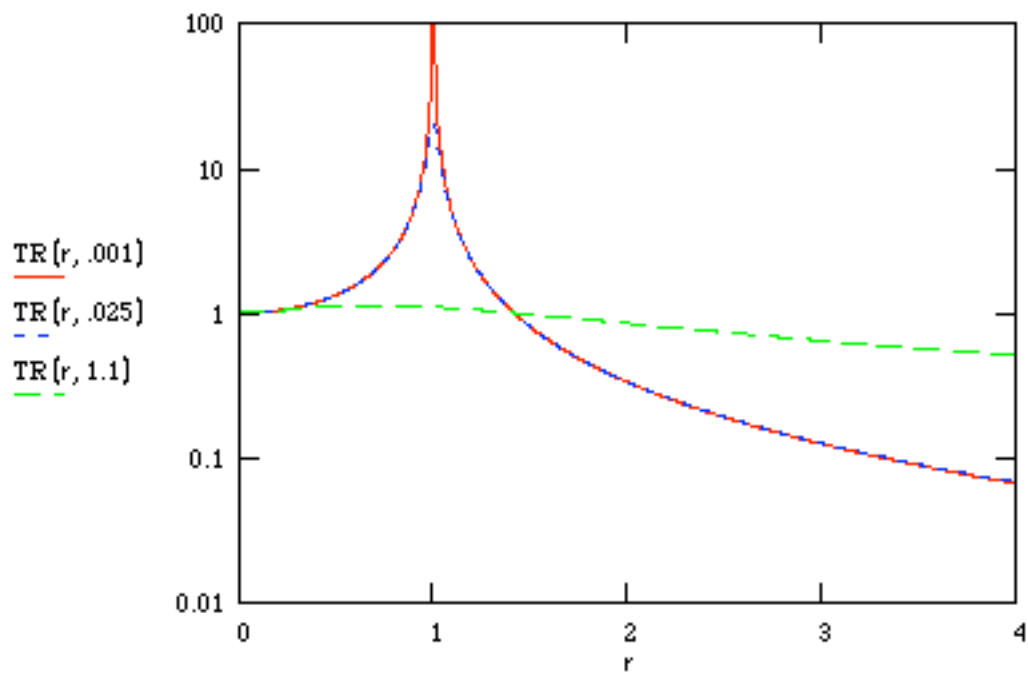
5.7   Plot the T.R. of Problem 5.6 for the cases $\zeta = 0.001$, $\zeta = 0.025$, and $\zeta = 1.1$.

   Solution:

$$\text{T.R.} = \sqrt{\frac{1 + \left(2\zeta r\right)^2}{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}}$$

   A plot of this is given for $\zeta = 0.001$, $\zeta = 0.025$, and $\zeta = 1.1$. The plot is given here from Mathcad:

$$\text{TR}\left(r, \zeta\right) := \sqrt{\frac{1 + \left(2 \cdot \zeta \cdot r\right)^2}{\left(1 - r^2\right)^2 + \left(2 \cdot \zeta \cdot r\right)^2}}$$

5.8 A simplified model of a washing machine is illustrated in Figure P5.8. A bundle of wet clothes forms a mass of 10 kg (m_b) in the machine and causes a rotating unbalance. The rotating mass is 20 kg (including m_b) and the diameter of the washer basket ($2e$) is 50 cm. Assume that the spin cycle rotates at 300 rpm. Let k be 1000 N/m and $\zeta = 0.01$. Calculate the force transmitted to the sides of the washing machine. Discuss the assumptions made in your analysis in view of what you might know about washing machines.

Top view

Frictionless
support
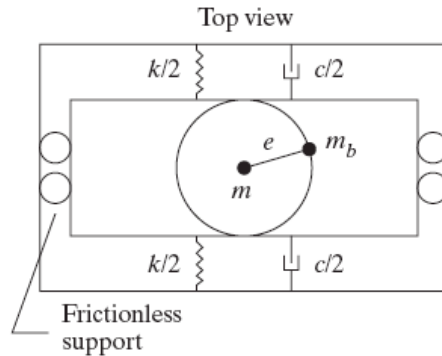
Solution: The transmitted force is given by $F_T = \sqrt{k^2 + c^2\omega_r^2}$ where

$$c = 2\zeta\omega_n, \quad \omega_n = \sqrt{\frac{k}{m}} = 7.071 \ \text{rad/s}, \ \omega_r = 300\frac{2\pi}{60} = 31.42 \ \text{rad/s},$$

and X is given by equation (2.84) as

$$X = \frac{m_0 e}{m} \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}}$$

Since $r = \dfrac{\omega_r}{\omega_n} = 4.443$, then $X = 0.1317$ m and

$$F_T = (0.1317)\sqrt{(1000)^2 + [2(0.01)(20)(7.071)]^2(31.42)^2} = 132.2 \ \text{N}$$

Two important assumptions have been made:
i) The out-of-balance mass is concentrated at a point and
ii) The mass is constant and distributed evenly (keep in mind that water enters and leaves) so that the mass actually changes.

5.9   Referring to Problem 5.8, let the spring constant and damping rate become variable. The quantities m, m_b, e and ω are all fixed by the previous design of the washing machine. Design the isolation system (i.e., decide on which value of k and c to use) so that the force transmitted to the side of the washing machine (considered as ground) is less than 100N.

Solution:

The force produced by the unbalance is $F_r = m_b a$ where a is given by the magnitude of equation (2.81):

$$F_r = m_0 \left| \ddot{x}_r \right| = e m_0 \omega_r^2 = (0.25)(10) \left[300 \left(\frac{2\pi}{60} \right) \right]^2 = 2467.4 \text{ N}$$

Since $F_T < 100$ N,

$$\text{T.R.} = \frac{F_T}{F_r} = \frac{100}{2467.4} = 0.0405$$

If the damping ratio is kept at 0.01, this becomes

$$\text{T.R.} = 0.0405 = \sqrt{\frac{1 + \left[2(0.01) r \right]^2}{\left(1 - r^2 \right)^2 + \left[2(0.01) r \right]^2}}$$

Solving for r yields $r = 5.079$.

Since $r = \dfrac{\omega_r}{\sqrt{k/m}}$,

$$k = \frac{m \omega_r^2}{r^2} = \frac{(20) \left[300 \left(\frac{2\pi}{60} \right) \right]^2}{5.079^2} = 765 \text{ N/m}$$

and

$$c = 2 \zeta \sqrt{km} = 2(0.01) \sqrt{(765)(20)} = 2.47 \text{ kg/s}$$

5.10 A harmonic force of maximum value of 25 N and frequency of 180 cycles/min acts on a machine of 25 kg mass. Design a support system for the machine (i.e., choose c, k) so that only 10% of the force applied to the machine is transmitted to the base supporting the machine.

Solution: From equation (5.7),

$$\text{T.R.} = 0.1 = \sqrt{\frac{1 + \left(2\zeta r\right)^2}{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}} \tag{1}$$

If we choose $\zeta = 0.1$, then solving the equation (1) numerically yields $r = 3.656$. Since $r = \dfrac{\omega}{\sqrt{k/m}}$ then:

$$k = \frac{m\omega^2}{r^2} = \frac{(25)\left[180\left(\dfrac{2\pi}{60}\right)\right]^2}{3.656^2} = 665 \text{ N/m}$$

and

$$c = 2\zeta\sqrt{km} = 2(0.1)\sqrt{(665)(25)} = 25.8 \text{ kg/s}$$

5.11 Consider a machine of mass 70 kg mounted to ground through an isolation system of total stiffness 30,000 N/m, with a measured damping ratio of 0.2. The machine produces a harmonic force of 450 N at 13 rad/s during steady-state operating conditions. Determine (a) the amplitude of motion of the machine, (b) the phase shift of the motion (with respect to a zero phase exciting force), (c) the transmissibility ratio, (d) the maximum dynamic force transmitted to the floor, and (e) the maximum velocity of the machine.

Solution:

(a) The amplitude of motion can be found from Window 5.2:

$$X = \frac{F_0 / m}{\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2\right]^{1/2}}$$

where $\omega_n = \sqrt{\dfrac{k}{m}} = 20.7$ rad/s. So,

$$X = 0.0229 \text{ m}$$

(b) The phase can also be found from Window 5.2:

$$\phi = \tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} = 22.5° = 0.393 \text{ rad}$$

(c) From Eq. 5.7, with $r = \dfrac{\omega}{\omega_n} = 0.628$

$$\text{T.R.} = \sqrt{\frac{1 + \left(2\zeta r\right)^2}{\left(1 - r^2\right)^2 + \left(2\zeta r\right)^2}} = 1.57$$

(d) The magnitude of the force transmitted to the ground is

$$F_T = \left(\text{T.R.}\right)F_0 = \left(450\right)\left(1.57\right) = 707.6 \text{ N}$$

(e) The maximum velocity would be

$$\omega A_0 = \left(13\right)\left(0.0229\right) = 0.298 \text{ m/s}$$

5.12   A small compressor weighs about 70 lb and runs at 900 rpm. The compressor is mounted on four supports made of metal with negligible damping.
(a) Design the stiffness of these supports so that only 15% of the harmonic force produced by the compressor is transmitted to the foundation.
(b) Design a metal spring that provides the appropriate stiffness using Section 1.5 (refer to Table 1.2 for material properties).

Solution:

(a) From Figure 5.9, the lines of 85% reduction and 900 rpm meet at a static deflection of 0.35 in. The spring stiffness is then

$$k = \frac{mg}{\delta_s} = \frac{70 \text{ lb}}{0.35 \text{ in}} = 200 \text{ lb/in}$$

The stiffness of each support should be $k/4 = 50$ lb/in.

(b) Try a helical spring given by equation (1.67):

$$k = 50 \text{ lb/in} = 8756 \text{ N/m} = \frac{Gd^4}{64nR^3}$$

Using $R = 0.1$ m, $n = 10$, and $G = 8.0 \times 10^{10}$ N/m² (for steel) yields

$$d = \left[\frac{64(8756)(10)(0.1)^3}{8.0 \times 10^{10}} \right]^{1/4} = 0.0163 \text{ m} = 1.63 \text{ cm}$$

5.13 Typically, in designing an isolation system, one cannot choose any continuous value of k and c but rather, works from a parts catalog wherein manufacturers list isolators available and their properties (and costs, details of which are ignored here). Table 5.3 lists several made up examples of available parts. Using this table, design an isolator for a 500-kg compressor running in steady state at 1500 rev/min. Keep in mind that as a rule of thumb compressors usually require a frequency ratio of $r = 3$.

Solution:

Since $r = \dfrac{\omega}{\sqrt{k/m}}$, then

$$k = \frac{m\omega^2}{r^2} = \frac{\left(500\left[1500\left(\dfrac{2\pi}{60}\right)\right]^2\right)}{3^2} = 1371 \times 10^3 \text{ N/m}$$

Choose isolator R-3 from Table 5.3. So, $k = 1000 \times 10^3$ N/m and $c = 1500$ N·s/m.

Check the value of r:

$$r = \frac{1500\left(\dfrac{2\pi}{60}\right)}{\sqrt{1000 \times 10^3 / 500}} = 3.51$$

This is reasonably close to $r = 3$.

5.14 An electric motor of mass 10 kg is mounted on four identical springs as indicated in Figure P5.14. The motor operates at a steady-state speed of 1750 rpm. The radius of gyration (see Example 1.4.6 for a definition) is 100 mm. Assume that the springs are undamped and choose a design (i.e., pick k) such that the transmissibility ratio in the vertical direction is 0.0194. With this value of k, determine the transmissibility ratio for the torsional vibration (i.e., using θ rather than x as the displacement coordinates).
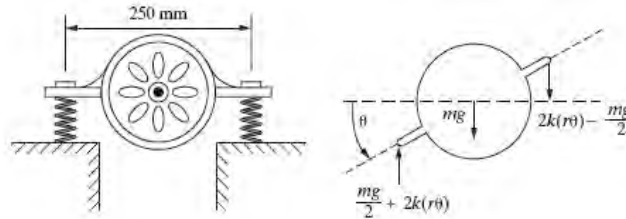
Solution:
TABLE 5.3 Catalog values of stiffness and damping properties of various off-the-shelf isolators

Part No.[a]	R-1	R-2	R-3	R-4	R-5	M-1	M-2	M-3	M-4	M-5
$k(10^3$N/m)	250	500	1000	1800	2500	75	150	250	500	750
c(N·s/m)	2000	1800	1500	1000	500	110	115	140	160	200

[a]The "R" in the part number designates that the isolator is made of rubber, and the "M" designates metal. In general, metal isolators are more expensive than rubber isolators.

With no damping, the transmissibility ratio is

$$\text{T.R.} = \frac{1}{r^2 - 1}$$

where

$$r = \frac{\omega}{\sqrt{4k/m}} = \frac{1750\left(\frac{2\pi}{60}\right)}{\sqrt{4k/10}} = \frac{579.5}{\sqrt{4k}}$$

$$0.0194 = \frac{1}{\dfrac{(579.5)^2}{4k} - 1}$$

$$4k = 6391 \text{ N/m}$$

For each spring, $k = 1598$ N/m.

For torsional vibration, the equation of motion is

$$I\ddot{\theta} = -\left[\frac{mg}{2} + 2kr\theta\right]r - \left[2kr\theta - \frac{mg}{2}\right]r$$

where $r = \dfrac{0.250 \text{ m}}{2} = 0.125$ m and from the definition of the radius of gyration and the center of percussion (see Example 1.4.6):

$$I = mk_0^2 = (10)(0.1)^2 = 0.1 \text{ kg·m}^2$$

So,

$$0.1\ddot{\theta} + 4(1598)(0.125)^2 \theta = 0$$

$$\ddot{\theta} + 998.6\theta = 0$$

The frequency ratio, r, is now

$$r = \frac{1750\left(\dfrac{2\pi}{60}\right)}{\sqrt{998.6}} = 5.80$$

$$\text{T.R.} = \frac{1}{r^2 - 1} = 0.0306$$

5.15 A large industrial exhaust fan is mounted on a steel frame in a factory. The plant manager has decided to mount a storage bin on the same platform. Adding mass to a system can change its dynamics substantially and the plant manager wants to know if this is a safe change to make. The original design of the fan support system is not available. Hence measurements of the floor amplitude (horizontal motion) are made at several different motor speeds in an attempt to measure the system dynamics. No resonance is observed in running the fan from zero to 500 rpm. Deflection measurements are made and it is found that the amplitude is 10 mm at 500 rpm and 4.5 mm at 400 rpm. The mass of the fan is 50 kg and the plant manager would like to store up to 50 kg on the same platform. The best operating speed for the exhaust fan is between 400 and 500 rpm depending on environmental conditions in the plant.

Solution:

A steel frame would be very lightly damped, so

$$\frac{X}{Y} = \frac{1}{1-r^2}$$

Since no resonance is observed between 0 and 500 rpm, $r < 1$.

When $\omega = 500\left(\frac{2\pi}{60}\right) = 52.36$ rad/s, $X = 10$ mm, so

$$10 = \frac{Y}{1-\left(\dfrac{52.36}{\omega_n}\right)^2}$$

Also, at $\omega = 400\left(\frac{2\pi}{60}\right) = 41.89$ rad/s, $X = 4.5$ mm, so

$$4.5 = \frac{Y}{1-\left(\dfrac{41.89}{\omega_n}\right)^2}$$

Solving for ω_n and Y yields

$\omega_n = 59.57$ rad/s
$Y = 2.275$ mm
The stiffness is $k = m\omega_n^2 = (50)(59.57)^2 = 177,453$ N/m. If an additional 50 kg is added so that $m = 100$ kg, the natural frequency becomes

$$\omega_n = \sqrt{\frac{177,453}{100}} = 42.13 \, \text{rad/s} = 402.3 \text{ rpm}$$

This would not be advisable because the normal operating range is 400 rpm to 500 rpm, and resonance would occur at 402.3 rpm.

5.16 A 350-kg rotating machine operates at 800 cycles/min. It is desired to reduce the transmissibility ratio by one-fourth of its current value by adding a rubber vibration isolation pad. How much static deflection must the pad be able to withstand?

Solution:

From equation (5.12), with $R = 0.25$:

$$r = \sqrt{\frac{2-0.25}{1-0.25}} = 1.528 = \frac{\omega}{\sqrt{k/m}} = \frac{800\left(\dfrac{2\pi}{60}\right)}{\sqrt{k/350}}$$

$$k = 1.053 \times 10^6 \text{ N/m}$$

The static deflection is

$$\delta_s = \frac{mg}{k} = \frac{(350)(9.81)}{1.053 \times 10^6} = 3.26 \text{ mm}$$

5.17 A 68-kg electric motor is mounted on an isolator of mass 1200 kg. The natural frequency of the entire system is 160 cycles/min and has a measured damping ratio of $\zeta = 1$. Determine the amplitude of vibration and the force transmitted to the floor if the out-of-balance force produced by the motor is $F(t) = 100 \sin(31.4t)$ in newtons.

Solution:

The amplitude of vibration is given in Window 5.2 as

$$A_0 = \frac{F_0/m}{\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2\right]^{1/2}}$$

where $F_0 = 100$ N, $m = 1268$ kg, $\omega = 31.4$ rad/s, and $\omega_n = 160\left(\dfrac{2\pi}{60}\right) = 16.76$ rad/s. So,

$$X = 6.226 \times 10^{-5} \text{ m}$$

The transmitted force is given by Eq. (5.6), with $r = \dfrac{31.4}{16.76} = 1.874$

$$F_T = F_0 \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}} = 85.97 \text{ N}$$

5.18 The force exerted by an eccentric ($e = 0.22$ mm) flywheel of 1000 kg, is $600 \cos(52.4t)$ in newtons. Design a mounting to reduce the amplitude of the force exerted on the floor to 1% of the force generated. Use this choice of damping to ensure that the maximum force transmitted is never greater than twice the generated force.

Solution:

Two conditions are given. The first is that T.R. = 2 at resonance ($r = 1$), and the second is that T.R. = 0.01 at the driving frequency. Use the first condition to solve for ζ. From equation (5.7),

$$T.R. = 2 = \left[\frac{1 + 4\zeta^2}{4\zeta^2} \right]^{1/2}$$

$$\zeta = 0.2887$$

At the frequency, $r = \dfrac{52.4}{\sqrt{k/1000}}$, so

$$T.R. = 0.01 = \left[\frac{1 + \left[2(0.2887)r \right]^2}{\left(1 - r^2\right)^2 + \left[2(0.2887)r \right]^2} \right]$$

$$r = 57.78 = \frac{52.4}{\sqrt{k/1000}}$$

$$k = 822.6 \text{ N/m}$$

Also,

$$c = 2\zeta\sqrt{km} = 523.6 \text{ kg/s}$$

5.19 A rotating machine weighing 4000 lb has an operating speed of 2000 rpm. It is desired to reduce the amplitude of the transmitted force by 80% using isolation pads. Calculate the stiffness required of the isolation pads to accomplish this design goal.

Solution:

Using Figure 5.9, the lines of 2000 rpm and 80% reduction meet at $\delta_s = 0.053$ in. The spring stiffness should be

$$k = \frac{mg}{\delta_s} = \frac{4000 \text{ lb}}{0.053 \text{ in}} = 75,472 \text{ lb/in}$$

5.20 The mass of a system may be changed to improve the vibration isolation characteristics. Such isolation systems often occur when mounting heavy compressors on factory floors. This is illustrated in Figure P5.20. In this case the soil provides the stiffness of the isolation system (damping is neglected) and the design problem becomes that of choosing the value of the mass of the concrete block/compressor system. Assume that the stiffness of the soil is about $k = 2.0 \times 10^7$ N/m and design the size of the concrete block (i.e., choose m) such that the isolation system reduces the transmitted force by 75%. Assume that the density of concrete is $\rho = 23,000$ N/m^3. The surface area of the cement block is 4 m^2. The steady-state operating speed of the compressor is 1800 rpm.
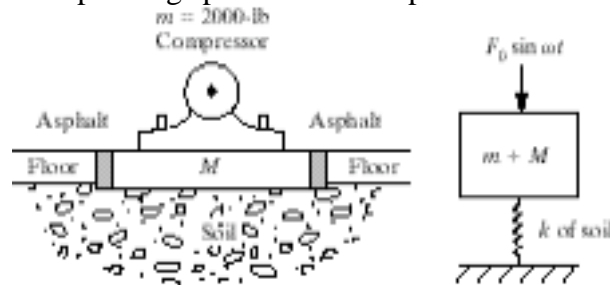
Solution:

Using Figure 5.9, the lines of 75% reduction and 1800 rpm cross at $\delta_s = 0.053$ in $= 0.1346$ cm. Thus the weight of the block should be

$$W_T = \left(m + M\right)g = k\delta_s = 2.0 \times 10^7 \left(0.1346 \times 10^{-2}\right) = 26,924 \text{ N}$$

The compressor weights $mg = (2000 \text{ lb})(4.448222 \text{ N/lb}) = 8896.4$ N. The concrete block should weight $W = W_T - 8896.4 = 18,028$ N. The volume of the block needs to be

$$V = \frac{W}{\rho} = \frac{18,028}{23,000} = 0.7838 \text{ m}^2$$

Assume the surface area is part exposed to the surface. Let the top be a meters on each side (square) and b meters deep. The volume and surface area equations are

$$A = 4\text{m}^2 = a^2$$

$$V = 0.7838 \text{ m}^3 = a^2 b$$

Solving for a and b yields

$$a = 2 \text{ m}$$

$$b = 0.196 \text{ m}$$

5.21 The instrument board of an aircraft is mounted on an isolation pad to protect the panel from vibration of the aircraft frame. The dominant vibration in the aircraft is measured to be at 2000 rpm. Because of size limitation in the aircraft's cabin, the isolators are only allowed to deflect 1/8 in. Find the percent of motion transmitted to the instrument pane if it weights 50 lb.

Solution:

From equation (2.71), with negligible damping,

$$\frac{X}{Y} = \frac{1}{\sqrt{\left(1-r^2\right)^2}}$$

This is the same as the equation that yields Figure 5.9. The lines of 2000 rpm and $\delta_s = 0.125$ in meet at 93%. So only 7% of the plane's motion is transmitted to the instrument panel.

5.22 Design a base isolation system for an electronic module of mass 5 kg so that only 10% of the displacement of the base is transmitted into displacement of the module at 50 Hz. What will the transmissibility be if the frequency of the base motion changes to 100 Hz? What if it reduces to 25 Hz?

Solution: Using Figure 5.9, the lines of 90% reduction and $\omega = (50 \text{ Hz})(60) = 3000$ rpm meet at $\delta_s = 0.042$ in $= 0.1067$ cm. The spring stiffness is then

$$k = \frac{mg}{\delta_s} = \frac{\left(5\right)\left(9.81\right)}{0.001067} = 45,979 \text{ N/m}$$

The natural frequency is $\omega = \sqrt{k/m} = 95.89$ rad/s.

At $\omega = 100$ Hz, $r = \dfrac{100\left(2\pi\right)}{95.89} = 6.552$, so the transmissibility ratio is

$$T.R. = \frac{1}{r^2 - 1} = 0.0238$$

At $\omega = 25$ Hz, $r = \dfrac{100\left(2\pi\right)}{95.89} = 1.638$, so the transmissibility ratio is

$$T.R. = \frac{1}{r^2 - 1} = 0.594$$

5.23 Redesign the system of Problem 5.22 such that the smallest transmissibility ratio possible is obtained over the range 50 to 75 Hz.

Solution:

If the deflection is limited, say 0.1 in, then the smallest transmissibility ratio in the frequency range of 50 to 75 Hz (3000 to 4500 rpm) would be 0.04 (96% reduction). The stiffness would be

$$k = \frac{mg}{\delta_s} = \frac{(5)(9.81)}{(0.1)(2.54)(0.01)} = 19{,}311 \text{ N/m}$$

5.24 A 2-kg printed circuit board for a computer is to be isolated from external vibration of frequency 3 rad/s at a maximum amplitude of 1 mm, as illustrated in Figure P5.24. Design an undamped isolator such that the transmitted displacement is 10% of the base motion. Also calculate the range of transmitted force.
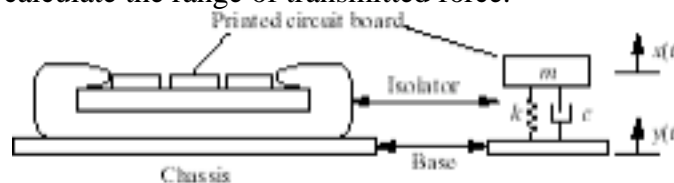
Solution:

Using Figure 5.9, the lines of 90% reduction and $\omega = 3(2\pi)(60)=1131$ rpm meet at $\delta_s = 0.3$ in $= 0.762$ cm. The stiffness is

$$k = \frac{mg}{\delta_s} = \frac{(2)(9.81)}{0.00762} = 2574.8 \text{ N/m}$$

From Window 5.1, the transmitted force would be

$$F_T = kYr^2 \left(\frac{1}{1-r^2} \right)$$

Since $Y = 0.001$ m and $r = \dfrac{3}{\sqrt{2574.8/2}} = 0.08361$

$$F_T = 0.0181 \text{ N}$$

5.25   Change the design of the isolator of Problem 5.24 by using a damping material with damping value ζ chosen such that the maximum T.R. at resonance is 2.

Solution:

At resonance, $r = 1$ and T.R. $= 2$, so

$$2 = \left[\frac{1 + 4\zeta^2}{4\zeta^2} \right]^{1/2}$$

Solving for ζ yields $\zeta = 0.2887$. Also T.R. $= 0.01$ at $\omega = 3$ rad/s, so

$$0.01 = \left[\frac{1 + 0.3333r^2}{\left(1 - r^2\right)^2 + 0.3333r^2} \right]$$

$$r = 6.134$$

Solving for k,

$$k = \frac{m\omega^2}{r^2} = \frac{(2)(3)^2}{6.134^2} = 0478 \text{ N/m}$$

The damping constant is

$$c = 2\zeta\sqrt{km} = 0.565 \text{ kg/s}$$

5.26   Calculate the damping ratio required to limit the displacement transmissibility to 4 at resonance for any damped isolation system.

Solution:

At resonance $r = 1$, so

$$T.R. = 4 = \left[\frac{1 + 4\zeta^2}{4\zeta^2} \right]^{1/2}$$

$$\zeta = 0.129$$

Problems and Solutions Section 5.3 (5.27 through 5.36)

5.27   A motor is mounted on a platform that is observed to vibrate excessively at an operating speed of 6000 rpm producing a 250-N force. Design a vibration absorber (undamped) to add to the platform. Note that in this case the absorber mass will only be allowed to move 2 mm because of geometric and size constraints.

   Solution:

   The amplitude of the absorber mass can be found from equation (5.22) and used to solve for k_a:

$$X_a = 0.002 \text{ m} = \frac{F_0}{k_a} = \frac{250}{k_a}$$

$$k_a = 125{,}000 \text{ N/m}$$

From equation (5.21),

$$\omega^2 = \frac{k_a}{m_a}$$

$$m_a = \frac{k_a}{\omega^2} = \frac{125{,}000}{\left[6000\left(\dfrac{2\pi}{60}\right)\right]^2} = 0.317 \text{ kg}$$

5.28 Consider an undamped vibration absorber with $\beta = 1$ and $\mu = 0.2$. Determine the operating range of frequencies for which $|Xk / F_0| \leq 0.5$.

Solution:

From equation (5.24), with $\beta = \dfrac{\omega_a}{\omega_p} = 1 (\text{i.e., } \omega_a = \omega_p)$ and $\mu = 0.2$,

$$\frac{Xk}{F_0} = \frac{1 - \left(\dfrac{\omega}{\omega_a}\right)}{\left[1 + 0.2(1)^2 - \left(\dfrac{\omega}{\omega_a}\right)^2\right]\left[1 - \left(\dfrac{\omega}{\omega_a}\right)^2\right] - 0.2(1)^2}$$

$$= \frac{1 - \left(\dfrac{\omega}{\omega_a}\right)^2}{\left(\dfrac{\omega}{\omega_a}\right)^4 - 2.2\left(\dfrac{\omega}{\omega_a}\right)^2 + 1}$$

For $\dfrac{Xk}{F_0} = 0.5$, this yields

$$0.5\left(\frac{\omega}{\omega_a}\right)^4 - 0.1\left(\frac{\omega}{\omega_a}\right)^2 - 0.5 = 0$$

Solving for the physical solution gives

$$\left(\frac{\omega}{\omega_a}\right) = 1.051$$

Solving for $\left(\dfrac{\omega}{\omega_a}\right)$ gives

$$\left(\frac{\omega}{\omega_a}\right) = 0.955, \ 1.813$$

Comparing this to the sketch in Figure 5.15, the values for which $\left| \dfrac{Xk}{F_0} \right| \leq 5$ are

$$0.955\omega_a \leq \omega \leq 1.051\omega_a \quad \text{and} \quad \omega \geq 1.813\omega_a$$

5.29 Consider an internal combustion engine that is modeled as a lumped inertia attached to ground through a spring. Assuming that the system has a measured resonance of 100 rad/s, design an absorber so that the amplitude is 0.01 m for a (measured) force input of 10^2 N.

Solution:

The amplitude of the absorber mass can be found from equation (5.22) and used to solve for k_a:

$$X_a = 0.01\text{m} = \frac{F_0}{k_a} = \frac{100}{k_a}$$

$$k_a = 10,000 \text{ N/m}$$

Choose $\omega = 2\omega_n = 200$ rad/s. From equation (5.21),

$$m_a = \frac{k_a}{\omega^2} = \frac{10,000}{200^2} = 0.25 \text{ kg}$$

5.30 A small rotating machine weighing 50 lb runs at a constant speed of 6000 rpm. The machine was installed in a building and it was discovered that the system was operating at resonance. Design a retrofit undamped absorber such that the nearest resonance is at least 20% away from the driving frequency.

Solution:

By observing Figure 5.15, the values of $\mu = 0.25$ and $\beta = 1$ result in the combined system's natural frequencies being 28.1% above the driving frequency and 21.8% below the driving frequency (since $\beta = \dfrac{\omega_a}{\omega_p} = 1$ and $\omega = \omega_p$). So the absorber should weigh

$$m_a = \mu m = \left(0.25\right)\left(50 \text{ lb}\right) = 12.5 \text{ lb}$$

and have stiffness

$$k_a = m_a \omega_a^2 = m_a \omega^2 = \left(12.5 \text{ lb}\right)\left(4.448222 \text{ N/lb}\right)\left(\frac{1}{9.81}\right)\left(6000\right)^2 \left(\frac{2\pi}{60}\right)^2$$

$$k_a = 2.24 \times 10^6 \text{ N/m} = 12,800 \text{ lb/in}$$

5.31   A 3000-kg machine tool exhibits a large resonance at 120 Hz. The plant manager attaches an absorber to the machine of 600 kg tuned to 120 Hz. Calculate the range of frequencies at which the amplitude of the machine vibration is less with the absorber fitted than without the absorber.

Solution:

For $\dfrac{Xk}{F_0} = 1$, equation (5.24) yields

$$\left[1 + \mu\left(\frac{\omega_a}{\omega_p}\right)^2 - \left(\frac{\omega}{\omega_a}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right] - \mu\left(\frac{\omega_a}{\omega_p}\right)^2 = 1 - \left(\frac{\omega}{\omega_a}\right)^2$$

Since $\mu = \dfrac{m_a}{m} = \dfrac{600}{3000} = 0.2,$ this becomes $\left(\dfrac{\omega}{\omega_a}\right) = 0,\ 1.0954.$

For $\dfrac{Xk}{F_0} = -1$, equation (5.24) yields

$$\left[1 + \mu\left(\frac{\omega_a}{\omega_p}\right)^2 - \left(\frac{\omega}{\omega_p}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right] - \mu\left(\frac{\omega_a}{\omega_p}\right)^2 = \left(\frac{\omega}{\omega_a}\right)^2 - 1$$

$$\left(\frac{\omega_a}{\omega_p}\right)^2\left(\frac{\omega}{\omega_a}\right)^4 - \left[2 + (\mu + 1)\left(\frac{\omega_a}{\omega_p}\right)^2\right]\left(\frac{\omega}{\omega_a}\right)^2 + 2 = 0$$

Since $\omega_a = \omega_p,$

$$\left(\frac{\omega}{\omega_a}\right)^4 - 3.2\left(\frac{\omega}{\omega_a}\right)^2 + 2 = 0$$

$$\left(\frac{\omega}{\omega_a}\right) = 0.9229, 1.5324$$

The range of frequencies at which $\left|\dfrac{Xk}{F_0}\right| > 1$ is

$$0 < \omega < 0.9229\omega_a \text{ and } 1.0954\omega_a < \omega < 1.5324\omega_a$$

Since $\omega_a = \omega_p$,

$$0 < \omega < 695.8 \text{ rad/s and } 825.9 < \omega < 1155.4 \text{ rad/s}$$

5.32 A motor-generator set is designed with steady-state operating speed between 2000 and 4000 rpm. Unfortunately, due to an imbalance in the machine, a large violent vibration occurs at around 3000 rpm. An initial absorber design is implemented with a mass of 2 kg tuned to 3000 rpm. This, however, causes the combined system natural frequencies that occur at 2500 and 3000 rpm. Redesign the absorber so that $\omega_1 < 2000$ rpm and $\omega_2 > 4000$ rpm, rendering the system safe for operation.

Solution: The mass of the primary system can be computed from equation (5.25). Since

$$\beta = \frac{\omega_a}{\omega_p} = 1 \text{ and } \left(\frac{\omega_1}{\omega_a}\right)^2 = \left(\frac{2500}{3000}\right)^2 = 0.6944, \text{ then}$$

$$\left(1\right)^2 \left(0.6944\right)^2 - \left[1 + \left(1\right)^2 \left(1 + \mu\right)\right]\left(0.6944\right) + 1 = 0$$

$$\mu = 0.1344$$

$$m = \frac{m_a}{\mu} = \frac{2}{0.1344} = 14.876 \text{ kg}$$

By increasing μ to 0.55 and decreasing β to 0.89, the design goal can be achieved. The mass and stiffness of the absorber should be

$$m_a = \mu m = \left(0.55\right)\left(14.876\right) = 8.18 \text{ kg}$$

$$k_a = m_a \omega_a^2 = m_a \beta^2 \omega_p^2 = \left(8.18\right)\left(0.89\right)^2 \left[3000\left(\frac{2\pi}{60}\right)\right]^2 = 639,600 \text{ N/m}$$

5.33 A rotating machine is mounted on the floor of a building. Together, the mass of the machines and the floor is 2000 lb. The machine operates in steady state at 600 rpm and causes the floor of the building to shake. The floor-machine system can be modeled as a spring-mass system similar to the optical table of Figure 5.14. Design an undamped absorber system to correct this problem. Make sure you consider the bandwidth.

Solution: To minimize the transmitted force, let $\omega_a = \omega = 600$ rpm. Also, since the floor shakes at 600 rpm, it is assumed that $\omega_p = 600$ rpm so that $\beta = 1$. Using equation (5.26) with $\mu = 0.1$ yields

$$\frac{\omega_n}{\omega_a} = 0.8543, \ 1.1705$$

So the natural frequencies of the combined system are $\omega_1 = 512.6$ rpm and $\omega_2 = 702.3$ rpm. These are sufficiently enough away from 600 rpm to avoid problems. Therefore the mass and stiffness of the absorber are

$$m_a = \mu m = (0.1)(2000 \text{ lbm}) = 200 \text{ lbm}$$

$$k_a = m_a \omega_a^2 = (200 \text{ lbm})\left(\frac{\text{slug}}{32.1174 \text{ lbm}}\right)\left[600\left(\frac{2\pi}{60}\right)\right]^2 = 25{,}541 \text{ lb/ft}$$

5.34   A pipe carrying steam through a section of a factory vibrates violently when the driving pump hits a speed of 300 rpm (see Figure P5.34). In an attempt to design an absorber, a trial 9-kg absorber tuned to 300 rpm was attached. By changing the pump speed it was found that the pipe-absorber system has a resonance at 207 rpm. Redesign the absorber so that the natural frequencies are 40% away from the driving frequency.
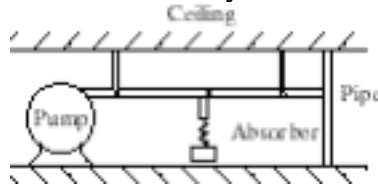
Solution:

The driving frequency is 300 rpm. 40% above and below this frequency is 180 rpm and 420 rpm. This is the design goal.

The mass of the primary system can be computed from equation (5.25).   Since

$\beta = \dfrac{\omega_a}{\omega_p} = 1$ and $\left(\dfrac{\omega_1}{\omega_a}\right)^2 = \left(\dfrac{207}{300}\right)^2 = 0.4761$, then

$$\left(1\right)^2\left(0.4761\right)^2 - \left[1 + \left(1\right)^2\left(1 + \mu\right)\right]\left(0.4761\right) + 1 = 0$$

$$\mu = 0.5765$$

$$m = \dfrac{m_a}{\mu} = \dfrac{9}{0.5765} = 15.611 \text{ kg}$$

By increasing μ to 0.9 and decreasing β to 0.85, the design goal can be achieved. The mass and stiffness of the absorber should be

$$m_a = \mu m = \left(0.9\right)\left(15.611\right) = 14.05 \text{ kg}$$

$$k_a = m_a\omega_a^2 = m_a\beta^2\omega_p^2 = \left(14.05\right)\left(0.85\right)^2\left[300\left(\dfrac{2\pi}{60}\right)\right]^2 = 10,020 \text{ N/m}$$

Note that μ is very large, which means a poor design.

5.35   A machine sorts bolts according to their size by moving a screen back and forth using a primary system of 2500 kg with a natural frequency of 400 cycle/min. Design a vibration absorber so that the machine-absorber system has natural frequencies below 160 cycles/min and above 320 rpm. The machine is illustrated in Figure P5.35.
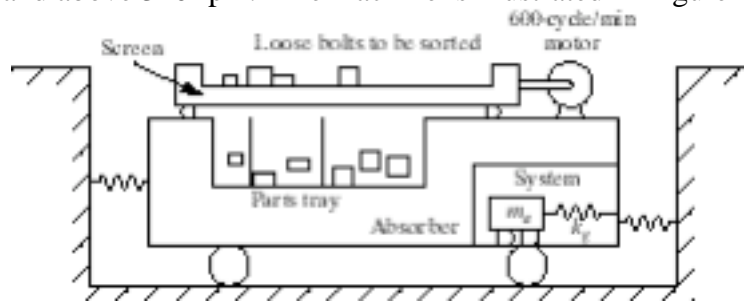
Solution:

Using Equation (5.26), and choose (by trial and error) $\beta = 0.4$ and $\mu = 0.01$, the design goal of $\omega_1 < 160$ rpm and $\omega_2 > 320$ rpm can be achieved. The actual values are $\omega_1 = 159.8$ rpm and $\omega_2 = 400.4$ rpm. The mass and stiffness of the absorber should be

$$m_a = \mu m = (0.01)(2500) = 25 \text{ kg}$$

$$k_a = m_a \omega_a^2 = m_a \beta^2 \omega_[^2 = (25)(0.2)^2 \left[400 \left(\frac{2\pi}{60} \right) \right]^2 = 1754.6 \text{ N/m}$$

5.36   A dynamic absorber is designed with $\mu = 1/4$ and $\omega_a = \omega_p$. Calculate the frequency range for which the ratio $\left| Xk \, / \, F_0 \right| < 1$.

Solution:

From Equation (5.24), with $\beta = \dfrac{\omega_a}{\omega_p} = 1$ and $\mu = 0.25$,

$$\frac{Xk}{F_0} = \frac{1 - \left(\dfrac{\omega}{\omega_a}\right)^2}{\left[1 + 0.25\left(1^2\right) - \left(\dfrac{\omega}{\omega_a}\right)^2\right]\left[1 - \left(\dfrac{\omega}{\omega_a}\right)^2\right] - 0.25\left(1\right)^2}$$

$$= \frac{1 - \left(\dfrac{\omega}{\omega_a}\right)^2}{\left(\dfrac{\omega}{\omega_a}\right)^4 - 2.25\left(\dfrac{\omega}{\omega_a}\right)^2 + 1}$$

For $\dfrac{Xk}{F_0} = 1$, this yields

$$\left(\frac{\omega}{\omega_a}\right)^4 - 1.25\left(\frac{\omega}{\omega_a}\right)^2 = 0$$

$$\left(\frac{\omega}{\omega_a}\right) = 0, \; 1.118$$

For $\dfrac{Xk}{F_0} = 1$, this yields

$$-\left(\frac{\omega}{\omega_a}\right)^4 + 3.25\left(\frac{\omega}{\omega_a}\right)^2 - 2 = 0$$

$$\left(\frac{\omega}{\omega_a} = 0.9081, \; 1.557\right)$$

Comparing this to the sketch in Figure 5.15, the values for which $\left|\dfrac{Xk}{F_0}\right| < 1$ are

$$0.9081\omega_a < \omega < 1.118\omega_a \text{ and } \omega > 1.557\omega_a$$

Problems and Solutions Section 5.4 (5.37 through 5.52)

5.37   A machine, largely made of aluminum, is modeled as a simple mass (of 100 kg) attached to ground through a spring of 2000 N/m. The machine is subjected to a 100-N harmonic force at 20 rad/s. Design an undamped tuned absorber system (i.e., calculate m_a and k_a) so that the machine is stationary at steady state. Aluminum, of course, is not completely undamped and has internal damping that gives rise to a damping ratio of about $\zeta = 0.001$. Similarly, the steel spring for the absorber gives rise to internal damping of about $\zeta_a = 0.0015$.   Calculate how much this spoils the absorber design by determining the magnitude X using equation (5.32).

Solution:

From equation (5.21), the steady-state vibration will be zero when

$$\omega^2 = \frac{k_a}{m_a}$$

Choosing $\mu = 0.2$ yields

$$m_a = \mu m = (0.2)(100) = 20 \text{ kg}$$
$$k_a = m_a \omega_a^2 = (20)(20)^2 = 8000 \text{ N/m}$$

With damping of $\zeta = 0.001$ and $\zeta_a = 0.0015$, the values of c and c_a are

$$c = 2\zeta\sqrt{km} = 2(0.001)\sqrt{(2000)(100)} = 0.894 \text{ kg/s}$$
$$c_a = 2\zeta_a\sqrt{k_a m_a} = 2(0.0015)\sqrt{(8000)(20)} = 1.2 \text{ kg/s}$$

From equation (5.32),

$$X = \frac{\left(k_a - m_a \omega^2\right)F_0 + c_a \omega F_0 j}{\det\left(K - \omega^2 M + \omega jC\right)}$$

Since

$$M = \begin{bmatrix} 100 & 0 \\ 0 & 20 \end{bmatrix} \quad C = \begin{bmatrix} 2.0944 & -1.2 \\ -1.2 & 1.2 \end{bmatrix} \quad K = \begin{bmatrix} 10,000 & -8000 \\ -8000 & 8000 \end{bmatrix}$$

the denominator is $-6.4 \times 10^7 - 1.104 \times 10^6 j$, so the value of X is

$$X = \frac{\left(k_a m_a \omega^2\right)\left(F_0 + c_a \omega F_0 j\right)}{\det\left(K - \omega^2 M + \omega jC\right)}$$

Using Window 5.4, the magnitude is

$$|X| = 3.75 \times 10^{-5} \text{ m}$$

This is a very small displacement, so the addition of internal damping will not affect the design very much.

5.38 Plot the magnitude of the primary system calculated in Problem 5.37 with and without the internal damping. Discuss how the damping affects the bandwidth and performance of the absorber designed without knowledge of internal damping.

Solution: From Problem 5.37, the values are

$$m = 100 \text{ kg} \qquad m_a = 20 \text{ kg}$$
$$c = 0.8944 \text{ kg/s} \quad c_a = 1.2 \text{ kg/s}$$
$$k = 2000 \text{ N/m} \qquad k_a = 8000 \text{ N/m}$$
$$F_0 = 100 \text{ N} \qquad \omega = 20 \text{ rad/s}$$

Using Equation (5.32), the magnitude of X is plotted versus ω with and without the internal damping (c). Note that X is reduced when $X < F_0/k = 0.05$ m and magnified when $X > 0.05$ m. The plots of the two values of X show that there is no observable difference when internal damping is added. In this case, knowledge of internal damping is not necessary.

$m := 100$ $ma := 20$ $ca := 1.2$ $k := 2000$
$ka := 8000$ $F0 := 100$

$$X(c, \omega) := \sqrt{\frac{\left(ka - ma \cdot \omega^2\right) \cdot F0^2 + \left[ca \cdot \left(\omega \cdot F0\right)\right]^2}{\left[\left(k - m \cdot \omega^2\right) \cdot \left(ka - ma \cdot \omega^2\right) - (ma \cdot ka + ca \cdot c) \cdot \omega^2\right]^2 + \left[ka \cdot c + k\ ca - \left[ca \cdot (m + ma) + c \cdot ma\right] \cdot \omega^2\right]^2 \cdot \omega^2}}$$

$\dfrac{X(0, \omega)}{X(.8944, \omega)}$
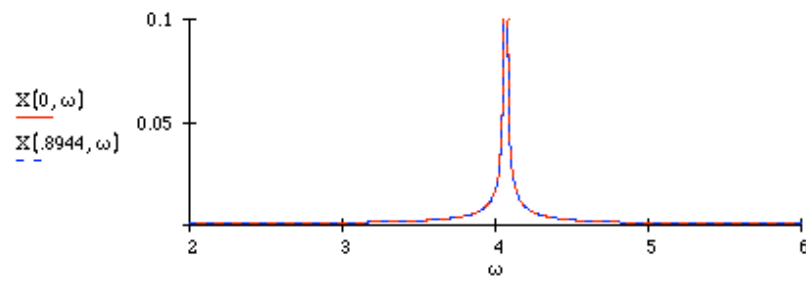
5.39 Derive Equation (5.35) for the damped absorber from Eqs. (5.34) and (5.32) along with Window 5.4. Also derive the nondimensional form of Equation (5.37) from Equation (5.35). Note the definition of ζ given in Equation (5.36) is not the same as the ζ values used in Problems 5.37 and 5.38.

Solution:

Substituting Equation (5.34) into the denominator of Equation (5.32) yields

$$\frac{X}{F_0} = \frac{\left(k_a - m_a\omega^2\right) + c_a\omega j}{\left[\left(-m\omega^2 + k\right)\left(-m_a\omega^2 + k_a\right)\right] + \left[\left(k - \left(m + m_a\right)\omega^2\right)c_a\omega\right]j}$$

Referring to Window 5.4, the value of $\left|\dfrac{X}{F_0}\right|$ can be found by noting that

$$A_1 = k_a - m_a\omega$$
$$B_1 = c_a\omega$$
$$A_2 = \left(-m\omega^2 + k\right)\left(-m_a\omega^2 + k_a\right) - m_a k_a \omega^2$$
$$B_2 = \left(k - \left(m + m_a\right)\omega^2\right)c_a\omega$$

Since

$$\left|\frac{X}{F_0}\right| = \sqrt{\frac{A_1^2 + B_1^2}{A_2^2 + B_2^2}}$$

then

$$\frac{X^2}{F_0^2} = \frac{\left(k_a - m_a\omega^2\right)^2 + c_a^2\omega^2}{\left[\left(-m\omega^2 + k\right)\left(-m_a\omega^2 + k_a\right) - m_a k_a \omega^2\right]^2 + \left[k - \left(m + m_a\right)\omega^2\right]^2 c_a^2 \omega 2}$$

which is Equation (5.35)

To derive Equation (5.37), substitute $c_a = 2\zeta m_a\omega_p, k_a = m_a\omega_a^2$, and $m_a = \mu m$, then multiply by k^2 to get

$$\frac{X^2 k^2}{F_0^2} = \frac{k^2 \left(\omega_a^2 - \omega^2\right)^2 + 4\zeta^2 \omega_p^2 \omega_{dr}^2 k^2}{\left[\left(k - m\omega^2\right)\left(\omega_a^2 - \omega^2\right) - \mu m_a^2 \omega^2\right]^2 + \left[k - \left(1 - \mu\right)m\omega^2\right]^2 \left(4\right)\zeta^2 \omega_p^2 \omega^2}$$

Substituting $k = m\omega_p^2, \omega = r\omega_p$, and $\omega_a = \beta\omega_p$ yields

$$\frac{X^2 k^2}{F_0^2} = \frac{m^2 \omega_p^4 \left(\beta^2 \omega_p^2 - r^2 \omega_p^2\right) + 4\zeta\ \omega_p^2 \omega_{dr}^2 k^2}{\left[\left(\omega_p^2 - r^2 \omega_p^2\right)\left(\beta^2 \omega_p^2 - r^2 \omega_p^2\right)m - \mu m\beta^2 r^2 \omega_p^4\right]^2 + \left[m\omega_p^2 - \left(1 - \mu\right)mr^2 \omega_p^2\right]^2 \left(4\right)\zeta^2 r^2 \omega_p^2}$$

Canceling m^2 and ω_p^8 yields

$$\frac{X^2 k^2}{F_0^2} = \frac{\left(\beta^2 - r^2\right)^2 + \left(2\zeta r\right)^2}{\left[\left(1 - r^2\right)\left(\beta^2 - r^2\right) - \mu r^2 \beta^2\right]^2 + \left(2\zeta r\right)^2 \left(1 - r^2 - \mu r^2\right)^2}$$

Rearranging and taking the square root gives the form of Equation (5.37):

$$\frac{Xk}{F_0} = \sqrt{\frac{\left(2\zeta r\right)^2 + \left(r^2 - \beta^2\right)^2}{\left(2\zeta r\right)^2 \left(r^2 - 1 + \mu r^2\right)^2 + \left[\mu r^2 \beta^2 - \left(r^2 - 1\right)\left(r^2 - \beta^2\right)\right]^2}}$$

5.40   (Project) If you have a three-dimensional graphics routine available, plot Equation (5.37) [i.e., plot (X/Δ) versus both r and ζ for $0 < \zeta < 1$ and $0 < r < 3$, and a fixed μ and β.] Discuss the nature of your results. Does this plot indicate any obvious design choices? How does it compare to the information obtained by the series of plots given in Figures 5.19 to 5.21? (Three-dimensional plots such as these are becoming commonplace and have not yet been taken advantage of fully in vibration absorber design.)

Solution: To compare to Figure 5.18, the values $\mu = 0.25$ and $\beta = 0.8$ in Equation (5.37) yield
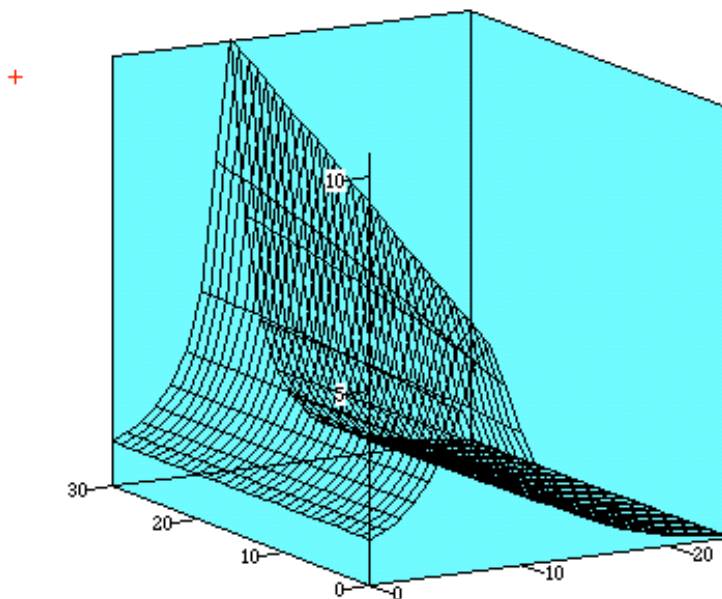
$$\frac{X}{\Delta} = \sqrt{\frac{\left(2\zeta r\right)^2 + \left(r^2 - 0.64\right)^2}{\left(2\zeta r\right)^2 \left(1.25 r^2 - 1\right)^2 + \left[0.16 r^2 - \left(r^2 - 1\right)\left(r^2 - 0.64\right)\right]^2}}$$

This is plotted for $0.5 < r < 2$ and $0.5 < \zeta < 1$. A Mathcad plot is given.

$N := 30 \quad i := 0 \ .. \ N \cdot .8 \quad j := 0 \ .. \ N \cdot 1 \quad r_i := 0.5 + i \cdot 0.05 \qquad \zeta_j := 0.5 + j \cdot 0.015$

$$X(r, \zeta) := \sqrt{\frac{\left(2 \cdot \zeta \cdot r\right)^2 + \left(r^2 - 0.64\right)^2}{\left(2 \cdot r \cdot \zeta\right)^2 \cdot \left(1.25 \cdot r^2 - 1^2\right)^2 + \left[0.16 \cdot r^2 - \left(r^2 - 1\right) \cdot \left(r^2 - 0.64\right)\right]^2}}$$

$M_{(i, j)} := X\left(r_i, \zeta_j\right)$

M

This supplies much more information than two-dimensional plots.

5.41 Repeat Problem 5.40 by plotting $|X / \Delta|$ versus r and β for a fixed ζ and μ.

Solution: Using Equation (5.37) with $\mu = 0.25$ and ζ 0.1 yields

$$\frac{X}{\Delta} = \sqrt{\frac{0.04r^2 + \left(r^2 - \beta^2\right)^2}{0.04r^2\left(1.25r^2 - 1\right)^2 + \left[0.25r^2\beta^2 - \left(r^2 - 1\right)\left(r^2 - \beta^2\right)\right]^2}}$$

This is plotted for $0.5 < r < 1.25$ and $0 < \beta < 3$.

$N := 30$ $i := 0 .. N \cdot .8$ $j := 0 .. N \cdot 1$ $r_i := 0.5 + i \cdot 0.05$ $\beta_j := 0 + j \cdot 0.1$

$$X(r, \beta) := \sqrt{\frac{0.04 \cdot r^2 + \left(r^2 - \beta^2\right)^2}{0.04 \cdot r^2 \cdot \left(1.25 \cdot r^2 - 1^2\right)^2 + \left[0.25 \cdot r^2 \cdot \beta^2 - \left(r^2 - 1\right) \cdot \left(r^2 - \beta^2\right)\right]^2}}$$

$+$

$M_{(i,j)} := X\left(r_i, \beta_j\right)$
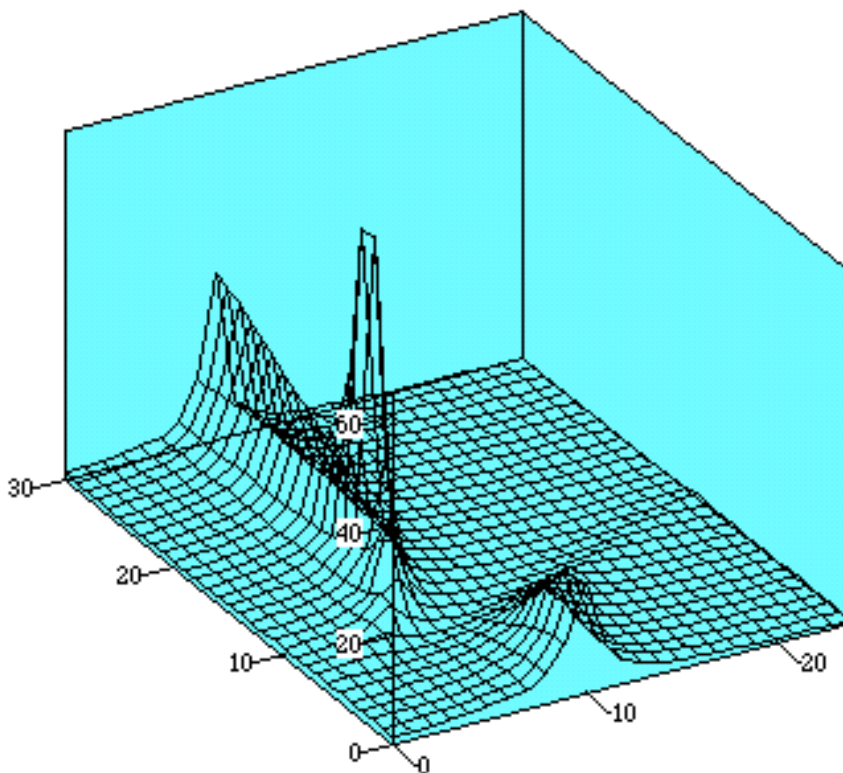
5.42 (Project) The full damped vibration absorber equations (5.32) and (5.33) have not historically been used in absorber design because of the complicated nature of the complex arithmetic involved. However, if you have a symbolic manipulation code available to you, calculate an expression for the magnitude X by using the code to calculate the magnitude and phase of Equation (5.32). Apply your results to the absorber design indicated in Problem 5.37 by using m_a, k_a and ζ_a as design variables (i.e., design the absorber).

Solution:

Equation (5.32):

$$X = \frac{\left(k_a - m_a\omega^2\right)F_0 + c_a\omega F_0}{\det\left(K - \omega^2 M + \omega jC\right)}$$

where M, C and K are defined above Equation (5.32).

Using Equation (5.34) for the denominator, then calculating the magnitude yields

$$|X| = \sqrt{\frac{\left(k_a - m_a\omega^2\right)F_0^2 + c_a^2\omega^2 F_0^2}{\left[\left(k - m\omega^2\right)\left(k_a - m_a\omega^2\right) - \left(m_a k_a + c_a c\right)\omega^2\right]^2 + \left[k_a c + kc_a - \left(c_a\left(m + m_a\right) + cm_a\right)\omega^2\right]^2 \omega^2}}$$

The phase is

$$\phi = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$

where the imaginary part, denoted Im, is
$$\mathrm{Im} = -ck_a^2 l + \left(2k_a m_a - 2k_a km_a - k_a^2 m_a\right)\omega^2$$
and the real part, denoted Re, is
$$\mathrm{Re} = k_a^2 k + \left(c_a^2 - k_a^2 m - 2k_a km_a - k_a^2 m_a\right)\omega^2$$
$$+ \left(\left(k + k_a\right)m_a^2 + 2k_a mm_a - c_a^2\left(m + m_a\right)\right)\omega^4 - mm_a^2\omega^6$$

From Problem 5.37 and its solution, the values are

$m = 100$ kg	$m_a = 20$ kg
$c = 0.8944$ kg/s	$c_a = 1.2$ kg/s
$k = 2000$ N/m	$k_a = 8000$ N/m
$F_0 = 100$ N	$\omega = 20$ rad/s

Substituting these values into the magnitude equation yields
$$|X| = 3.75\times10^{-5} \text{ m}$$

This is the same result as given in Problem 5.37.

5.43 A machine of mass 200 kg is driven harmonically by a 100-N force at 10 rad/s. The stiffness of the machine is 20,000 N/m. Design a broadband vibration absorber [i.e., Equation (5.37)] to limit the machine's motion as much as possible over the frequency range 8 to 12 rad/s. Note that other physical constraints limit the added absorber mass to be at most 50 kg.

Solution:

Since $\omega_p = \sqrt{\dfrac{k}{m}} = 10$ rad/s, then r ranges from

$$\frac{8}{10} \leq r \leq \frac{12}{10}$$

$$0.8 \leq r \leq 1.2$$

By observing Figure 5.21, the values of $\mu = 0.25$, $\beta = 0.8$, and $\zeta = 0.27$ yield a reasonable solution for the required range of r. So the values of m_a, c_a, and k_a are

$m_a = \mu m = (0.25)(200) = 50$ kg
$c_a = 2\zeta m_a \omega_a = 2(0.27)(50)(10) = 270$ kg/s
$k_a = m_a \omega_a \beta^2 \omega_p^2 = (50)(10)(0.8)^2(10)^2 = 32000$ N/m

Note that an extensive optimization could have been used to solve for μ, β, and ζ, but this is not covered until section 5.5.

5.44 Often absorber designs are afterthoughts such as indicated in example 5.3.1. Add a damper to the absorber design of Figure 5.17 to increase the useful bandwidth of operation of the absorber system in the event the driving frequency drifts beyond the range indicated in Example 5.3.2.

Solution:

From Examples 5.3.1 and 5.3.2,

$$m = 73.16 \text{ kg} \quad m_a = 18.29 \text{ kg}$$
$$k = 2600 \text{ N/m} \quad k_a = 6500 \text{ N/m}$$

$$7.4059 < \omega < 21.0821 \text{ rad/s}$$

The values μ and β are

$$\mu = \frac{m_a}{m} = 0.25$$

$$\beta = \frac{\omega_a}{\omega_p} = \frac{\sqrt{k_a / m_a}}{\sqrt{k / m}} = 3.1623$$

Choosing $\zeta = 0.2$ (by trial and error) will allow ω to go beyond 21.0821 rad/s without $\frac{X_k}{F_0}$ going above 1. However, it will not prevent $\frac{Xk}{F_0}$ from going above 1 when $\omega <$ 7.4089 rad/s. The value of c_a is

$$c_a = 2\zeta m_a \omega_p = 2(0.2)(18.29)\sqrt{\frac{2600}{73.16}} = 43.61 \text{ kg/s}$$

5.45 Again consider the absorber design of Example 5.3.1. If the absorber spring is made of aluminum and introduces a damping ratio of $\zeta = 0.001$, calculate the effect of this on the deflection of the saw (primary system) with the design given in Example 5.3.1.

Solution:

From Examples 5.3.1 and 5.3.2,

$$X = \frac{\left(k_a - m_a \omega^2\right)F_0 + c_a \omega F_0 j}{\det\left(K - \omega^2 M + \omega jC\right)}$$

where $c_a = 2\zeta\sqrt{k_a m_a} = 2(0.001)\sqrt{(6500)(18.29)} = 0.6896$ kg/s

Since

$$M = \begin{bmatrix} 73.16 & 0 \\ 0 & 18.29 \end{bmatrix} \quad C = \begin{bmatrix} 0.6896 & -0.6896 \\ -0.6896 & 0.6896 \end{bmatrix} \quad K = \begin{bmatrix} 9100 & -6500 \\ -6500 & 6500 \end{bmatrix}$$

The denominator is $-1.4131 \times 10^7 - 12,363j$ when $\omega = 7.4089$ rad/s,

$$\left|X_1\right| = 0.00499 \text{ m}$$

and when $\omega = 21.0821$ rad/s,

$$\left|X_2\right| = 0.00512 \text{ m}$$

The nondimensional values become

$$\left|\frac{X_j k}{F_0}\right| = 0.999$$

$$\left|\frac{X_2 k}{F_0}\right| = 1.023$$

There is very little effect on the saw deflection since the values of $\left|\dfrac{Xk}{F_0}\right|$ are still approximately 1 at the endpoints of the driving frequency range.

5.46   Consider the undamped primary system with a viscous absorber as modeled in Figure 5.22 and the rotational counterpart of Figure 5.23. Calculate the magnification factor $\left| Xk / M_o \right|$ for a 400 kg compressor having a natural frequency of 16.2 Hz if driven at resonance, for an absorber system defined by $\mu = 0.133$ and $\zeta = 0.025$.

Solution:

From Eqs. (5.39), with $\mu = 0.133$, $\zeta = 0.025$, and $r = 1$:

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2\left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2}} = 150.6$$

The design with $\zeta = 0.1$ produces the smallest displacement.

5.47   Recalculate the magnification factor $\left| Xk / M_o \right|$ for the compressor of Problem 5.46 if the damping factor is changed to $\zeta = 0.1$. Which absorber design produces the smallest displacement of the primary system $\zeta = 0.025$ or $\zeta = 0.1$?
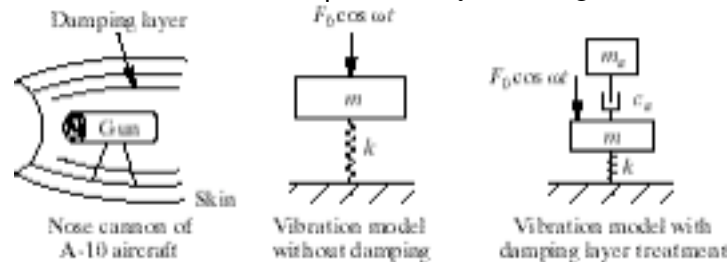
Solution:

From Equation (5.39), with $\mu = 0.133$, $\zeta = 0.1$, and $r = 1$:

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2\left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2}} = 38.34$$

The design with $\zeta = 0.1$ produces the smallest displacement.

5.48 Consider a one-degree-of-freedom model of the nose of an aircraft (A-10) as illustrated in Figure P5.48. The nose cracked under fatigue during battle conditions. This problem has been fixed by adding a viscoelastic material to the inside of the skin to act as a damped vibration absorber as illustrated in Figure P5.48. This fixed the problem and the vibration fatigue cracking disappeared in the A-10's after they were retrofitted with viscoelastic damping treatments. While the actual values remain classified, use the following data to calculate the required damping ratio given $M = 100$ kg, $f_a = 3$ Hz, and $k = 3.533 \times 10^6$ N/m, such that the maximum response is less than 0.25 mm. Note that since mass always needs to be limited in an aircraft, use $\mu = 0.1$ in your design.

Dumping layer

$F_0 \cos \omega t$

m_a

$F_0 \cos \omega t$

c_a

Gun

m

m

k

k

Skin

Nose cannon of
A-10 aircraft

Vibration model
without damping

Vibration model with
damping layer treatment

Solution:

From Equation (5.39), with $\mu = 0.1$, and $r = \dfrac{30(2\pi)}{\sqrt{k/m}} = 1.885$, and M_0 replaced by F_0,

$$\frac{Xk}{F_0} = \sqrt{\frac{4\zeta^2 + (1.885)^2}{4\zeta^2\left[(1.1)(1.885)^2 - 1\right]^2\left[(1.885)^2 - 1\right]^2 (1.885)^2}}$$

$$= \sqrt{\frac{4\zeta^2 + 3.553}{33.834\zeta + 23.159}}$$

With no damping $\dfrac{Xk}{F_0} = 0.392$. This value must be reduced. Choose a "high" damping ratio of $\zeta = 0.7$ so that

$$\frac{Xk}{F_0} = 0.372$$

The value of c_a is

$$c_a = 2\zeta\mu m\sqrt{\frac{k}{m}} = 2(0.7)(0.1)(100)\sqrt{\frac{10^6}{100}} = 1400 \text{ kg/s}$$

5.49 Plot an amplification curve such as Figure 5.24 by using Equation (5.39) for $\zeta = 0.02$ after several values of μ ($\mu = 0.1, 0.25, 0.5$, and 1). Can you form any conclusions about the effect of the mass ratio on the response of the primary system? Note that as μ gets large $|Xk / M_o|$ gets very small. What is wrong with using very large μ in absorber design?
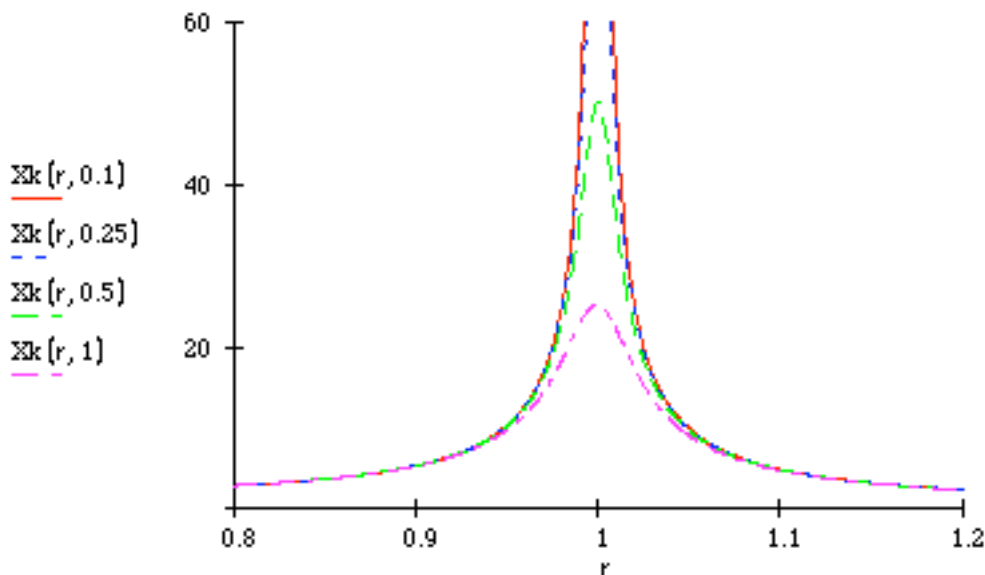
Solution:

From Equation (5.39), with $\zeta = 0.1$:

$$\frac{Xk}{M_0} = \sqrt{\frac{0.0016 + r^2}{0.0016\left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2 r^2}}$$

The following plot shows amplitude curves for $\mu = 0.1, 0.25, 0.5$, and 1.

$$Xk(r,\mu) := \sqrt{\frac{0.0016 + r^2}{0.0016 \cdot \left(r^2 + \mu \cdot r^2 - 1\right)^2 + \left(r^2 - 1\right)^2 \cdot r^2}}$$

Note that as the mass ratio, μ, increases, the response of the primary system decreases, particularly in the region near resonance. A higher mass ratio, however, indicates a poor design (and can be quite expensive).

5.50 A Houdaille damper is to be designed for an automobile engine. Choose a value for ζ and μ if the magnification $\left| Xk / M_o \right|$ is to be limited to 4 at resonance. (One solution is $\mu = 1$, $\zeta = 0.129$.)

Solution:

From Equation (5.39), with $r = 1$:

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + 1}{4\zeta^2 \mu^2}}$$

For $\dfrac{Xk}{M_0} = 4$,

$$64\zeta^2 \mu^2 = 4\zeta^2 + 1$$

If μ is limited to 0.3, then the value of ζ is

$$64\zeta^2 (0.3)^2 = 4\zeta^2 + 1$$

$$\zeta = 0.754$$

5.51 Determine the amplitude of vibration for the various dampers of Problem 5.46 if $\zeta = 0.1$, and $F_0 = 100$ N.

Solution:

From Problem 5.46,

$$k = m\omega_n^2 = (400)\left[(16.2)(2\pi)\right]^2 = 4.144 \times 10^6 \, \text{N/m}$$

Also, $\mu = 0.1$, $r = 1$, and $F_0 = 100$ N. So, from Equation (5.39), with M_0 replaced by F_0,

$$X = \frac{F_0}{k} \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2 \left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2 r^2}} = 0.00123 \, \text{m}$$

5.52   (Project) Use your knowledge of absorbers and isolation to design a device that will protect a mass from both shock inputs and harmonic inputs. It may help to have a particular device in mind such as the module discussed in Figure 5.6.

Solution:

One way to approach this problem would be to design an isolator to protect the mass from shock inputs, and an absorber to protect the mass from harmonic disturbances. An absorber would be particularly useful if the frequency of the harmonic disturbance(s) is well known.

This is a very general approach to such a problem, and solutions will vary greatly depending on the particular parameters involved in an actual system.

Problems and Solutions Section 5.5 (5.53 through 5.66)

5.53 Design a Houdaille damper for an engine modeled as having an inertia of 1.5 kg·m² and a natural frequency of 33 Hz.   Choose a design such that the maximum dynamic magnification is less than 6:

$$\left|\frac{Xk}{M_0}\right| < 6$$

The design consists of choosing J_2 and c_a, the required optimal damping.

Solution:

From Equation (5.50),

$$\left(\frac{Xk}{M_0}\right)_{max} = 1 + \frac{2}{\mu}$$

Since $\left|\dfrac{Xk}{M_0}\right| < 6$, then

$$6 > 1 + \frac{2}{\mu}$$

$$\mu > 0.4$$

Choose $\mu = 0.4$. From Equation (5.49), the optimal damping is

$$\zeta_{op} = \frac{1}{\sqrt{2(\mu + 1)(\mu + 2)}} = 0.3858$$

The values of J_2 and c_a are

$$J_2 = \mu J_1 = (0.4)(1.5 \text{ kg} \cdot \text{m}^2 \text{ / rad}) = 0.6 \text{ kg} \cdot \text{m}^2 \text{ / } rad$$

$$c_a = 2\zeta_{op} J_2 \omega_p = 2(0.3858)(0.6)(33)(2\pi) = 95.98 \text{ N} \cdot \text{m} \cdot \text{s/rad}$$

5.54   Consider the damped vibration absorber of equation (5.37) with β fixed at β = 1/2 and μ fixed at μ = 0.25. Calculate the value of ζ that minimizes $|X / \Delta|$. Plot this function for several values of $0 < \zeta < 1$ to check your design. If you cannot solve this analytically, consider using a three-dimensional plot of $|X / \Delta|$ versus r and ζ to determine your design.

Solution:

From equation (5.37), with β = 0.5 and μ = 0.25, let

$$f\left(r,\zeta\right) = \frac{X}{\Delta}\sqrt{\frac{4\zeta^2 r^2 + \left(r^2 - 0.25\right)^2}{4\zeta^2 r^2\left(1.25r^2 - 1\right)^2 + \left[0.065r^2 - \left(r^2 - 1\right)\left(r^2 - 0.25\right)\right]^2}}$$

From equataions (5.44) and (5.45), with $f = \dfrac{A^{1/2}}{B^{1/2}}$,

$$\frac{\partial f}{\partial \zeta} = 0$$

becomes

$$BdA - AdB$$

Since $B = 4\zeta^2 r^2\left(1.25r - 1\right)^2 + \left[0.0625r^2 - \left(r^2 - 1\right)\left(r^2 - 0.25\right)\right]^2$ and

$A = 4\zeta^2 r^2 + \left(r^2 - 0.25\right)^2$, then

$$dA = \frac{\partial A}{\partial \zeta} = 8\zeta r^2$$

$$dB = \frac{\partial B}{\partial \zeta} = 8\zeta r^2\left(1.25r^2 - 1\right)^2$$

So,

$$\left\{4\zeta^2 r^2\left(1.25r^2 - 1\right)^2 + \left[0.0625r^2 - \left(4^2 - 1\right)\left(r^2 - 0.25\right)\right]^2\right\}\left(8\zeta r^2\right)$$

$$= \left\{4\zeta^2 r^2 + \left(r^2 - 0.25\right)^2\right\}\left(8\zeta r^2\right)\left(1.25r^2 - 1\right)^2$$

$$\left[0.0625r^2 - \left(r^2 - 1\right)\left(r^2 - 0.25\right)^2\right] = \left(r^2 - 0.25\right)^2\left(1.25r^2 - 1\right)^2$$

Taking the square root yields

$$0.625r^2 - \left(r^2 - 1\right)\left(r^2 - 0.25\right) = \pm\left(r^2 - 0.25\right)\left(1.25r^2 - 1\right)$$

Solving for r yields

$$r = 0.4896, \ 0.9628$$

Now take the derivative

$$\frac{\partial f}{\partial r} = 0$$

becomes

$$BdA = AdB$$

Since $B = 4\zeta^2 r^2 \left(1.25r^2 - 1\right)^2 + \left[0.0625r^2 - \left(r^2 - 1\right)\left(r^2 - 0.25\right)\right]^2$ and

$A = 4\zeta^2 r^2 + \left(r^2 - 0.25\right)^2$, then

$$dA \equiv \frac{\partial A}{\partial \zeta} = 8\zeta^2 r + 2\left(r^2 - 0.25\right)\left(2r\right)$$

$$dB \equiv \frac{\partial B}{\partial \zeta} = 8\zeta^2 r \left(1.25r^2 - 1\right)^2 + 8\zeta^2 r^2 \left(1.25r^2 - \left(2r\right)\right)\left(2.5r\right)$$

$$+2\left[0.0625r^2 - \left(4^2 - 1\right)\left(r^2 - 0.25\right)\right]\left[0.125r - \left(2r\right)\left(r^2 - 0.25\right) - \left(r^2 - 1\right)\left(2r\right)\right]$$

Solving $B \ dA = A \ dB$ for ζ yields

$$r = 0.4896 \rightarrow \zeta = 0.1145 \rightarrow \frac{X}{\delta_{st}} = 1.4279$$

$$r = 0.9628 \rightarrow \zeta = 0.3197 \rightarrow \frac{X}{\delta_{st}} = 6.3029$$

To determine the optimal damping ratio, make a plot of $|X \ / \ \Delta|$ versus r for $\zeta = 0.01$, 0.1145, 0.3197, and 0.7.

$$f(r,\zeta) := \sqrt{\frac{(2\cdot\zeta\cdot r)^2 + (r^2 - 0.25)^2}{(2\cdot r\cdot\zeta)^2 \cdot (1.25\cdot r^2 - 1^2)^2 + \left[0.0625\cdot r^2 - (r^2 - 1)\cdot(r^2 - 0.25)\right]^2}}$$
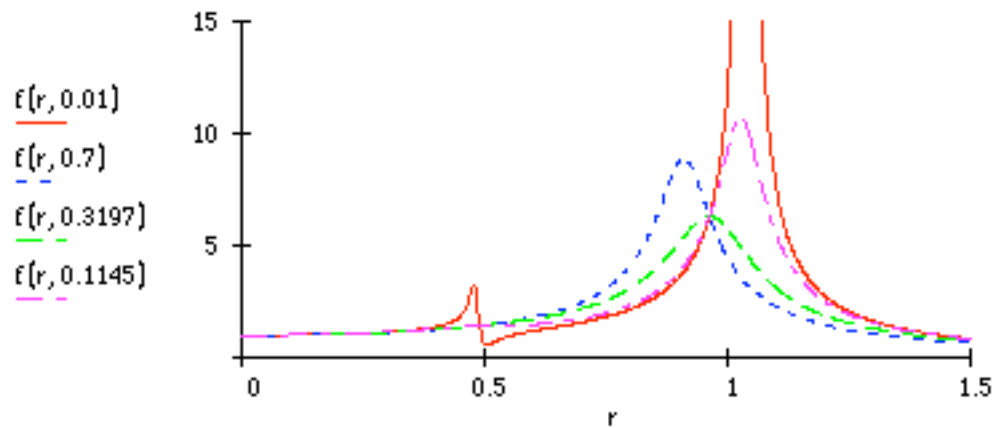
The value of $\zeta = 0.3197$ yields the best overall response (i.e., the lowest maximum).

5.55 For a Houdaille damper with mass ratio $\mu = 0.25$, calculate the optimum damping ratio and the frequency at which the damper is most effective at reducing the amplitude of vibration of the primary system.

Solution:

From equation (5.49), with $\mu = 0.25$,

$$\zeta_{op} = \frac{1}{\sqrt{2(\mu + 1)(\mu + 2)}} = 0.422$$

From equation (5.48),

$$r = \sqrt{\frac{2}{2 + \mu}} = 0.943$$

The damper would be most effective at $\omega = r\omega_n = 0.943\omega_n$, i.e., where the amplitude is greatest:

5.56 Consider again the system of Problem 5.53. If the damping ratio is changed to $\zeta = 0.1$, what happens to $\left| Xk / M_0 \right|$?

Solution:

If $\zeta_{op} = 0.1$, the value of μ becomes

$$0.1 = \frac{1}{\sqrt{2(\mu + 1)(\mu + 2)}}$$

$$0.02\mu^2 + 0.06\mu = 0.96 = 0$$

$$\mu = -8.589,\ 5.589$$

Clearly $\mu = 5.589$ is the physical solution. The maximum value of $\left| \dfrac{Xk}{M_0} \right|$ would be

$$\left(\frac{Xk}{M_0} \right)_{\text{max}} = 1 + \frac{2}{\mu} = 1.358$$

which is less than 6 (the requirement of Problem 5.53). Note that the value of μ is extremely large.

5.57 Derive Equation (5.42) from Equation (5.35) and derive Equation (5.49) for the optimal damping ratio.

Solution:

Equation (5.37) is derived from Equation (5.35) in Problem 5.39.

Start with Equation (5.37):

$$\frac{Xk}{F_0} = \sqrt{\frac{(2\zeta r)^2 + (r^2 - \beta^2)^2}{(2\zeta r)^2 (r^2 - 1 + \mu r^2)^2 + \left[\mu r^2 \beta^2 - (r^2 - 1)(r^2 - \beta^2)\right]^2}}$$

To derive Equation (5.42), which is the same as Equation (5.39), note that $c = k_a = \omega_a = 0$, which also means $\beta = 0$. Since this is a moment equation, F_0 is replaced by M_0. Therefore,

$$\frac{Xk}{F_0} = \sqrt{\frac{(2\zeta r)^2 + r^4}{(2\zeta r)^2 (r^2 - 1 + \mu r^2)^2 + (r^2 - 1)^2 r^4}} = \sqrt{\frac{4\zeta^2 + r^4}{4\zeta^2 (r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2}}$$

which is Equation (5.42) after canceling r^2.

To derive Equation (5.49), first let Equation (5.42) be $f(r, \zeta)$. Since $f = \dfrac{A^{1/2}}{B^{1/2}}$, where

$A = 4\zeta^2 + r^2$ and $B = 4\zeta^2 (r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2$, then

$$\frac{\partial f}{\partial \zeta} = 0$$

becomes
$$B dA = A dB$$

where
$$dA \equiv \frac{\partial A}{\partial \zeta} = 8\zeta$$

$$dB \equiv \frac{\partial B}{\partial \zeta} = 8\zeta \left(r^2 + \mu r^2 - 1\right)^2$$

So,

$$\left\{4\zeta^2\left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2 r^2\right\}(8\zeta) = \left\{4\zeta^2 + r^2\right\}(8\zeta)\left(r^2 + \mu r^2 - 1\right)^2$$

$$\left(r^2 - 1\right)^2 = \left(r^2 + \mu r^2 - 1\right)^2$$

$$\left(r^2 - 1\right) = \pm\left(r^2 + \mu r^2 - 1\right)$$

Taking the minus sign (the plus sign yields $r = 0$).

$$\left(2 + \mu\right)r^2 - 2 = 0$$

$$r = \sqrt{\frac{2}{\left(2 + \mu\right)}}$$

Now take the other partial derivative $\dfrac{\partial f}{\partial r} = 0$, which becomes

$$BdA = AdB$$

$$dA \equiv \frac{\partial A}{\partial r} = 2r$$

$$dB \equiv \frac{\partial B}{\partial r} = 16\zeta^2 r\left(1 + \mu\right)\left(r^2 + \mu r^2 - 1\right) + 4r^3\left(r^2 - 1\right) + 2r\left(r^2 - 1\right)^2$$

So,

$$\left\{4\zeta^2\left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2 r^2\right\}(2r)$$

$$= \left\{4\zeta^2 + r^2\right\}\left[16\zeta^2 r\left(1 + \mu\right)\left(r^2 + \mu r^2 - 1\right) + 4r^3(r^2 - 1) + 2r\left(r^2 - 1\right)^2\right]$$

Substituting $r = \sqrt{\dfrac{2}{\left(2 + \mu\right)}}$ yields, after rearranging

$$4\zeta^2\left[\frac{2}{2 + \mu} + \frac{2\mu}{2 + \mu} - 1\right]^2 + \left[\frac{2}{2 + \mu} - 1\right]^2\left(\frac{2}{2 + \mu}\right)$$

$$= \left[4\zeta^2 + \frac{2}{2 + \mu}\right]\left[8\zeta^2\left(1 - \mu\right)\left(\frac{2}{2 + \mu} + \frac{2\mu}{2 + \mu} - 1\right) + 2\left(\frac{2}{2 + \mu}\right)\left(\frac{2}{2 + \mu} - 1\right) + \left(\frac{2}{2 + \mu} - 1\right)^2\right]$$

Expanding and canceling terms yields

$$4\zeta^4 \left(1 + \mu\right)\left(2 + \mu\right) + 2\zeta^2\mu - \frac{2}{2 + \mu} = 0$$

The physical solution for ζ is

$$\zeta = \frac{1}{\sqrt{2\left(1 + \mu\right)\left(2 + \mu\right)}}$$

which is Equation (5.49).

5.58 Consider the design suggested in Example 5.5.1. Calculate the percent change in the maximum deflection if the damping constant changes 10% from an optimal value. If the optimal damping is fixed but the mass of the absorber changes by 10%, what percent change in $\left| Xk / M_0 \right|_{max}$ results? Is the optimal absorber design more sensitive to changes in c_a or m_a?

Solution:

From Problems 5.51 and 5.46, $F_0 = 100$ N, $k = 4.144 \times 10^6$ N/m, and $\mu = 0.133$. The optimal damping is

$$\zeta_{op} = \frac{1}{\sqrt{2(1 + \mu)(2 + \mu)}} = 0.4549$$

The deflection is given by Equation (5.42), and M_0 replaced by F_0,

$$X = \frac{F_0}{k} \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2 \left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2 r^2}}$$

Also, the maximum displacement will occur at $r = \sqrt{\dfrac{2}{(2 + \mu)}} = 0.9683$. If the damping constant changes by 10%, ζ will also change by 10% since $\zeta = \dfrac{c_a}{2m\omega_p}$. The value of X for 0.9 ζ_{op}, ζ_{op}, and 1.1 ζ_{op} is

$$\zeta = 0.9\zeta_{op} \quad \rightarrow \quad X = 3.870 \times 10^{-4} \text{ m}$$
$$\zeta = \zeta_{op} \quad \quad \rightarrow \quad X = 3.870 \times 10^{-4} \text{ m}$$
$$\zeta = 1.1\zeta_{op} \quad \rightarrow \quad X = 3.870 \times 10^{-4} \text{ m}$$

There is no change in X with a 10% change in ζ_{op}.

If m_a changes by 10%, μ will also change by 10% since $\mu = \dfrac{m_a}{m}$. The value of $\left(\dfrac{Xk}{F_0}\right)_{max}$ for 0.9μ, μ, and 1.1μ is

$$0.9\mu \quad \rightarrow \quad r = 0.9714 \quad \rightarrow \quad \left(\frac{Xk}{F_0}\right)_{max} = 17.708(+10.4\%)$$

$$\mu \quad \rightarrow \quad r = 0.9683 \quad \rightarrow \quad \left(\frac{Xk}{F_0}\right)_{max} = 16.038$$

$$1.1\mu \quad \rightarrow \quad r = 0.9318 \quad \rightarrow \quad \left(\frac{Xk}{F_0}\right)_{max} = 14.671(-8.5\%)$$

The displacement is more sensitive to changes in m_a than c_a.

5.59 Consider the elastic isolation problem described in Figure 5.26. Derive equations (5.57) and (5.58) from equation (5.53).

Solution:

Rewrite equation (5.53) in matrix form as

$$\begin{bmatrix} k_1 - m\omega^2 + jc\omega & -jc\omega \\ -jc\omega & -(k_2 + jc\omega) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

The inverse of the matrix on the left is

$$\frac{1}{-k_2\left(k_1 - m\omega^2\right) - jc\omega\left(k_1 + k_2 m\omega^2\right)} \begin{bmatrix} -\left(k_2 + jc\omega\right) & jc\omega \\ jc\omega & k_1 m\omega^2 + jc\omega \end{bmatrix}$$

Solving for X_1 and X_2 yields

$$X_1 = \frac{\left(k_2 + jc\omega\right) F_0}{k_2\left(k_1 - m\omega^2\right) + jc\omega\left(k_1 + k_2 - m\omega_{dr}^2\right)}$$

$$X_2 = \frac{c\omega_{dr} F_0 j}{k_2\left(k_1 - m\omega^2\right) + jc\omega\left(k_1 + k_2 - m\omega_{dr}^2\right)}$$

which are equations (5.54) and (5.55).

5.60 Use the derivative calculation for finding maximum and minimum to derive equations (5.57) and (5.58) for the elastic damper system.

Solution:

From equation (5.56)

$$T.R. = \frac{\sqrt{1 + 4(1 + \gamma)^2 \zeta^2 r^2}}{\sqrt{(1 - r^2)^2 + 4\zeta^2 r^2 (1 + \gamma - r^2\gamma)^2}}$$

Equation (5.45) is applicable here, so that

$$BdA = AdB$$

where $A = 1 + 4(1 + \gamma)^2 \zeta^2 r^2$ and $B = (1 - r^2)^2 + 4\zeta^2 r^2 (1 + \gamma - r^2\gamma)^2$ differentiating with respect to ζ yields

$$dA \equiv \frac{\partial A}{\partial \zeta} = 8(1 + \gamma)^2 \zeta r^2$$

$$dB \equiv \frac{\partial B}{\partial \zeta} = 8\zeta r^2 (1 + \gamma - r^2\gamma)^2$$

So,

$$\left\{ (1 - r^2)^2 + 4\zeta^2 r^2 (1 + \gamma - r^2\gamma)^2 \right\} (8)(1 + \gamma)^2 \zeta r^2$$

$$= \left\{ 1 + 4(1 + \gamma)^2 \zeta^2 r^2 \right\} (8\zeta r^2)(1 + \gamma - r^2\gamma)^2$$

$$(1 - r^2)^2 (1 + \gamma)^2 = (1 + \gamma - r^2\gamma)^2$$

$$(1 - r^2)(1 + \gamma) = \pm(1 + \gamma - r^2\gamma)$$

The minus sign yields the physical result

$$r^2(2\gamma + 1) = 2(1 + \gamma)$$

$$r = \sqrt{\frac{2(1 + \gamma)}{1 + 2\gamma}}$$

which is equation (5.57)

Differentiating with respect to r yields

$$dA \equiv \frac{\partial A}{\partial r} = 8(1+\gamma)^2 \zeta^2 r$$

$$dB \equiv \frac{\partial B}{\partial r} = 2(1-r^2)(-2r) + 8\zeta^2 r(1+\gamma-r^2\gamma)^2 + 8\zeta^2 r^2(1+\gamma-r^2\gamma)(-2r\gamma)$$

So,

$$\left\{ (1-r^2)^2 + 4\zeta^2 r^2 (1+\gamma-r^2\gamma)^2 \right\} (8\zeta^2 r)(1+\gamma)^2$$

$$= \left\{ 1 + 4(1+\gamma)^2 \zeta^2 r^2 \right\} \left[-4r(1-r^2) + 8\zeta^2 r(1+\gamma-r^2\gamma)^2 - 16\gamma\zeta^2 r^3 (1+\gamma-r^2\gamma) \right]$$

Substituting for r and manipulating yields

$$\left[64\gamma(1+\gamma)^5 \left(\frac{1}{1+2\gamma} \right) \right] \zeta^4 + 8 \left[\gamma(1+\gamma)^2 + (1+\gamma)^3 (1+2\gamma) - 2(1+\gamma)^4 \right] \zeta^2 - (1+2\gamma) = 0$$

Solving for ζ yields the physical result

$$\zeta = \frac{\sqrt{2(1+2\gamma)/\gamma}}{4(1+\gamma)}$$

which is Equation (5.58).

5.61 A 1000-kg mass is suspended from ground by a 40,000-N/m spring. A viscoelastic damper is added, as indicated in Figure 5.26. Design the isolator (choose k_2 and c) such that when a 70-N sinusoidal force is applied to the mass, no more than 100 N is transmitted to ground.

Solution:

From equation (5.59),

$$\left(T.R.\right)_{max} = 1 + 2\gamma$$

$$\frac{F_T}{F_0} = \frac{100}{70} = 1.429 = 1 + 2\gamma$$

$$\gamma = 0.2143$$

The isolator stiffness should be

$$k_2 = \gamma k_1 = \left(0.2143\right)\left(40,000\right) = 8571 \text{N/m}$$

From equation (5.58),

$$\zeta_{op} = \frac{\sqrt{2\left(1 + 2\gamma\right)/\gamma}}{4\left(1 + \gamma\right)} = 0.7518$$

The isolator damping should be

$$c = 2\zeta_{op}\sqrt{\frac{k_1}{m}} = 2\left(0.7518\right)\sqrt{\frac{40,000}{1000}} = 9.51 \text{kg/s}$$

5.62 Consider the isolation design of Example 5.5.2. If the value of the damping coefficient changes 10% from the optimal value (of 188.56 kg/s), what percent change occurs in $(T.R.)_{max}$? If c remains at its optimal value and k_2 changes by 10%, what percent change occurs in $(T.R.)_{max}$? Is the design of this type of isolation more sensitive to changes in damping or stiffness?

Solution:

From Example 5.5.2, $c = 188.56$ kg/s and $k_2 = 200$ N/m. If the value of c changes by 10%, the value of T.R. becomes (with $r = 5$ and $\gamma = 0.5$),

$$0.9c \rightarrow \zeta_{op} = 0.4243 \rightarrow T.R. = 0.1228(-1.78\%)$$
$$c \rightarrow \zeta_{op} = 0.4714 \rightarrow T.R. = 0.1250$$
$$1.1c \rightarrow \zeta_{op} = 0.5185 \rightarrow T.R. = 0.1267\left(+1.39\%\right)$$

If the value of k_2 changes by 10%, the value of T.R. becomes (with $r = 5$ and $\zeta = 0.4714$),

$$0.9k_2 \rightarrow \gamma = 0.45 \rightarrow T.R. = 0.1327\left(+6.17\%\right)$$
$$k_2 \rightarrow \gamma = 0.5 \rightarrow T.R. = 0.1250$$
$$1.1k_2 \rightarrow \gamma = 0.55 \rightarrow T.R. = 0.1183\left(-5.31\%\right)$$

This design is more sensitive to changes in stiffness.

5.63 A 3000-kg machine is mounted on an isolator with an elastically coupled viscous damper such as indicated in Figure 5.26. The machine stiffness (k_1) is 2.943×10^6 N/m, $\gamma = 0.5$, and $c = 56.4 \times 10^3$ N·s/m. The machine, a large compressor, develops a harmonic force of 1000 N at 7 Hz. Determine the amplitude of vibration of the machine.

Solution:

The amplitude of vibration is given by Equation (5.54) as

$$X_1 = \frac{\left(k_2 + jc\omega\right)F_0}{k_2\left(k_1 - m\omega^2\right) + jc\omega\left(k_1 + k_2 - m\omega_{dr}^2\right)}$$

Since $F_0 = 1000$ N, $\omega = 7(2\pi) = 43.98$ rad/s, $m = 3000$ kg, $c = 56.4 \times 10^3$ N·s/m, $k_1 = 2.943 \times 10^6$ N/m, and $k_2 = \gamma k_1 = 1.4715 \times 10^6$ N/m, then

$$X_1 = -4.982 \times 10^{-4} - 1.816 \times 10^{-4} j$$

The magnitude is

$$\left|X_1\right| = 5.303 \times 10^{-4} \text{ m}$$

5.64 Again consider the compressor isolation design given in Problem 5.63. If the isolation material is changed so that the damping in the isolator is changed to $\zeta = 0.15$, what is the force transmitted? Next determine the optimal value for the damping ratio and calculate the resulting transmitted force.

Solution:

From Problem 5.63, $\gamma = 0.5$, $F_0 = 1000$ N, and $r = \dfrac{\omega}{\sqrt{k_1/m}} = \dfrac{7(2\pi)}{\sqrt{2.943 \times 10^6 / 3000}} =$ 1.404. Since $\zeta = 0.15$, the transmitted force is [from Equation (5.56)],

$$F_T = F_0 \sqrt{\frac{1 + 4(1+\gamma)^2 \zeta^2 r^2}{\left(1 - r^2\right)^2 + 4\zeta^2 r^2 \left(1 + \gamma - r^2\gamma\right)^2}} = 1188 \text{ N}$$

The optimal value for the damping ratio is found from equation (5.58):

$$\zeta_{op} = \frac{\sqrt{2(1+2\gamma)/\gamma}}{4(1+\gamma)} = 0.4714$$

The transmitted force is then

$$F_T = 1874 \text{ N}$$

5.65 Consider the optimal vibration isolation design of Problem 5.64. Calculate the optimal design if the compressor's steady-state driving frequency changes to 24.7 Hz. If the wrong optimal point is used (i.e., if the optimal damping for the 7-Hz driving frequency is used), what happens to the transmissibility ratio?

Solution:

From Problems 5.63 and 5.64, $\gamma = 0.5$, $F_0 = 1000$ N, $k_1 = 2.943 \times 10^6$ N, and $m = 3000$ kg.

The optimal damping is

$$\zeta_{op} = \frac{\sqrt{2(1+2\gamma)/\gamma}}{4(1+\gamma)} = 0.4714$$

The value of c and k_2 would be

$$c = 2\zeta_{op}\sqrt{k_1 m} = 88.589 \text{ kg/s}$$
$$k_2 = \gamma k_1 = 1.472 \times 10^6 \text{ N/m}$$

The isolation design is independent of the driving frequency in this problem, so the transmissibility ratio would not change.

5.66   Recall the optimal vibration absorber of Problem 5.53. This design is based on a steady-state response. Calculate the response of the primary system to an impulse of magnitude M_0 applied to the primary inertia J_1. How does the maximum amplitude of the transient compare to that in steady state?

Solution:

The response of the system given in Problem 5.53 cannot be solved by the means of modal analysis given in Chapter 4 because the system is not proportionally damped. However, the steady-state response of a damped system to an impulse is simply zero. Therefore, the maximum amplitude of the transient will be of interest. For a sinusoidal input, a numerical simulation might be necessary to determine the effects of the transient response.

Problems and Solutions Section 5.6 (5.67 through 5.73)

5.67 Compare the resonant amplitude at steady state (assume a driving frequency of 100 Hz) of a piece of nitrite rubber at 50°F versus the value at 75°F. Use the values for η from Table 5.2.

Solution:

From equation (5.63),

$$X = \frac{F_0}{k\left(1 + \eta j\right) - m\omega^2}$$

At resonance $\omega = \sqrt{\dfrac{k}{m}}$ so

$$X = \frac{F_0}{k\left(1 - \eta j\right) - 1} = \frac{F_0}{k\eta j}$$

The magnitude is

$$\left|X\right| = \frac{1}{\eta}\left(\frac{F_0}{k}\right)$$

At 50°, $\eta = 0.5$ and at 75°, $\eta = 0.28$, so

$$\left|X\right|_{50°} = \frac{2F_0}{k}$$

$$\left|X\right|_{75°} = \frac{3.57 F_0}{k}$$

5.68 Using Equation (5.67), calculate the new modulus of a $0.05 \times 0.01 \times 1$, piece of pinned-pinned aluminum covered with a 1-cm-thick piece of nitrite rubber at 75°F driven at 100 Hz.

Solution:

From Table 1.2, $E_1 = 7.1 \times 10^{10}$ N/m² for aluminum. From Table 5.2, $E_2 = 2.758 \times 10^7$ N/m² for nitrate rubber, Also,

$$I = I_1 = \frac{1}{3}(0.05)(1)^3 = 0.01667 \text{ m}^4$$

$$e_2 = \frac{E_2}{E_1} = \frac{2.758 \times 10^7}{7.1 \times 10^{10}} = 3.885 \times 10^{-4}$$

$$h_2 = \frac{H_2}{H_1} = \frac{0.01}{0.01} = 1$$

From Equation (5.67),

$$E = \frac{E_1 I_1}{I}\left[1 + e_2 h_2^2 + 3(1 + h_2)^2 \frac{e_2 h_2}{1 + e_2 j_2}\right] = 7.136 \times 10^{10} \text{ N/m}^2$$

5.69 Calculate Problem 5.68 again at 50°F. What percent effect does this change in temperature have on the modulus of the layered material?

Solution:

From Problem 5.68, with $E_2 = 4.137 \times 10^7$ N/m²,

$$I = I_1 = 0.01667 \text{ m}^4$$

$$e_2 = \frac{E_2}{E_1} = \frac{4.137 \times 10^7}{7.1 \times 10^{10}} = 5.827 \times 10^{-4}$$

$$h_2 = \frac{H_2}{H_1} = \frac{0.01}{0.01} = 1$$

From Equation (5.67),

$$E = \frac{E_1 I_1}{I}\left[1 + e_2 h_2^2 + 3\left(1 + h_2\right)^2 \frac{e_2 h_2}{1 + e_2 h_2}\right] = 7.154 \times 10^{10} \text{ N/m}^2$$

This is an increase of 0.25% of the layered material's modulus.

5.70   Repeat the design of Example 5.6.1 by
(a) changing the operating frequency to 1000 Hz, and
(b) changing the operating temperature to 50°F.
Discuss which of these designs yields the most favorable system.

Solution:

From Ex. 5.6.1, $E_1 = 7.1 \times 10^{10}$ N/m^2 and $h_2 = 1$.

(a) 75°, 1000 Hz

$$\eta_2 = 0.55$$
$$E_2 = 4.826 \times 10^7 \, \text{N/m}^2$$
$$e_2 = \frac{E_2}{E_1} = 6.797 \times 10^{-4}$$

From Equation (5.68),

$$\eta = \frac{e_2 h_2 \left(3 + 6h_2 + 4h_2^2 + 2e_2 h_2^2 + e_2^2 h_2^4\right)}{\left(1 + e_2 h_2\right)\left(1 + 4e_2 h_2 + 6e_2 h_2^2 + 4e_2 h_2^3 + e_2^2 h_2^4\right)} \eta_2 = 0.00481$$

(b) 50°, 1000 Hz

$$\eta_2 = 0.5$$
$$E_2 = 4.137 \times 10^7 \ \text{N/m}^2$$
$$e_2 = \frac{E_2}{E_1} = \frac{4.137 \times 10^7}{7.1 \times 10^{10}} = 5.827 \times 10^{-4}$$

From Equation (5.68),

$$\eta = 0.00375$$

Increasing the driving frequency results in a higher loss factor compared to the effects of lowering the temperature.

5.71   Reconsider Example 5.6.2. Make a plot of thickness of the damping treatment versus loss factor.

Solution:

From Ex. 5.6.2, $\eta_2 = 0.261$, $e_2 = 0.01$, and $H_1 = 1$ cm. So, from Equation (5.69),

$$\eta = 14e_2 \frac{H_2^2}{H_1^2}\eta_2 = 0.03654 H_2^2 \qquad \left(H_2 \text{ in cm}\right)$$

$\eta\,(H2) := 0.03654 \cdot H2^2$
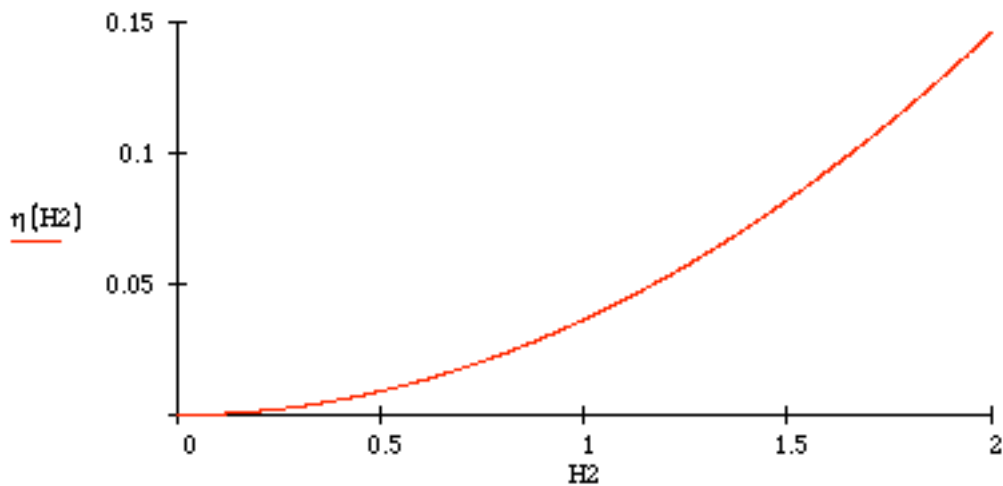
A plot of η versus H_2 in centimeters

5.72 Calculate the maximum transmissibility coefficient of the center of the shelf of Example 5.6.1. Make a plot of the maximum transmissibility ratio for this system frequency, using Table 5.2 for each temperature.

Solution: If the system is modeled as shown in Figure 5.18, then the maximum transmissibility occurs at (from Equation (5.50)),

$$\left(\frac{Xk}{F_0}\right)_{\text{max}} = 1 + \frac{2}{\mu}$$

where μ is found from Equation (5.49) as the solution to

$$\zeta = \frac{1}{\sqrt{2(\mu+1)(\mu+2)}}$$

The value of ζ is $\frac{\eta}{2}$ at resonance. So, at 75° and 100 Hz,

$$\zeta = \frac{\eta}{2} = \frac{0.00151}{2} = 0.000755 = \frac{1}{\sqrt{2(\mu+1)(\mu+2)}}$$

$$\Rightarrow \mu = 935$$

$$\frac{Xk}{F_0} = 1 + \frac{2}{935} = 1.002$$

For 50° and 100 Hz, $\eta = 0.00375$ (from Problem 5.70), so

$$\zeta = \frac{\eta}{2} = \frac{0.00375}{2} = 0.001875 = \frac{1}{\sqrt{2(\mu+1)(\mu+2)}}$$

$$\mu = 375.6$$

$$\frac{Xk}{F_0} = 1 + \frac{2}{375.6} = 1.005$$

$i := 1 .. 2$ $x_1 := 1.002$ $x_2 := 1.005$

$y_1 := 75$ $y_2 := 50$

This gives some idea of the relationship, but not a very good one as it includes only two points
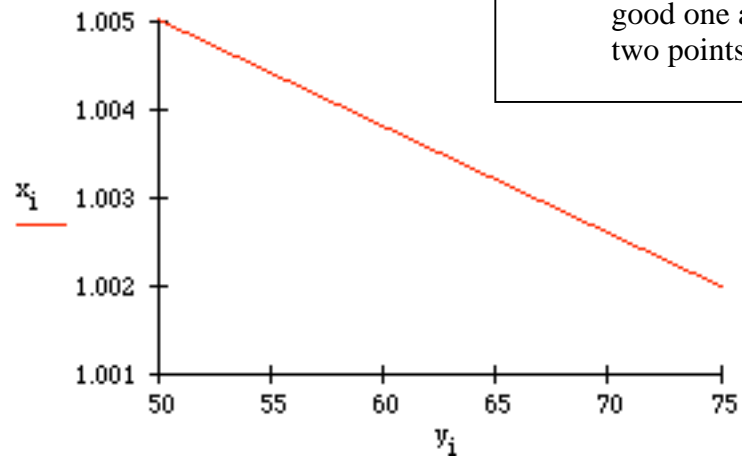
5.73 The damping ratio associated with steel is about $\zeta = 0.001$. Does it make any difference whether the shelf in Example 5.6.1 is made out of aluminum or steel? What percent improvement in damping ratio at resonance does the rubber layer provide the steel shelf?

Solution:

If the shelf in Ex. 5.6.1 is made out of steel, $E_1 = 2.0 \times 10^{11}$ N/m^2. Therefore,

$$e_2 = \frac{E_2}{E_1} = \frac{2.758 \times 10^7}{2.0 \times 10^{11}} = 0.0001379$$

Also, $\eta_2 = 0.55$ and $h_2 = 1$. From Equation (5.68),

$$\eta = \frac{e_2 h_2 \left(3 + 6h_2 + 4h_2^2 + 2e_2 h_2^2 + e_2^2 H_2^4\right)}{\left(1 + e_2 h_2\right)\left(1 + 4e_2 h_2 + 6e_2 h_2^2 + 4e_2 h_2^3 + e_2 h_2^4\right)} \eta_2 = 0.0005$$

At resonance, $\zeta = \dfrac{\eta}{2} = 0.00025$. The rubber actually reduced the damping of the steel shelf by 75%.

Problems and Solution Section 5.7 (5.74 through 5.80)

5.74 A 100-kg compressor rotor has a shaft stiffness of 1.4×10^7 N/m. The compressor is designed to operate at a speed of 6000 rpm. The internal damping of the rotor shaft system is measured to be $\zeta = 0.01$.
(a) If the rotor has an eccentric radius of 1 cm, what is the rotor system's critical speed?
(b) Calculate the whirl amplitude at critical speed. Compare your results to those of Example 5.7.1.

Solution:
(a) The critical speed is the rotor's natural frequency, so

$$\omega_c = \sqrt{\frac{k}{m}} = \sqrt{\frac{1.4 \times 10^7}{100}} = 374.2 \text{ rad/s} = 3573 \text{ rpm}$$

(b) At critical speed, $r = 1$, so from Equation (5.81),

$$X = \frac{\alpha}{2\zeta} = \frac{0.01}{2(0.01)} = 0.5 \text{ m}$$

So a system with higher eccentricity and lower damping has a greater whirl amplitude (see Example 5.7.1).

5.75 Redesign the rotor system of Problem 5.74 such that the whirl amplitude at critical speed is less than 1 cm by changing the mass of the rotor.

Solution: From Problem 5.74, $k = 1.4 \times 10^7$ N/m, $m = 100$ kg, $\zeta = 0.01$, and $\alpha = 0.01$m. Since the whirl amplitude at critical speed must be less than 0.01 m, the value of ζ that would satisfy this is, from equation (5.81),

$$X = \frac{\alpha}{2\zeta}$$

$$\zeta = \frac{\alpha}{2X} = \frac{0.01}{2(0.01)} = 0.5$$

The original damping ratio was 0.01, so the value of c is

$$c = 2\zeta m\omega = 2(0.01)(100)\sqrt{\frac{1.4 \times 10^7}{100}} = 784.33 \text{ kg/s}$$

So, the new mass should be, with $\zeta = 0.5$,

$$748.33 = 2(0.5)m\sqrt{\frac{k}{m}} = \sqrt{km} = \sqrt{1.4 \times 10^7 m}$$

$$\Rightarrow m = 0.04 \text{ kg}$$

This is not practical.

5.76 Determine the effect of the rotor system's damping ratio on the design of the whirl amplitude at critical speed for the system of Example 5.7.1 by plotting X at $r = 1$ for ζ between $0 < \zeta < 1$.

Solution:

From Example 5.7.1, with $r = 1$ and $\alpha = 0.001$ m,

$$X = \frac{0.001}{2\zeta} = \frac{0.0005}{\zeta}$$

$$\text{x}(\zeta) := \frac{0.0005}{\zeta}$$
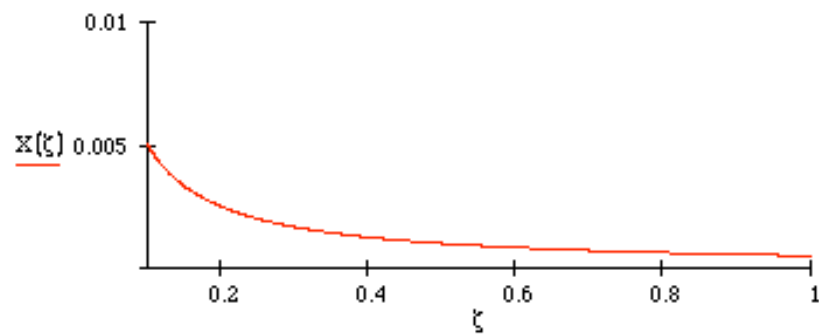
5.77 The flywheel of an automobile engine has a mass of about 50 kg and an eccentricity of about 1 cm. The operating speed ranges from 1200 rpm (idle) to 5000 rpm (red line). Choose the remaining parameters so that whirling amplitude is never more than 1 mm.

Solution:

From Equation (5.81),

$$X = 0.001 = \frac{0.01r^2}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}}$$

Choosing $\zeta = 0.1$, the physical solution is

$$r = 0.3018$$

By observing Figure 5.34, $r = 0.3018$ is the maximum value of r. So at $\left(\omega_r\right)_{max} = 500$ rpm, the stiffness must be

$$r = 0.3018 = \frac{5000\left(\dfrac{2\pi}{60}\right)}{\sqrt{k/50}}$$

$$k = 1.505\times10^8 \text{ N/m}$$

5.78 Consider the design of the compressor rotor system of Example 5.7.1. The amplitude of the whirling motion depends on the parameters α, ζ, m, k and the driving frequency. Which parameter has the greatest effect on the amplitude? Discuss your results.

Solution:

From Example 5.7.1, $\alpha = 0.001$ m, $\zeta = 0.05$, $m = 55$ kg, $\omega_r = 6000$ rpm, and $k = 1.4 \times 10^7$ N/m. To find out what effect each parameter has on this system, each value will be varied by 10%.

The original system has $r = 1.2454$ and $X = 0.002746$ m.

$0.9a = 0.009$m $\quad \rightarrow \quad r = 1.2454 \quad \rightarrow \quad X = 0.002471$ m (-10.0%)

$1.1a = 0.0011$ m $\quad \rightarrow \quad r = 1.2454 \quad \rightarrow \quad X = 0.003020$ m $\left(+10.0\%\right)$

$1.9\zeta = 0.045$ $\quad \rightarrow \quad r = 1.2454 \quad \rightarrow \quad X = 0.002759$ m $\left(+0.465\%\right)$

$1.1\zeta = 0.055$ $\quad \rightarrow \quad r = 1.2454 \quad \rightarrow \quad X = 0.002732$ m $\left(-0.507\%\right)$

$0.9m = 49.5$ kg $\quad \rightarrow \quad r = 1.1815 \quad \rightarrow \quad X = 0.003379$ m $\left(+23.1\%\right)$

$1.1m = 60.5$ kg $\quad \rightarrow \quad r = 1.3062 \quad \rightarrow \quad X = 0.002376$ m $\left(-13.5\%\right)$

$0.9k = 1.26 \times 10^7$ N/m $\quad \rightarrow \quad r = 1.3127 \quad \rightarrow \quad X = 0.002344$ m $\left(-14.6\%\right)$

$1.1k = 1.54 \times 10^7$ N/m $\quad \rightarrow \quad r = 1.1874 \quad \rightarrow \quad X = 0.003304$ m $\left(+20.3\%\right)$

The mass and stiffness values have the greatest effect on the amplitude, while the damping ratio has the smallest effect.

5.79 At critical speed the amplitude is determined entirely by the damping ratio and the eccentricity. If a rotor has an eccentricity of 1 cm, what value of damping ratio is required to limit the deflection to 1 cm?

Solution:

Since $X = 0.01$ m, $a = 0.01$ m, and at critical speed $r = 1$, then from Equation (5.81),

$$X = 0.01 \text{ m} = \frac{a}{2\zeta} = \frac{0.01}{2\zeta}$$

$$\zeta = 0.5$$

5.80 A rotor system has damping limited by $\zeta < 0.05$. What is the maximum value of eccentricity allowable in the rotor design if the maximum amplitude at critical speed must be less than 1 cm?

Solution:

Since $X = 0.01$ m, $\zeta < 0.05$, and at critical speed $r = 1$, then from Equation (5.81),

$$X = 0.01 \text{ m} = \frac{a}{2\zeta} = \frac{a}{2(0.05)}$$

$$a = 0.001 \text{ m} = 1 \text{ mm}$$

Problems and Solutions Section 5.8 (5.81 through 5.85)

5.81 Recall the definitions of settling time, time to peak, and overshoot given in Example 3.2.1 and illustrated in Figure 3.6. Consider a single-degree-of-freedom system with mass $m = 2$ kg, damping coefficient $c = 0.8$ N·s/m, and stiffness 8 N/m. Design a PD controller such that the settling time of the closed-loop system is less than 10 s.

Solution: The settling time is

$$t_s = \frac{3}{\zeta\omega}$$

Since $t_s = 10$ s,

$$\zeta\omega = 0.3$$

The equation of motion with a PD controller is

$$m\ddot{x} + \left(c + g_2\right)\dot{x} + \left(k + g_1\right)x = 0$$

So,

$$\omega = \sqrt{\frac{k + g_1}{m}} = \sqrt{\frac{8 + g_1}{2}}$$

$$\zeta = \frac{c + g_2}{2m\omega} = \frac{0.8 + g_2}{2(2)\omega}$$

Therefore,

$$\zeta\omega = \left(\frac{0.8 + g_2}{4\omega}\right)\omega = 0.3$$

$$g_2 = 0.4 \text{ N} \cdot \text{s/m}$$

The gain g_1 can take on any value (including 0).

5.82 Redesign the control system given in Example 5.8.1 if the available internal damping is reduced to 50 N·s/m.

Solution: If the value of c is limited to 50 N·s/m, then g_2 becomes

$$g_2 = 180 - c = 180 - 50 = 130 \text{ N} \cdot \text{s/m}$$

5.83 Consider the compressor rotor-shaft system discussed in Problem 5.74. Modern designers have considered using electromagnetic bearings in such rotor systems to improve their design. Use a derivative feedback control law on the design of this compressor to increase the effective damping ratio to $\zeta = 0.5$. Calculate the required gain. How does this affect the answer to parts (a) and (b) of Problem 5.74?

Solution: From Problem 5.74, $m = 100$ kg, $k = 1.4 \times 10^7$ N/m, $a = 0.01$, $\zeta_{old} = 0.01$. The value of c is

$$c = 2\zeta_{old}\sqrt{km} = 2(0.01)\sqrt{(1.4\times 10^7)(100)} = 748.3 \text{ kg/s}$$

With derivative feedback, the coefficient of $\dot{x}$ in the equation of motion is $c + g_2$. For $\zeta = 0.5$,

$$c + g_2 = 748.3 + g_2 = 2(0.5)\sqrt{(1.4\times 10^7)(100)} = 37,416.6$$

$$g_2 = 36,668.2 \text{ kg/s}$$

(a) The rotor's critical speed remains the same because it is only dependent upon the mass stiffness.
(b) The whirl amplitude becomes

$$X = \frac{a}{2\zeta} = \frac{0.01}{2(0.5)} = 0.01 \text{ m}$$

It is reduced by 80% because of the increased damping.

5.84 Calculate the magnitude of the force required of the actuator used in the feedback control system of Example 5.8.1. See if you can find a device that provides this much force.

Solution: The magnitude of the actuator force would be

$$F = g_2|\dot{x}| = g_2\omega_n X$$

where X is, from Equation (2.26), at steady-state,

$$X = \frac{F_0 / m}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}}$$

A large value of X would occur at resonance, for example, where $\omega = \omega_{dr} = 10$ rad/s, so

$$F = (80)(10)\frac{F_0 / 10}{2(0.9)(10)(10)} = 0.444F_0$$

5.85 In some cases the force actuator used in a control system also introduces dynamics. In this case a system of the form given in Equation (5.27) may result where m_a, c_a and k_a are values associated with the actuator (rather than an absorber). In this case the absorber system indicated in Figure 5.18 can be considered as the control system and the motion of the mass m is the object of the control system. Let $m = 10$ kg, $k = 100$ N/m, and $c = 0$. Choose the feedback control law to be

$$u = -g_1 x - g_2 \dot{x}$$

and assume that $c_a = 20$ N·s/m, $k_a = 100$ N/m and $m_a = 1$ kg. Calculate g_1 and g_2 so that x is as small as possible for a driving frequency of 5 rad/s. [Hint: Replace k with $k + g$, and c with $c + g$ in Equation (5.27)]

Solution:

Let the control law be called position feedback, applied to the mass m. The equation of motion then becomes Equation (5.27) with k replaced by $k + g_1$. Then the amplitude X can be expressed as Equation (5.35) with k replaced by $k + g_1$ and given values of m, m_a, k_a and c_a. This yields

$$\frac{X^2}{F_0^2} = \frac{(100-25)^2 + (25)(400)}{\left\{\left[100 + g_1 - (10)(25)\right]\left[100-25\right] - 2500\right\}^2 + \left[100 - (11)(25)\right]^2 (25)(400)}$$

$$\frac{X^2}{F_0^2} = \frac{2.78}{g_1^2 - 366.7 g_1 + 88{,}055.6}$$

Clearly X is a minimum if $g_1^2 - 366.7 g_1 + 88{,}055.6$ is a minimum. Thus consider the derivatives of the quadratic form with respect to g_1 to find the max value per the discussion on the top of page 265.

$$\frac{d}{dg_1}\left(g_1 - 366.7 g_1 + 88{,}055.6\right) = 2g_1 - 366.7 = 0$$

so that $g_1 = 183.35$

Note that $d^2 / dg_1^2 = 2 > 0$ so that this is a maximum and X is a minimum for this gain.

Problems Section 5.9 (5.86 through 5.88)

5.86   Reconsider Example 5.2.1, which describes the design of a vibration isolator to protect an electronic module. Recalculate the solution to this example using equation (5.92).

Solution: If data sheets are not available use $G'_\omega = G'/2$. One of many possible designs is given. From the example we have T.R. = 0.5, $m = 3$ kg and $\omega = 35$ rad/s = 5.57 Hz. From equation (5.92):

$$T.R. = \frac{\sqrt{1 + \eta^2}}{\sqrt{\left(1 - r^2 \dfrac{G'}{G'_\omega}\right)^2 + \eta^2}} = 0.5$$

From Table 5.2 for 75°F and frequency of 10 Hz (the closest value listed), the value of E and η are:

$$E = 2.068 \times 10^7 \text{ N/m}^2 \qquad \text{and} \qquad \eta = 0.21$$

Thus $G' = E/3 = 6.89 \times 10^9 \text{ N/m}^2$ using the approximation suggested after equation (5.86).   They dynamic shear modulus is estimated from plots such as Figure 5.38 to be $G'_\omega = G'/2$. Thus equation 5.92 becomes

$$0.5^2 = \frac{1 + (0.21)^2}{\left(1 - r^2 \dfrac{G'}{G'/2}\right)^2 + (0.21)^2}$$

This is solved numerically in the following Mathcad session:

$$TR(a, b, r) := \frac{\sqrt{1 + a^2}}{\sqrt{\left(1 - b \cdot r^2\right)^2 + a^2}}$$

$$TM(r) := .5^2$$
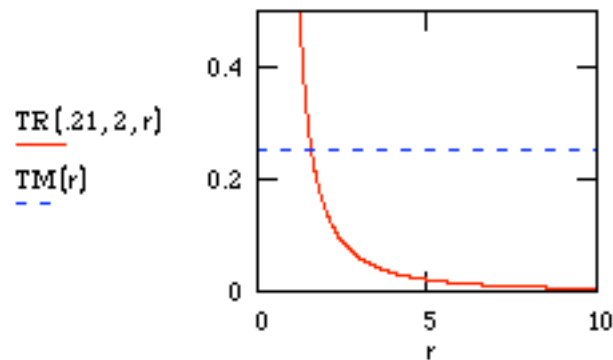
From the plot, any value of r greater then about 2.5 will do the trick. Choosing r
= 2.5 yields $\omega_n = \dfrac{\omega}{3.5} = \dfrac{35}{3.5} \Rightarrow \sqrt{\dfrac{k}{m}} = 10 \Rightarrow k = 100(3) = 300$ N/m

5.87   A machine part is driven at 40 Hz at room temperature. The machine has a mass of 100 kg.   Use Figure 5.42 to determine an appropriate isolator so that the transmissibility is less than 1.

Solution: Given $f = 40$ Hz, $m = 100$ kg or about 220 lbs. and *T.R.* <1. The maximum static load per mount is 3 lbs. Therefore the system would require a minimum of 73 mounts. Assume then that 75 mounts are used. Thus

$$\frac{220\#}{75} = 2.9\# \text{ per mount}$$

For the isolator, f_n <0.5 $f = 0.5(40) = 20$ Hz. Therefore the f_n of the isolator must be less then 20 Hz. Referring to the performance characteristics of the table in Figure 5.42 yields 4 possible isolator choices:

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5.88   Make a comparison between the transmissibility ratio of Window 5.1 and that of equation (5.92).

Solution: Comparing equation (5.92) with Window 5.1 yields:

Window 5.1: $\qquad T.R. = \dfrac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$

Equation (5.92): $\qquad T.R. = \dfrac{\sqrt{1 + \eta^2}}{\sqrt{\left(1 - r^2 \dfrac{G'}{G'_\omega}\right)^2 + \eta^2}}$

Comparing the two equations yields

$$\eta = 2\zeta r \quad \text{and} \quad \frac{G'}{G'_\omega} \approx 1$$

Chapter 6

Problems and Solutions Section 6.2 (6.1 through 6.7)

6.1   Prove the orthogonality condition of equation (6.28).

Solution:

Calculate the integrals directly. For $n = n$, let $u = n\pi x/l$ so that $du = (n\pi/l)dx$ and the integral becomes

$$\frac{l}{n\pi}\int_0^{n\pi} \sin^2 u\, du = \frac{l}{n\pi}\left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right)\Bigg|_0^{n\pi}$$

$$= \frac{l}{n\pi}\left(\frac{1}{2}n\pi - \frac{1}{4}\sin 4n\pi\right) - 0 = \frac{l}{2}$$

where the first step used a table of integrals. For $n \neq m$ let $u = \pi x/l$ so that $du = (\pi/l)dx$ and

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \frac{l}{\pi}\int_0^l \sin mu \sin nu\, du$$

which upon consulting a table of integrals is

$$\frac{l}{\pi}\left\{\frac{\sin(m-n)\pi}{2(m-n)} - \frac{\sin(n+m)\pi}{2(n+m)}\right\} = 0 .$$

6.2   Calculate the orthogonality of the modes in Example 6.2.3.

Solution:

One needs to show that $\int_0^1 X_n(x) X_m(x) dx = 0$ for $m \neq n$, where $X_m(t) = a_n \sin \sigma_n x$. But each mode $X_n(x)$ must satisfy equation (6.14), i.e.

$$X_n'' = -\sigma_n^2 X_n \tag{1}$$

Likewise

$$X_m'' = \sigma_m^2 X_m \tag{2}$$

Multiply (1) by X_m and integrate from 0 to l. Then multiply (2) by $X_n(x)$ and integrate from 0 to l. This yields

$$\int_0^l X_n'' X_m dx = -\sigma_n^2 \int_0^l X_n X_m dx$$

$$\int_0^l X_m'' X_n dx = -\sigma_m^2 \int_0^l X_m X_n dx$$

Subtracting these two equations yields

$$\int_0^l \left(X_n'' X_m - X_m'' X_n \right) dx = \left(\sigma_n^2 - \sigma_m^2 \right) \int_0^l X_n(x) X_m(x) dx$$

Integrate by parts on the left side to get

$$\int_0^l X_n' X_m' dx - \int_0^l X_m' X_n' dx + X_n' X_m \Big|_0^l - X_m X_n' \Big|_0^l$$
$$= X_m(l) k X_n(l) - X_n(l) k X_m(l) = 0$$

from the boundary condition given by eq. (6.50). Thus

$$\left(\sigma_n^2 - \sigma_m^2 \right) \int_0^l X_n X_m dx = 0.$$

But from fig. 6.4, $\sigma_n \neq \sigma_m$ for $m \neq n$ so that

$$\int_0^l X_n X_m dx = a_n^2 \int_0^l \sin \sigma_n x \sin \sigma_m x dx = 0$$

and the modes are orthogonal.

6.3. Plot the first four modes of Example 6.2.3, for the case $l = 1$ m, $k = 800$ N/m and $\tau = 800$ N/m.

Solution:

The mode shapes are given as $\sin\sigma_n x$ where σ_n satisfies eq. (6.51). To solve this numerically values of l, k and τ must be given. For example chose $l = 1$ m, $k = 800$ N/m, and $\tau = 800$ N/m the equation (6.51) becomes

$$\tan \sigma = -\sigma$$

Solving using MATLAB for the first 4 values yields

$$\sigma_1 = 2.029, \sigma_2 = 4.913, \sigma_3 = 7.979, \sigma_4 = 11.0855$$

So that the mode shapes are $\sin(2.029)x$, $\sin(4.913)x$, $\sin(7.979)x$ and $\sin(11.0855)x$. These are plotted below using Mathcad.

$$\sigma := \begin{bmatrix} 2.029 \\ 4.913 \\ 7.979 \\ 11.0855 \end{bmatrix}$$

$$X1(x) := \sin(\sigma_0 \cdot x) \quad X2(x) := \sin(\sigma_1 \cdot x) \quad X3(x) := \sin(\sigma_2 \cdot x) \quad X4(x) := \sin(\sigma_3 \cdot x)$$
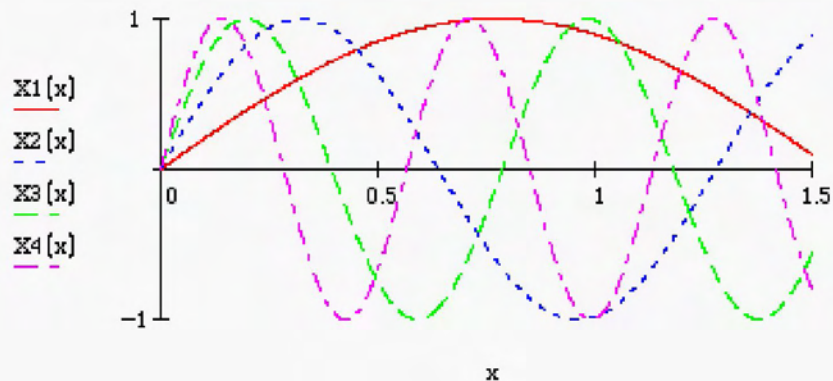
6.4 Consider a cable that has one end fixed and one end free. The free end cannot support a transverse force, so that $w_x(l,t) = 0$. Calculate the natural frequencies and mode shapes.

Solution:

The cable equation results in (6.17). The boundary conditions are

$$w(x,t) = X(x)T(t) = 0 \text{ at } x = 0 \text{ (fixed end)}$$

so that $X(0) = 0$ and

$$w_x(x,t) = X'(x)T(t) = 0 \text{ at } x = l \text{ (free end)}$$

so that $X(l) = 0$. Applying these to equation (6.17) yields

$$0 = a_1 \sin(0) + a_2 \cos(0) \text{ so that } a_2 = 0$$
$$0 = a_1 \sigma \cos \sigma(l)$$

so that $\cos \sigma l = 0$ or $\sigma l = n$ for odd n and the natural frequency $\sigma_n = \dfrac{n\pi}{2l}$, $n = 1, 3,$

5... or $\sigma_n = \dfrac{2n - 1\pi}{2l}$, $n = 1, 2, 3...$Since $a_2 = 0$, and a_1 is arbitrary the mode shapes are

$$a_n \sin\left(\frac{(2n-1)\pi x}{2l}\right), \quad n = 1,2,3...$$

the natural frequencies are from (6.15) and (6.24):

$$\omega_n = \sqrt{\sigma_n^2 c^2} = c\sigma_n = \frac{(2n-1)\pi c}{2l} = \frac{(2n-1)\pi}{2l}\sqrt{\tau / \rho}$$

6.5 Calculate the coefficients c_n and d_n of equation (6.27) for the system of a clamped-clamped string to the initial displacement given in Figure P6.5 and an initial velocity of $w_t(x,0) = 0$.

Solution:

For the clamped-clamped string the solution is given by eq. (6.27) as

$$w(x,t) = \sum_{i=1}^{\infty}(c_n \sin\sigma_n x \sin\sigma_n ct + d_n \sin\sigma_n x \cos\sigma_n ct)$$

Series $w_t(x,0) = 0$, equation (6.33) yields that $c_n = 0$ for all n. The coefficients d_n are given by eq. (6.31) as

$$d_n = \frac{2}{l}\int_0^l \omega_0(x)\sin\frac{m\pi x}{l}dx \quad m = 1,2,...$$

From fig. 6.16 $\omega_0(x) = \begin{cases} 2x/l & 0 \le x \le l/2 \\ 2(l-x)/l & l/2 \le x \le l \end{cases}$ cm. Calculation yields

$$d_n = \frac{2}{l}\left\{\int_0^{l/2}\frac{2x}{l}\sin\frac{n\pi x}{l}dx + \int_{l/2}^l\frac{2}{l}(l-x)\sin\frac{n\pi x}{l}dx\right\}$$

$$= \frac{8}{\pi^2 n^2}\sin\frac{n\pi}{2} \quad n = 1,3,5...$$

and d_n is zero for even values of n.

6.6   Plot the response of the string in Problem 6.5 for the piano string of Example
6.2.2 ($l = 1.4$ m, $m = 110$ g, $\tau = 11.1 \times 10^4$ N) at $x = l/4$ and $x = l/2$, using 3, 5, and
10 terms in the solution.

Solution:

For the piano string of example 6.22, $l = 1.4$m and $c = 11.89$. From problem 6.5
the solution has the form

$$w(x,t) = \frac{8}{\pi^2} \left\{ \sum_{m,\text{odd}=1}^{\infty} \frac{1}{m^2} \sin\frac{m\pi}{2} \sin\frac{m\pi x}{l} \cos\frac{m\pi c}{l} t \right\}$$

For 3 terms at $x = l/4 = 3.5$, this series becomes

$$w_3(3.5,t) = 0.81 \left\{ 0.24\cos 26.68t + 0.07858\cos 80.04t - 0.02828\cos 133.40t \right\}$$

for 5 terms this becomes

$$w_5(3.5,t) = w_3 + 0.01442\cos 182t + 0.00873\cos 240.13t$$

The next terms have coefficients 0.00584, 0.00418, 0.00314, 0.00244 and 0.00195
respectively. Any of the codes can be used to easily plot these. Plot of w_3 and w_5
at $l/4$ are given below in Mathcad:

$\text{w3}(t) := 0.81 \cdot (0.24 \cos(26.68 \cdot t) + 0.07858 \cdot \cos(80.84 \cdot t) - 0.2828 \cdot \cos(133.4 \cdot t))$

$\text{w5}(t) := \text{w3}(t) + 0.01441 \cdot \cos(182 \cdot t) + 0.00873 \cdot \cos(240.13 \cdot t)$
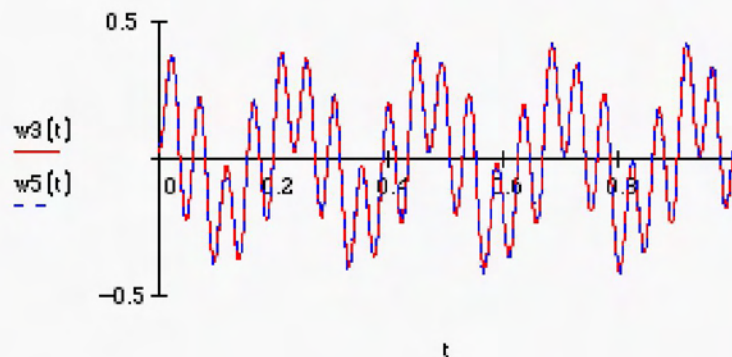
6.7   Consider the clamped string of Problem 6.5. Calculate the response of the string to the initial condition

$$w(x,0) = \sin\frac{3\pi x}{l} \quad w_t(x,0) = 0$$

Plot the response at $x = l/2$ and $x = l/4$, for the parameters of Example 6.2.2.

Solution:

Since $w_t = 0$ each if the coefficients c_n is zero in equation (6.33). Thus the solution is of the form

$$w(x,t) = \sum_{i=1}^{\infty} d_n \sin\frac{n\pi x}{l}\cos\frac{n\pi c}{l}t$$

as given in problem 6.5. Equation (6.31) for the initial position yields

$$d_n = \frac{2}{l}\int_0^l \sin\frac{3\pi x}{l}\sin\frac{m\pi x}{l}dx \quad m = 1,2,...$$

Because of the orthogonality all the $d_n = 0$ except d_3 and from the above integral $d_3 = 1$. Hence the solution collapses to the single term

$$w(x,t) = \sin\frac{3\pi x}{l}\sin\frac{3\pi c}{l}t$$

At $x = l/2$ this becomes

$$w\left(\frac{l}{2},t\right) = \sin\frac{3\pi}{2}\cos\frac{3\pi c}{l}t = -\cos\frac{3\pi c}{l}t$$

At $x = l/4$

$$w\left(\frac{l}{4},t\right) = \sin\frac{3\pi}{4}\cos\frac{3\pi c}{l}t = 0.707\cos\frac{3\pi c}{l}t$$

Using the values for the piano string ($l = 1.4$, $c = 1188$ m/s) $w(l/4,t)$ is simply a cosine of frequency 8000 rad/s and amplitude 0.707.

Problems and Solutions Section 6.3 (6.8 through 6.29)

6.8   Calculate the natural frequencies and mode shapes for a free-free bar. Calculate the temporal solution of the first mode.

Solution:

Following example 6.31 (with different B.C.'s), the spatial response of the bar will be

$$X(x) = a \sin \sigma x + b \cos \sigma x$$

The boundary conditions are $X'(0) = X'(l) = 0$. The expression for X' is $X'(x) = \sigma a \cos \sigma x - \sigma b \sin \sigma x$ so at 0:

$$0 = \sigma a \Rightarrow a = 0$$

at l

$$0 = -\sigma b \sin \sigma l, \quad b \neq 0$$

so that $\sigma l = n\pi$ or $\sigma = n\pi/l$ where n starts a *zero*. Hence the mode shapes are of the form

$$X_n(x) = b_n \cos \frac{n\pi x}{l} \text{ for } n = 1, 2, 3, \ldots \text{ and for } n = 0,$$

$$X_0(x) = b_0 \cos\left(\frac{0\pi}{l} x\right) = b_0 \text{ a constant.}$$

The temporal solution is given by eq. (6.15) to be

$$\frac{\ddot{T}_n(t)}{c^2 T_n(t)} = -\sigma^2$$

so that the temporal solution of the first mode:

$$\ddot{T}_0(t) + 0c^2 T_0(t) = 0 = \ddot{T}_0(t) \quad \underline{T_0(t) = b + ct}$$

6.9   Calculate the natural frequencies and mode shapes of a clamped-clamped bar.

Solution: The calculation of the natural frequencies and mode shapes of a clamped-clamped bar is identical to that of the fixed-fixed string since the equations of motion are mathematically the same. The solution of this problem is thus given at the beginning of section 6.2, but is repeated here: Applying separation of variable to eq. (6.56) yields that the spatial variable must satisfy eq. (6.59) of example 6.3.1, i.e., $X(x) = a \sin \sigma x + b \cos \sigma x$ where a and b are constants to be determined. The clamped boundary conditions require that $X(0) = X(l) = $ or

$$0 = b \text{ or } X = a\sin\sigma x$$
$$0 = a\sin\sigma l \text{ or } \sigma = n\pi/l$$

Hence the mode shapes will be of the form

$$X_n = a_n\sin\sigma_n x$$

Where $\sigma_v = n\pi/l$. The frequencies are determined from the temporal solution and become

$$\omega_n = \sigma_n c = \frac{n\pi}{l}\sqrt{\frac{E}{\rho}}, \quad n = 1,2,3,...$$

6.10   It is desired to design a 4.5 m, clamped-free bar such that the first natural frequency is 1878 Hz. Of what material should it be made?

Solution: First change the frequency into radians:
$$1878 \text{ Hz} = 1878 \times 2\pi \text{ rad/s} = 11800 \text{ rad/s}$$
The first natural frequency is given computed in Example 6.3.1, Equation (6.63) as

$$\omega_1 = \frac{2\pi}{l}\sqrt{\frac{E}{\rho}} \Rightarrow \frac{E}{\rho} = \omega_1^2 \frac{4l^2}{\pi^2} = (11800)^2 \frac{4l^2}{\pi^2}$$

$$\Rightarrow \frac{E}{\rho} = 7.143 \times 10^7$$

in Nm/kg. Examining the ratios from Table 2.1 for the values given yields that for Steel:

$$\frac{E}{\rho} = \frac{2 \times 10^{11}}{2.8 \times 10^3} = 7.143 \times 10^7 \text{ Nm/kg}$$

Thus a **steel** bar with a length 4.5 meters will have a first natural frequency of 1878 Hz. This is something like a truck chassis.

6.11 Compare the natural frequencies of a clamped-free 1-m aluminum bar to that of a 1-m bar made of steel, a carbon composite, and a piece of wood.

Solution:

For a clamped-free bar the natural frequencies are given by eq. (6.6.3) as

$$\omega_n = \frac{(2n-1)\pi}{2l}\sqrt{\frac{E}{\rho}}$$

Referring to values of r and E from table 1.2 yields (for ω_1):
Steel

$$\frac{\pi}{(2)(1)}\sqrt{\frac{2.0 \times 10^{11}}{7.8 \times 10^3}} = 7,954 \text{ rad/s (1266Hz)}$$

Aluminum

$$\frac{\pi}{(2)(1)}\sqrt{\frac{7.1 \times 10^{10}}{2.7 \times 10^3}} = 8,055 \text{ rad/s (1282 Hz)}$$

Wood

$$\frac{\pi}{(2)(1)}\sqrt{\frac{5.4 \times 10^9}{6.0 \times 10^2}} = 4,712 \text{ rad/s (750 Hz)}$$

Carbon composite (student must hunt for E/ρ and guess a little) from Vinson and Sierakowski's book on composites $\sqrt{E/\rho} = 3118$ and

$$\frac{\pi}{2}(3118) = 4897 \text{ rad/s (780 Hz)}$$

6.12 Derive the boundary conditions for a clamped-free bar with a solid lumped mass, of mass M attached to free end.

Solution: At the clamped end, $x = 0$, the boundary condition is $w(0,t) = 0$ or $X(x) = 0$. At the end $x = l$ the tensile force in the bar must be equal to the inertia force of the attached mass. For an attached mass of value M, this becomes

$$EA\frac{\partial w(x,t)}{\partial x}\bigg|_{x=l} = -M\frac{\partial^2 w(x,t)}{\partial t^2}\bigg|_{x=l}$$

6.13 Calculate the mode shapes and natural frequencies of the bar of Problem 6.12. State how the lumped mass affects the natural frequencies and the mode shapes.

Solution: Via separation of variables [i.e., $w(x,t) = X(x)T(t)$], the spatial equation becomes (following example 6.3.1 for instance)

$$X(x) = a\sin\sigma x + b\cos\sigma x$$

Applying the boundary condition at $x = 0$ yields

$$X(0) = 0 = a\sin(0) + b\cos(0) \Rightarrow b = 0 \cdot 0 = b$$

so the spatial solution reduces to $X(x) = a\sin\sigma x$. Now the second boundary condition (see 6.12) involves time deviates so that $w(x,t) = X(x)T(t)$ substituted into the boundary condition $EAW_x = -Mw_{tt}(l,t)$ becomes:

$$EAX'(l)T(t) = -MX(l)\ddot{T}(t) \Rightarrow \frac{EAX'(l)}{MX(l)} = -\frac{\ddot{T}(t)}{T(t)}$$

From equation (6.15) $\ddot{T}/T = -\sigma^2 c^2$, so this boundary condition becomes

$$\frac{EA}{M} \cdot \frac{X'(l)}{X(l)} = \sigma^2 c^2 \qquad\qquad (1)$$

Substitution of $X(x) = a\sin\sigma x$ and $X'(x) = a\sigma\cos\sigma x$ into (1) yields

$$\frac{EA}{M}\frac{a\sigma\cos\sigma l}{a\sin\sigma l} = \sigma^2 c^2$$

or

$$\cot\sigma l = \frac{\sigma c^2 M}{EA}$$

which describes multiple values of $\sigma = \sigma_n$, $n = 1, 2, 3,\dots$ The frequency of oscillation is related to σ_n by $\omega_n = \sigma_n c$, where $c = \sqrt{E/\rho}$. Let $\rho Al = m$ be the mass of the beam and rewrite $\cot(\sigma l)$ as

$$\cot\sigma l = \cot\left(\frac{\omega_n l}{c}\right) = \frac{\sigma(E/\rho)M}{EA} = \frac{(\omega_n l/c)}{A\rho l} \cdot M = \frac{\omega_n l}{c}\frac{M}{m}.$$

This can be rewritten as

$$\alpha\cot\alpha = \beta$$

where $\beta = m/M$ and $\alpha = \omega_n l/c$. As the mass ratio β increases (tip mass increases) the frequency increases. The mode shapes are proportional to $\sin\sigma_n x$, where σ_n is calculated numerically from $\cot(\sigma l) = (M/m)\sigma l$, similar to the calculation showing in Figure 6.4. This is illustrated in the following Mathcad session.

$$f(\sigma l) := \cot(\sigma l) \qquad g(\beta, \sigma l) := \beta \cdot \sigma l$$

Here, β is the end mass - to - bar mass ratio ($\beta = M/\rho AL$). Hence the transcendental equation $f(\sigma l)=g(\sigma l)$ is totally nondimensional which makes it possible to study the effect of end mass - to - bar mass ratio in a nondimesional basis. That is what you see in the following figure where 5 different end mass - to - bar mass ratios are investigated.
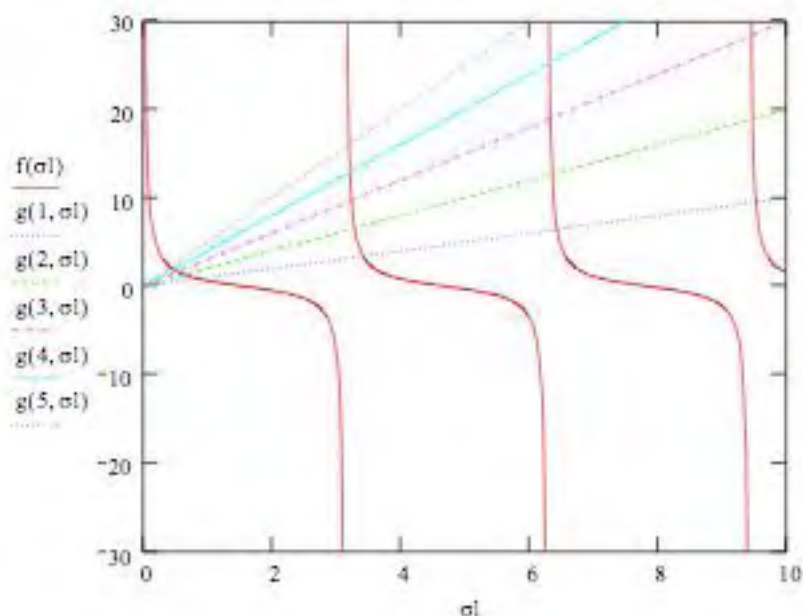
For each value of β (from 1 to 5), the intersection of f and g gives the frequency parameter σL which is directly proportional to the natural frequency of the mode of interest. Therefore, it is obvious from the figure that **increasing the end mass reduces the natural frequencies of the bar** (and looks like the first natural frequency is the most sensitive one). This makes perfect physical sense: if you add a large end mass, the structure becomes much more flexible.

The limiting behavoirs are $\beta=0$ and $\beta=inf$.. The former case ($\beta=0$) gives the natural frequencies of a clamped-free bar without tip mass (from the roots of $\cot(\sigma l)=0$ or just $\cos(\sigma l)=0$ if you like) whereas the latter case ($\beta=inf$.) gives the natural frequencies of a clamped-clamped bar (from the roots of $\sin(\sigma l)=0$) which is also evident from the above graph.

6.14 Calculate and plot the first three mode shapes of a clamed-free bar.

Solution: The second entry of Table 6.1 yields the solution

$$X_n(x) = \sin\frac{(2n-1)}{2\ell}\pi x$$

which is calculated following the procedures out lined in Example 6.3.1. The plot is given in Mathcad for the case $\ell = 1$m.

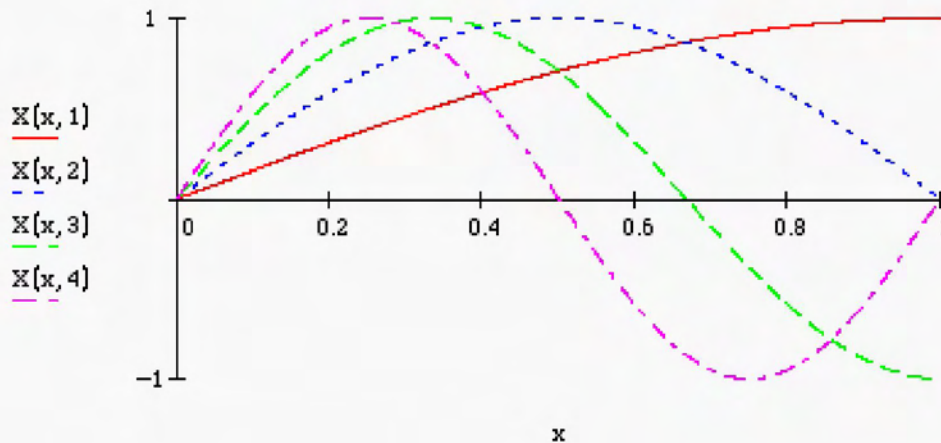$$X(x,n) := \sin\left[\frac{\left[(2\,n - 1)\cdot\pi\cdot x\right]}{2}\right]$$
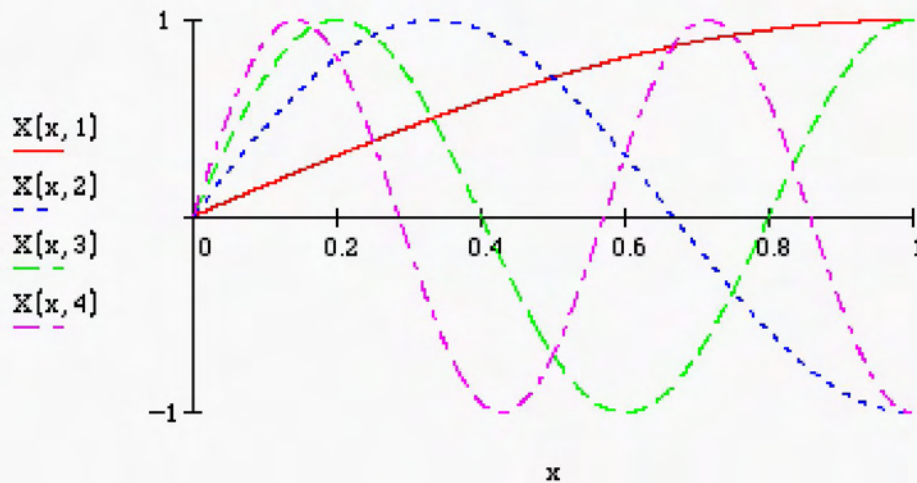
6.15 Calculate and plot the first three mode shapes of a clamed-clamped bar and compare them to the plots of Problem 6.14.

Solution: As in problem 6.14 the solution is given in table 6.1. The important item here is to notice the difference between mode shapes from the plots of $\sin \dfrac{(2n-1)\pi}{2l} x$ and $\sin (n\pi x/l)$. In particular notice the difference at the free end.

$$X(x,n) := \sin\left(\frac{n \cdot \pi \cdot x}{2}\right)$$

6.16 Calculate and compare the eigenvalues of the free-free, clamped-free, and the clamped-clamed bar. Are the related? What does this state about the system's natural frequencies?

Solution:

Students can calculate these or just use the results listed in table 6.1. Note for $l = 1$

> free-free $0, \pi c, 2\pi c\ldots$
>
> clamped-free $\dfrac{\pi c}{2}, \dfrac{3\pi c}{2}, \dfrac{5\pi c}{2}\ldots$
>
> clamped-clamped $\pi c, 2\pi c, 3\pi c\ldots$

so that the free-free and clamped-clamped values are a π shift from one another with the clamped-free values falling in between: as the number of constraints increases, the frequency increases.

6.17 Consider the nonuniform bar of Figure P6.17, which changes cross-sectional area as indicated in the figure. In the figure A_1, E_1, ρ_1, and l_1 are the cross-sectional area, modulus, density and length of the first segment, respectively, and A_2, E_2, ρ_2, and l_2 are the corresponding physical parameters of the second segment. Determine the characteristic equation.

Solution: Let the subscript 1 denote the first part of the beam and 2 the second part of the beam. The bar equation must be satisfied in each part so that equation of motion is in two parts:

$$E_1 \frac{\partial^2 w_1(x,t)}{\partial x^2} = \rho_1 \frac{\partial^2 w_1(x,t)}{\partial t^2} \quad 0 < x < \ell_1$$

$$E_2 \frac{\partial^2 w_2(x,t)}{\partial x^2} = \rho_2 \frac{\partial^2 w_2(x,t)}{\partial t^2} \quad \ell_1 < x < \ell_1 + \ell_2 = \ell$$

The boundary conditions are the two from the clamped-free configuration then there are two more conditions expressing force and displacement continuity at the point where the two beams join ($x = \ell_1$). Follow the procedure of separation of variables but this time keep the constant c in the spatial equation so that we may write: $w_1(x,t) = X_1(x)T(t)$ and $w_2(x,t) = X_2(x)T(t)$ where the function of time is common to both beams. Then denoting σ^2 as the separation constant and substituting the separated forms into the equation of motion yields:

$$\frac{c_1^2 X_1''(x)}{X_1(x)} = \frac{\ddot{T}(t)}{T(t)} = -\sigma^2 \quad 0 < x < l_1 \quad \text{and } c_1 = \sqrt{\frac{E_1}{\rho_1}} \quad (1)$$

$$\frac{c_2^2 X_2''(x)}{X_2(x)} = \frac{\ddot{T}(t)}{T(t)} = -\sigma^2 \quad l_1 < x < l \quad \text{and } c_2 = \sqrt{\frac{E_2}{\rho_2}} \quad (2)$$

In this way the temporal equation for both parts is the same (σ does not depend on which part of the beam and will show up in the characteristic equation). Solving the two spatial equations yields:

$$(1) \Rightarrow X_1 = a_1 \sin\frac{\sigma}{c_1} x + a_2 \cos\frac{\sigma}{c_1} x \quad 0 < x < \ell_1$$

$$(2) \Rightarrow X_2 = a_3 \sin\frac{\sigma}{c_2} x + a_4 \cos\frac{\sigma}{c_2} x \quad \ell_1 < x < \ell$$

There are now 4 boundary conditions (one at each end and two in the middle) which will yield 4 equations in the 4 coefficients a_i. This set of equations must be singular yielding the characteristic equation for σ.
From the clamped end:

$$X_1(0) = 0 \Rightarrow a_1 \sin(0) + a_2 \cos(0) = 0 \tag{3}$$

From the free end:

$$X_2'(\ell) = 0 \Rightarrow \frac{\sigma}{c_2} a_3 \cos\frac{\sigma\ell}{c_2} - \frac{\sigma}{c_2} a_4 \sin\frac{\sigma\ell}{c_2} = 0 \tag{4}$$

From the middle and enforcing displacement continuity at $x = \ell_1$:

$$a_1 \sin\frac{\sigma}{c_1}\ell_1 + a_2 \cos\frac{\sigma}{c_1}\ell_1 = a_3 \sin\frac{\sigma}{c_2}\ell_1 + a_4 \cos\frac{\sigma}{c_2}\ell_1 \tag{5}$$

From the middle and enforcing force, equation (6.54) continuity at $x = \ell_1$:

$$E_1 A_1 X_1'(\ell_1) = E_2 A_2 X'(\ell_1)$$

$$\Rightarrow E_1 A_1 \frac{\sigma}{c_1}(a_1 \cos\frac{\sigma\ell_1}{c_1} - a_2 \sin\frac{\sigma\ell_1}{c_1}) = E_2 A_2 \frac{\sigma}{c_2}(a_3 \cos\frac{\sigma\ell_1}{c_2} - \frac{\sigma}{c_2} a_4 \sin\frac{\sigma\ell_1}{c_2}) \qquad (6)$$

Equations (3) through (6) are 4 equations in the 4 unknowns a_i. Writing these in matrix form as a homogeneous algebraic equation yields:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\[6pt] 0 & 0 & \cos\dfrac{\sigma l}{c_2} & -\sin\dfrac{\sigma l}{c_2} \\[10pt] \sin\dfrac{\sigma}{c_1} l_1 & \cos\dfrac{\sigma}{c_1} l_1 & -\sin\dfrac{\sigma}{c_2} l_1 & -\cos\dfrac{\sigma}{c_2} l_1 \\[10pt] \dfrac{E_1 A_1}{c_1}\cos\dfrac{\sigma l_1}{c_1} & -\dfrac{E_1 A_1}{c_1}\sin\dfrac{\sigma l_1}{c_1} & -\dfrac{E_2 A_2}{c_2}\cos\dfrac{\sigma l_1}{c_2} & \dfrac{E_2 A_2}{c_2}\sin\dfrac{\sigma l_1}{c_2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In order for the vector **a** to be nonzero, the determinant of the matrix coefficient must be zero (recall chapter 4). This yields the characteristic equation (computed using Mathcad):

$$E_2 A_2 c_1 \sin\frac{\sigma l_1}{c_1}\left[\sin\frac{\sigma l}{c_2}\cos\frac{\sigma l_1}{c_2} - \sin\frac{\sigma l_1}{c_2}\cos\frac{\sigma l}{c_2} \right]$$

$$= E_1 A_1 c_2 \cos\frac{\sigma l_1}{c_1}\left[\sin\frac{\sigma l_1}{c_2}\sin\frac{\sigma l}{c_2} + \cos\frac{\sigma l_1}{c_2}\cos\frac{\sigma l}{c_2} \right] \qquad (7)$$

$$\Rightarrow$$

$$\frac{E_2 A_2 c_1}{E_1 A_1 c_2}\tan\frac{\sigma l_1}{c_1}\left[\sin\frac{\sigma l}{c_2}\cos\frac{\sigma l_1}{c_2} - \cos\frac{\sigma l}{c_2}\sin\frac{\sigma l_1}{c_2} \right]$$

$$= \sin\frac{\sigma l}{c_2}\sin\frac{\sigma l_1}{c_2} + \cos\frac{\sigma l}{c_2}\cos\frac{\sigma l_1}{c_2} \qquad (8)$$

Further simplifying yields

$$\frac{E_2 A_2 c_1}{E_1 A_1 c_2}\tan\frac{\sigma l_1}{c_1}\sin\frac{\sigma(l - l_1)}{c_2} = -\cos\frac{\sigma(l - l_1)}{c_2}$$

$$\Rightarrow \frac{E_2 A_2 c_1}{E_1 A_1 c_2}\tan\frac{\sigma l_1}{c_1}\tan\frac{\sigma(l - l_1)}{c_2} = -1$$

Given the parameter values, equation (9) must be solved numerically for σ, yielding the natural frequencies.

6.18 Show that the solution obtained to Problem 6.17 is consistent with that of a uniform bar.

Solution:

If the bar is the same, then $E_1 = E_2 = E$, $\rho_1 = \rho_2 = \rho$ etc. and the characteristic equation from (1) in the solution to Problem 6.17 becomes ($l = l_1$)

$$\sin\frac{\sigma\ell}{c}\left[\sin\frac{\sigma\ell}{c}\cos\frac{\sigma\ell}{c} - \sin\frac{\sigma\ell}{c}\cos\frac{\sigma\ell}{c}\right] = \cos\frac{\sigma\ell}{c}\left[\sin\frac{\sigma\ell}{c}\sin\frac{\sigma\ell}{c} + \cos\frac{\sigma\ell}{c}\cos\frac{\sigma\ell}{c}\right]$$

$$\Rightarrow \sin\frac{\sigma\ell}{c}(0) = \cos\frac{\sigma\ell}{c}\left[\sin^2\frac{\sigma\ell}{c} + \cos^2\frac{\sigma\ell}{c}\right]$$

$$\Rightarrow 0 = \cos\frac{\sigma\ell}{c}(1) \Rightarrow \frac{\sigma\ell}{c} = \frac{2n-1}{2}\pi$$

so that $\sigma_n = \omega_n = \dfrac{(2n-1)\pi}{2l}\sqrt{\dfrac{E}{\rho}}$ which according to table 6.1 entry 2 is the frequency of a clamped-free bar of length l .

6.19 Calculate the first three natural frequencies for the cable and spring system of Example 6.2.3 for $l = 1$, $k = 100$, $\tau = 100$ (SI units).

Solution:

For $l = 1$, $k = 100$ and $\tau = 100$ the frequency equation (6.51) becomes

$$\tan \sigma = -\sigma$$

Using MATLAB the first 3 solutions are

$\sigma_1 = 0$, $\sigma_2 = 2.029$, $\sigma_3 = 4.913$. But zero is not allowed because of the boundary conditions.

6.20   Calculate the first three natural frequencies of a clamped-free cable with a mass of value m attached to the free end. Compare these to the frequencies obtained in Problem 6.17.

Solution:

Recall example 6.1.1. The force balance at the boundary $x = l$ yields

$$\tau w_x(x,t)\big|_{x=l} = -m w_{tt}(l,t)$$

The boundary condition at $x = 3$ remains $w(0,t) = 0$. The equation of motion is (6.8) or

$$c^2 w_{xx}(x,t) = w_{tt}(x,t)$$

Again, separation of variable $w(x,t) = X(x)T(t)$ yields eq. (6.12) or

$$\frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{c^2 T(t)} = -\sigma^2$$

The spatial equation is

$$X'' + \sigma^2 X(x) = 0$$

which has solution $X(x) = a_1 \sin \sigma x + a_2 \cos \sigma x$. Applying the boundary conditions yields $X(0) = 0$ or $a_2 = 0$. Substitution of $X(x) = a_1 \sin \sigma_2 x$ into the boundary condition at $x = l$ yields

$$[a_1 \tau \sigma \cos \sigma l]T(t) = -m \ddot{T}(t) a_1 \sin \sigma l$$

But $\ddot{T}(t) / T(t) = \sigma^2 c^2$ so this becomes

$$\tau \sigma \cos \sigma l = m \sigma^2 c^2$$

or that

$$\tan \sigma l = \frac{\tau}{m \sigma c^2} \quad (\text{or } \cot \sigma l = \frac{n\sigma}{\rho})$$

is the characteristic equation (see also table 6.1) with mode shape $\sin \sigma_n x$. A plot of their characteristic equation $\cos(\sigma l) = \frac{mc^2}{lr}\sigma l = \frac{m}{lp}(\sigma l)$ yields the value of the frequencies relative to those of problem 6.16.

6.21 Calculate the boundary conditions of a bar fixed at $x = 0$ and connected to ground through a mass and a spring as illustrated in Figure P6.21.

Solution:

A free body diagram of the boundary is shown in Figure 1.
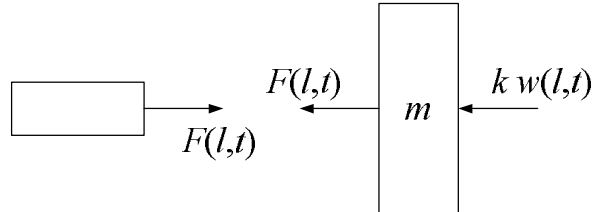
Figure 1

Consider first the end of the rod, the force is related to the axial extension of the rod though

$$F(l,t) = \sigma A\big|_{x=l} = EA\frac{\partial w(x,t)}{\partial x}\bigg|_{x=l}$$

On the other hand, applying Newton's second law to the mass yields

$$-F(l,t) - kw(x,t)\big|_{x=l} = m\frac{\partial w(x,t)}{\partial t^2}\bigg|_{x=l}$$

Hence, this yields the following boundary condition

$$m\frac{\partial w(x,t)}{\partial t^2}\bigg|_{x=l} = -EA\frac{\partial w(x,t)}{\partial x}\bigg|_{x=l} - kw(x,t)\big|_{x=l}$$

6.22 Calculate the natural frequency equation for the system of Problem 6.21.

Solution:

The boundary condition at $x = 0$ is just $w(x,t)|_{x=0} = 0$. Again from separation of variables

$$\ddot{T}(t)/T(t) = -c^2\sigma^2, \quad X(x) = a\sin\sigma x + b\cos\sigma x$$

Applying the boundary condition at 0 yields $X(0) = 0 = b$, so the spatial solution will be of the form $X(x) = a\sin\sigma x$. Substitution of the separated form $w(x,t) = X(x)T(t)$ into the boundary condition at l yields (from problem 6.21)

$$mX(l)\ddot{T}(t) = -kX(l)T(t) - EAX'(l)T(t)$$

Dividing by $T(t)$, and substitution of $\ddot{T}/T = -\sigma^2 c^2$ and $X = a \sin \sigma l$ yields

$$- EA\sigma \cos \sigma l = (-m\sigma^2 c^2 + k)\sin \sigma l \quad \text{or} \quad \tan \sigma l = -\frac{EA\sigma}{k - m\sigma^2 c^2} \quad \text{is the}$$

frequency or characteristic equation. Note that this reduces to the values given in Table 6.1 for the special case $m = 0$ and for the case $k = 0$.

6.23   Estimate the natural frequencies of an automobile frame for vibration in its longitudinal direction (i.e., along the length of the car) by modeling the frame as a (one-dimensional) steel bar.

Solution:

Note: The fundamental frequency of an automobile is of primary importance in assuming the quality of an automobile. While an automobile certainly has numerous modes, its fundamental frequency apparently has a large correlation with the occupants perception of quality. The fundamental frequency of a Mercedes 300 series is 25 Hz. Infinity and Lexus have frequencies in the low twenties. This problem has no straightforward answer. Students should think about their own cars or that of their family. For steel $\rho = 7.8 \times 10^3$ kg/m^2, $E = 2.0 \times 10^{11}$ N/m. For a Ford Taurus $l = 4.5$ m and assume the width to be 1 meter. The frequency equation in Hertz of a free-free beam is (excluding the rigid body mode)

$$f_n = \frac{n}{2\pi} \frac{\pi}{l} \sqrt{\frac{E}{\rho}} = 562 \text{ Hz, } 1125 \text{Hz} \dots$$

where $n = 1,2,\dots$ The frequency measured by auto engineers is from a 3 dimensional finite element model and modal test data. The frequency most felt is probably a transverse frequency.

6.24 Consider the first natural frequency of the bar of Problem 6.21 with $k = 0$ and Table 6.2, which is fixed at one end and has a lumped-mass, M, attached at the free end. Compare this to the natural frequency of the same system modeled as a single-degree-of-freedom spring-mass system given in Figure 1.21. What happens to the comparison as M becomes small and goes to zero?

Solution:

From figure 1.21, $k = EA/l$ is the stiffness of a cantilevered bar. Hence the frequency is

$$\omega_n = \sqrt{k/m} = \sqrt{\frac{EA}{lm}}$$

for the bar with tip mass m modeled as a single degree of freedom system. Now consider the first natural frequency of the distributed mass model of the same structure given in the last entry of table 6.1.

$$\omega_1 = \frac{\lambda_1 c}{l} = \frac{\lambda_1}{l}\sqrt{\frac{E}{\rho}}$$

where λ_1 satisfies $\cot \lambda_1 = \left(\dfrac{m}{\rho A l}\right)\lambda_1$. This last expression can be written as $\lambda_1 \tan \lambda_1 \left(\dfrac{\rho c l}{m}\right)$ since $\lambda_1 = \omega_1 l/c$,

$$\frac{\omega_1 l}{c}\tan\left(\frac{\omega_1 l}{c}\right) = \frac{\rho A l}{m}$$

Now for small, or negligible beam mass, c becomes very large $\left(c = \sqrt{E/\rho}\right)$ and $\omega_1 l/c$ becomes small so that $\tan\theta$ can be approximated as θ. Then this last expression becomes

$$\left(\frac{\omega_1 l}{c}\right)^2 = \frac{\rho A l}{m}, \text{ or } \omega_1 = \sqrt{\frac{EA}{lm}}$$

in agreement with the single degree of freedom values of figure 1.21. As the tip mass goes to zero, the equation for figure 1.21 does not appear to make sense. The equation for ω_1 however reduces to that of a cantilevered beam, i.e., $\omega_1 = \pi c/2l$ since the frequency equation returns to $\omega_1(l/c) = 0$.

6.25 Following the line of thought suggested in Problem 6.24, model the system of Problem 6.21 as a lumped-mass single-degree-of-freedom system and compare this frequency to the first natural frequency obtained in Problem 6.22.

Solution: Note that the system of figure P6.21 is a mass connected to two springs in parallel if the bar is modeled as spring. The stiffness of a bar is given in Chapter 1 to be

$$k_{bar} = \frac{EA}{\ell}$$

The equivalent stiffness is just the sum, so that the equation of motion is

$$m\ddot{x} + \left(\frac{EA}{\ell} + k\right)x = 0$$

Thus the natural frequency of the bar and spring of figure P6.21 modeled as a single degree of freedom system is just

$$\omega_n = \sqrt{\frac{EA}{m\ell} + \frac{k}{m}}$$

The first natural frequency of the system treated as a distributed mass systems is given by the characteristic equation given in the solution to problem 6.22. To make a comparison, chose some specific values. For a 4 m aluminum beam connected to 1000 kg mass through a 100,000 N/m spring the value is given in the following Mathcad session:

$k := 100000 \qquad m := 1000 \qquad \rho := 7.8 \cdot 10^3 \qquad E := 2 \cdot 10^{11}$

$L := 4 \qquad A := 0.2 \cdot 0.5 \qquad \rho \cdot A = 780$

$$c := \sqrt{\frac{E}{\rho}} \qquad \omega n := \sqrt{\frac{E \cdot A}{m \cdot L} + \frac{k}{m}}$$

$$\omega n = 2.236 \cdot 10^3$$

$$\sigma := \frac{\pi}{6}$$

$$f(\sigma) := \tan(\sigma \cdot L) + \frac{E \cdot A \cdot \sigma}{\left(m \cdot \sigma^2 \cdot c^2\right) + k}$$

$$root(f(\sigma), \sigma) = 0.545$$

Note for the parameter values chose the frequency of the lumped mass model is a little less then the actual value.

$$\omega 1 := c \cdot \sigma$$

$$\omega 1 = 2.651 \cdot 10^3$$

6.26 Calculate the response of a clamped-free bar to an initial displacement 1 cm at the free end and a zero initial velocity. Assume that $\rho = 7.8 \times 10^3$ kg/m^3, $A = 0.001$ m^2, $E=10^{10}$ N/m^2, and $l = 0.5$ m. Plot the response at $x = l$ and $x = l/2$ using the first three modes.

Solution:

The initial conditions are $w(x,t) = 0.01\delta(x\text{-}l)$ and $w_t(x,0) = 0$ and the boundary conditions are $w(0,t) = 0$ and $w_x(l,t) = 0$. From example 6.3.1 the mode shapes are $\sin\left(\dfrac{2n-1}{2l}\right)\pi x$ and the natural frequencies are

$$\omega_n = \left(\frac{2n-1}{2l}\right)\sqrt{\frac{E}{\rho}} = (2n-1)(1132.38)$$

The solution is given in example 6.3.2 as

$$w(x,t) = \sum(c_n \sin\omega_n t + d_n \cos\omega_n t)\sin\left(\frac{2n-1}{2l}\right)\pi x$$

so that the velocity is

$$w_t(x,t) = \sum_{n=1}^{\infty}(\omega_n c_n \cos\omega_n t - d_n\omega_n \sin\omega_n t)\sin\left(\frac{2n-1}{2l}\right)\pi x$$

Using $w_t(x,0) = 0$ then yields $c_n = 0$ for $n = 1, 2, \ldots$, so that

$$0.01\delta(x-l) = \sum d_n \cos\omega_n t\sin\frac{2n-1}{2l}\pi x$$

Multiplying by $\sin\dfrac{2m-1}{2l}\pi x$ and integrating from 0 to l yields

$$0.01\int_0^l \delta(x-l)\sin\left(\frac{2m-1}{2l}\right)\pi x dx = c_m\int_0^l \sin^2\left(\frac{2m-1}{2l}\right)\pi x dx$$

using the orthogonality of $\sin \sigma_n x$..

$$0.01\sin\frac{2m-1}{2}\pi = c_m\frac{l}{2}, \ m = 1,2,3...$$

so that $c_m = (.02)(-1)^{m+1}/l = (.004)(-1)^{m+1}$ and the solution is

$$w(x,t) = \sum_{n=1}^{\infty} (.004)(-1)^{n+1} \sin[(2n-1)(1132.28)t]\sin(2n-1)\pi x$$

For $n = 3$ and $x = 0.5$,

$$w(0.5,t) = (.004)[\sin 1132.28t - 0 - \sin 33968t]$$

For $n = 3$ and $x = l/2 = 0.25$

$$w(.25,t) = (.004)[.707\sin 1132.28 - \sin 2264.56t + .707\sin 339684t]$$
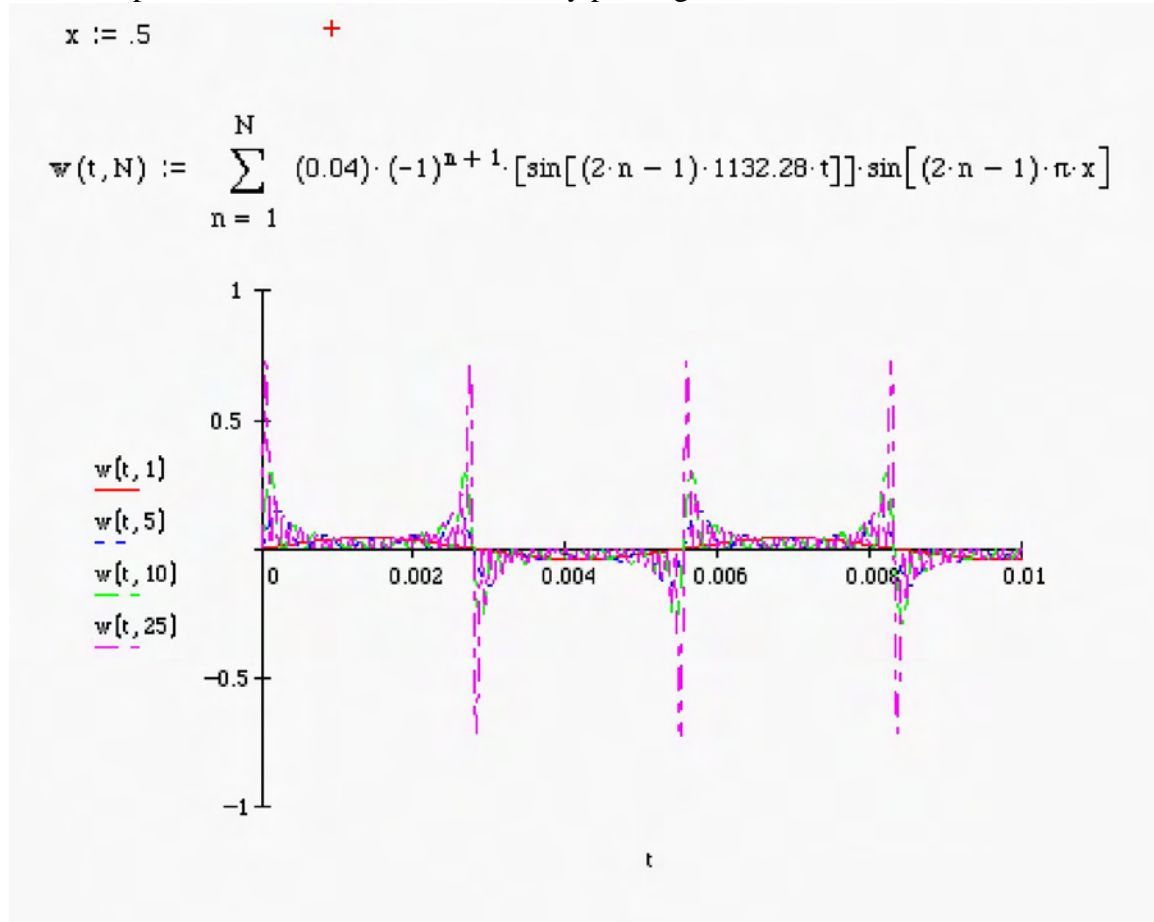
These are plotted below using Mathcad:

w1 (t) := 0.004· (sin (1132.28· t) − sin (339684· t))

w2 (t) := 0.004· (0.707· sin (1132.28· t) − sin (2264.56· t) + 0.707· sin (339684· t))
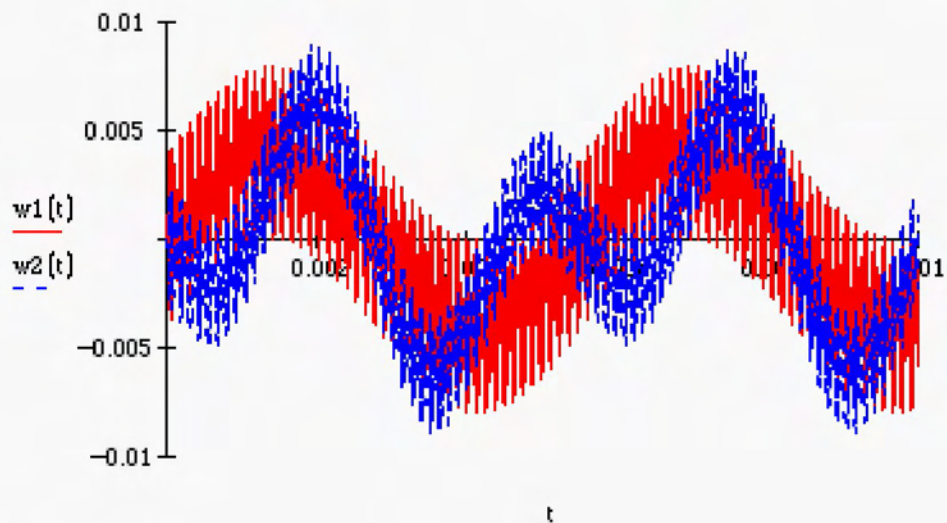
6.27   Repeat the plots of Problem 6.26 for 5 modes, 10 modes, 15 modes, and so on, to answer the question of how many modes are needed in the summation of equation (6.27) in order to yield an accurate plot of the response for this system.

Solution: The following plots in Mathcad illustrate that it takes 10 modes to capture the behavior of this series, by plotting the formula of 6.26.

$x := .5$ +

$$w(t,N) := \sum_{n=1}^{N} (0.04) \cdot (-1)^{n+1} \cdot \left[\sin\left[(2 \cdot n - 1) \cdot 1132.28 \cdot t\right]\right] \cdot \sin\left[(2 \cdot n - 1) \cdot \pi \cdot x\right]$$

$w(t, 1)$
$w(t, 5)$
$w(t, 10)$
$w(t, 25)$

6.28   A moving bar is traveling along the x axis with constant velocity and is suddenly stopped at the end at $x = 0$, so that the initial conditions are $(x,0) = 0$ and $w(x,0) = v$. Calculate the vibration response.

Solution:

Model the bar as a free-free bar. Then from Table 6.2 the natural frequencies are $n\pi c/l$ and the mode shapes are $\cos(n\pi x/l)$. Thus the solution is of the form

$$w(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t)\cos(n\pi x/l)$$

Using the initial condition $w(x,t) = 0$ yields that $B_n = 0$ for $n = 1, 2, 3,\ldots$, i.e.

$$w(x,0) = 0 = \sum B_n \cos(n\pi x/l)$$

which is multiplied by $\cos(n\pi x/l)$ and integrated over $(0,l)$ using orthogonality to get $B_n = 0$. Next differentiate

$$w(x,t) = \sum A_n \sin \omega_n t \cos n\pi x/l$$

to get $w_z(x,t)$, then set $t = 0$ to use the second initial condition.

$$w_t(x,0) = \sum A_n \omega_n \cos(0) \cos(n\pi x/l)$$

Modeling the initial velocity as $v\delta(x)$, multiplying by $\cos m\pi x/l$ and integrating yields

$$\int_0^l \delta(x)v\cos(n\pi x/l)dx = \omega_n\left(\frac{l}{2}\right)A_n, \quad \text{or} \quad A_n = \frac{\partial V}{l\omega_n}$$

so that

$$w(x,t) = \frac{2v}{\pi c}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

Note that Thomson uses a form of this problem as example 3 of section 5.3, but he models the moving beam as having a clamped free rather than free-free boundary. What do you think?

6.29 Calculate the response of the clamped-clamped string of Section 6.2 to a zero initial velocity and an initial displacement of $w_0(x) = \sin(2\pi x/l)$. Plot the response at $x = l/2$.

Solution:

The clamped-clamped string has eigenfunction $\sin n\pi x/l$ and solution given by equation (6.27) where the unknown coefficients c_n and d_n are given by equation (6.31) and (6.33) respectively. Since $\dot{w}_0 = 0$, equation 6.33 yields $c_n = 0$, $n = 1,2,3..$ with $w_0 = \sin(2\pi x/l)$,

$$d_n = \frac{2}{l}\int_0^l \sin(2\pi x/l)\sin(n\pi x/l)dx$$

which is zero for each n except $n = 2$, in which case $d_n = 1$. Hence

$$w(x,t) = \sin(2\pi ct/l)\sin(2\pi x/l)$$

For $x = l/2$

$$w(l/2,t) = \sin(2\pi ct/l)$$

which has a well known plot given in the following Mathcad session using the values for a piano wire.

$$c := 11.89 \quad L := 1.4$$

$$w(x,t) := \left[\sin\left[(2)\cdot\pi\cdot c\cdot\frac{t}{L}\right]\right]\cdot\sin\left(2\cdot\pi\cdot\frac{x}{L}\right)$$
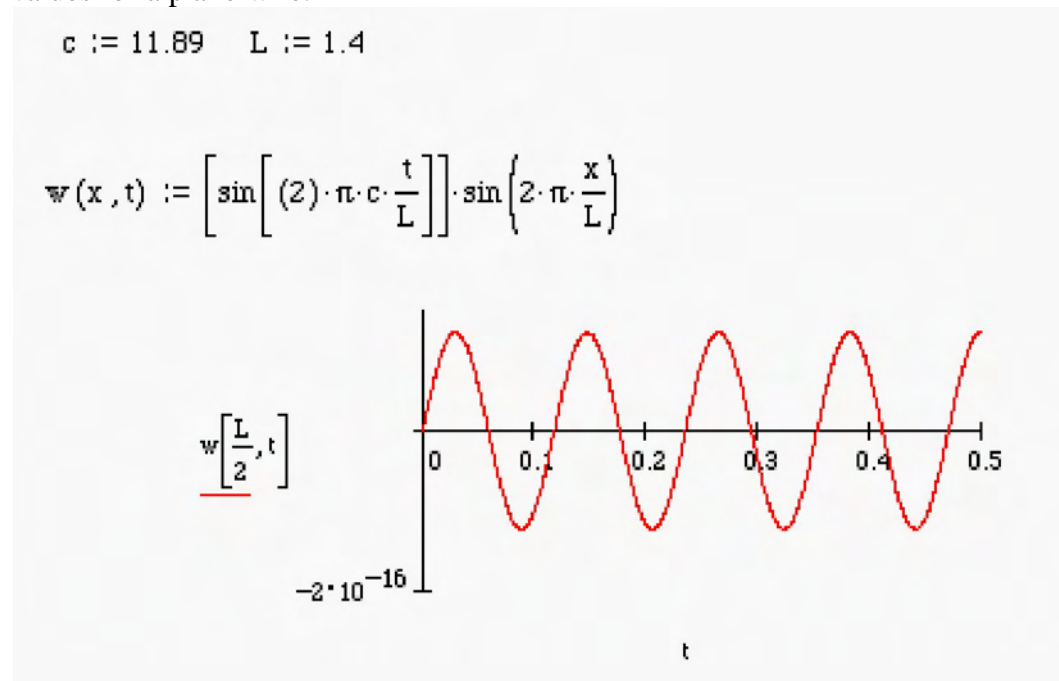
Problems and Solutions Section 6.4 (6.30 through 6.39)

6.30   Calculate the first three natural frequencies of torsional vibration of a shaft of Figure 6.7 clamped at $x = 0$, if a disk of inertia $J_0 = 10$ kg m^2/rad is attached to the end of the shaft at $x = l$. Assume that $l = 0.5$ m, $J = 5$ m^4, $G = 2.5 \times 10^9$ Pa, $\rho = 2700$ kg/m^3.

Solution: The equation of motion is $\ddot{\theta} = \dfrac{G}{\rho}\theta''$. Assume separation of variables:

$\theta = \phi(X)q(t)$ to get $\phi\ddot{q} = \dfrac{G}{\rho}\phi''q$ or $\dfrac{\rho}{G}\dfrac{\ddot{q}}{q} = \dfrac{\phi''}{\phi} = -\sigma^2$ so that

$$\ddot{q} + \dfrac{G}{\rho}\sigma^2 q = 0 \text{ and } \phi'' + \sigma^2\phi = 0$$

where $\omega^2 = \dfrac{G}{\rho}\sigma^2$. The clamped-inertia boundary condition is $\theta(0,t) = 0$, and

$-GJ\theta'(l,t) = J_0\ddot{\theta}(l,t)$. This yields that $\phi(0) = 0$ and

$$GJ\phi'(l)q(t) = J_0\phi(l)\ddot{q}(t) = J_0\phi(l)\dfrac{G}{\rho}\sigma^2 q(t)$$

or $J\phi'(l) = J_0\dfrac{\sigma^2}{\rho}\phi(l)$

The solution of the spatial equation is of the form

$$\phi(x) = A\sin\sigma x + B\cos\sigma x$$

but the clamped boundary condition yields $B = 0$. The inertia boundary condition

$$JA\sigma\cos\sigma l = J_0\dfrac{\sigma^2}{\rho}A\sin\sigma l$$

yields

$$\tan\sigma l = \dfrac{J}{J_0}\dfrac{\rho l}{\sigma l} = \dfrac{1}{\sigma l}\left(\dfrac{5\text{ m}^4}{10\text{kg m}^2}\right) \qquad (2700\text{kg/m}^3)(0.5\text{m})$$

So the frequency equation is

$$\tan\sigma l = \dfrac{675}{\sigma l}$$

Using the MATLAB function **fsolve**; this has the solutions

$\left.\begin{array}{l}\sigma_1 l = 1.5685\\ \sigma_2 l = 4.7054\\ \sigma_3 l = 7.8424\end{array}\right\}$ or $\left\{\begin{array}{l}\sigma_1 = 3.1369\\ \sigma_2 = 9.4108\\ \sigma_3 = 15.6847\end{array}\right.$

Thus $\omega_1 = 3018.5$ rad/s $\Rightarrow$ $f_1 = 480.4$ Hz

$\omega_2 = 9055.6$ rad/s $\Rightarrow$ $f_2 = 1441.2$ Hz

$\omega_3 = 15092.6$ rad/s $\Rightarrow$ $f_3 = 2402.1$ Hz

6.31 Compare the frequencies calculated in the previous problem to the frequencies of the lumped-mass single-degree-of-freedom approximation of the same system.

Solution:

First calculate the equivalent torsional stiffness of the rod.

$$k = \frac{GJ}{l} = \frac{(2.5 \times 10^9)(5)}{0.5} = 2.5 \times 10^{10}$$

$$J_0 \ddot{\theta} = -k\theta$$

$$J_0 \ddot{\theta} + k\theta = 0$$

$$10\ddot{\theta} + 2.5 \times 10^{10}\theta = 0 \quad \text{or} \quad \ddot{\theta} + 2.5 \times 10^9 \theta = 0$$

so that $\omega^2 = 2.5 \times 10^9$, $\omega = 5 \times 10^5$ rad/s or about 80,000 Hz, far from the 482 Hz of problem 6.30.

6.32 Calculate the natural frequencies and mode shapes of a shaft in torsion of shear modulus G, length l, polar inertia J, and density ρ that is free at $x = 0$ and connected to a disk of inertia J_0 at $x = l$.

Solution:

Assume zero initial conditions, i.e. $\theta(x,0) = \dot{\theta}(x,0) = 0$. From equation 6.66

$$\frac{\partial^2 \theta(x,t)}{\partial t^2} = \left(\frac{G}{\rho}\right) \frac{\partial^2 \theta(x,t)}{\partial x^2} \tag{1}$$

The boundary condition at $x = l$ and at $x = 0$ is

$$GJ\frac{\partial \theta(l,t)}{\partial x} = -J_0 \frac{\partial^2 \theta(l,t)}{\partial t^2} \qquad \frac{\partial \theta(0,t)}{\partial x} = 0$$

Using separation of variable in (1) of form $\theta(x,t) = \Theta(x)\mathrm{T}(t)$ yields:

$$\frac{\Theta''(x)}{\Theta(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = -\sigma^2 \tag{2}$$

where $c^2 = \dfrac{G}{\rho}$ and σ^2 is a separation constant. (2) can now be rewritten as 2 equations

$$\Theta''(x) + \sigma^2 \Theta(x) = 0$$

$$\ddot{T}(t) + c^2\sigma^2 T(t) = 0 \rightarrow \omega = \sigma = \sigma\sqrt{\frac{G}{\rho}}$$

from the boundary condition at $x = l$

$$GJ\Theta'(l)T(t) = -J_0 \Theta(l)\ddot{T}(t)$$

$$-\frac{GJ}{J_0}\frac{\Theta'(l)}{\Theta(l)} = \frac{\ddot{T}(t)}{T(t)} = -c^2\sigma^2$$

$$\Theta'(l) = \frac{J_0}{GJ}\frac{G}{\rho}\sigma^2\Theta = \frac{J_0\sigma^2}{J\rho}\Theta(l)$$

The boundary condition at $x = 0$ yields simply $\Theta'(0) = 0$. The general solution is of the form

$$\Theta(x) = a_1 \sin \sigma x + a_2 \cos \sigma x \text{ so that } \Theta'(x) = a_1 \sigma \cos \sigma x - a_2 \sigma \sin \sigma x$$

The boundary conditions applied to these solutions yield:

$$\Theta'(l) = a_1 \sigma \cos \sigma l - a_2 \sigma \sin \sigma l = \frac{J_0 \sigma^2}{J\rho}[a_1 \sin \sigma l + a_2 \cos \sigma l]$$

$$a_1\left[\cos \sigma l - \frac{J_0 \sigma^2}{J\rho}\sin \sigma l\right] = a_2\left[\sin \sigma l + \frac{J_0 \sigma}{J\rho}\cos \sigma l\right]$$

$$\Theta'(0) = a_1 \sigma = 0 \rightarrow a_1 = 0$$

$$a_2\left[\sin \sigma l + \frac{J_0 \sigma}{J\rho}\cos \sigma l\right] = 0$$

For the non-trivial solution of this last expression, the coefficients of a_2 must vanish, which yields

$$\tan \sigma l = -\frac{J_0}{J\rho}\sigma$$

This must be solved numerically for σ (except for the rigid body case of $\sigma = 0$) and the frequency is calculated from $\omega = \sigma\sqrt{\dfrac{G}{\rho}}$. The mode shapes are $\Theta(x) = a_2$ cos σx. Note the solution for σ is illustrated in figure 6.4 page 479 of the text.

6.33 Consider the lumped-mass model of Figure 4.21 and the corresponding three-degree-of-freedom model of Example 4.8.1. Let $J_1 = k_1 = 0$ in this model and collapse it to a two-degree-of-freedom model. Comparing this to Example 6.4.1, it is seen that they are a lumped-mass model and a distributed mass model of the same physical device. Referring to Chapter 1 for the effects of lumped stiffness on a rod in torsion (k_2), compare the frequencies of the lumped-mass two-degree-of-freedom model with those of Example 6.4.1.

Solution: From Mathcad:

$$ro := 7800 \qquad J := 5 \qquad J_1 := 10 \qquad J_2 := 10 \qquad L := 0.425 \qquad G := 80 \cdot 10^9$$

$$a := \frac{ro \cdot J \cdot L}{J_1 + J_2} \qquad\qquad b := \frac{J_1 \cdot J_2}{(J_1 + J_2) \cdot ro \cdot J \cdot L} \qquad \text{from p.492}$$

$$a = 828.75 \qquad\qquad\qquad b = 3.017 \times 10^{-4}$$

$$f(x) := (b \cdot x^2 - a) \cdot \tan(x) \qquad\qquad g(x) := x$$

$$x_1 := 0 \qquad\qquad x_2 := 3.138$$

$$\omega_1 := x_1 \cdot \sqrt{\frac{G}{ro \cdot L^2}} \qquad\qquad \boxed{\omega_1 = 0}$$
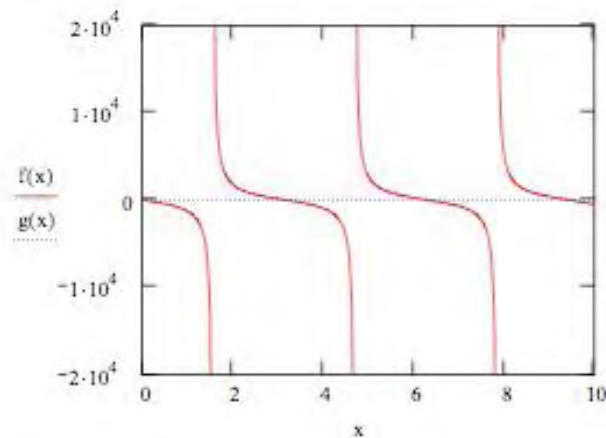
$$\omega_2 := x_2 \cdot \sqrt{\frac{G}{ro \cdot L^2}} \qquad \omega_2 = 2.365 \times 10^4 \qquad f_2 := \frac{\omega_2}{2\pi} \qquad \boxed{f_2 = 3.763 \times 10^3}$$

2 dof model

$$k := \frac{G \cdot J}{L} \qquad k = 9.412 \times 10^{11}$$

$$M := \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \qquad K := \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

$$M = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \qquad K = \begin{pmatrix} 9.412 \times 10^{11} & -9.412 \times 10^{11} \\ -9.412 \times 10^{11} & 9.412 \times 10^{11} \end{pmatrix}$$

$$\text{sort}\left(\text{eigenvals}\left(M^{-1} \cdot K\right)\right) = \begin{pmatrix} 1.526 \times 10^{-5} \\ 1.882 \times 10^{11} \end{pmatrix}$$

Natural frequencies: $\qquad \text{nat\_freqs} := \text{sort}\left(\sqrt{\text{eigenvals}\left(M^{-1} \cdot K\right)}\right)$

$$\frac{\text{nat\_freqs}}{2\pi} = \begin{pmatrix} 6.217 \times 10^{-4} \\ 6.905 \times 10^{4} \end{pmatrix}$$

6.34 The modulus and density of a 1-m aluminum rod are $E = 7.1 \times 10^{10}$ N/m^2, $G = 2.7 \times 10^{10}$ N/m^2, and $\rho = 2.7 \times 10^3$ kg/m^2. Compare the torsional natural frequencies with the longitudinal natural frequencies for a free-clamped rod.

Solution:

The appropriate boundary conditions are: $\theta'(0,t) = 0$ and $\theta(l,t) = 0$ for the rod and $w'(0,t) = 0 = w(l,t)$ for the bar. The separated equations are

$$\ddot{\theta} = \left(\frac{G}{\rho}\right)\theta'' \quad \text{and} \quad \phi\ddot{q} = \left(\frac{G}{\rho}\right)\phi''q$$

$$\ddot{q} + \left(\frac{G}{\rho}\right)\sigma^2 q = 0 \quad \text{and} \quad \phi'' + \sigma^2\phi = 0$$

Solutions are

$$q_n = A_n \sin\omega_n t + B_n \cos\omega_n t \quad \text{and} \quad \phi_n = C_n \sin\sigma_n x + D_n \cos\sigma_n x$$

where $\omega_n^2 = \dfrac{G}{\rho}\sigma_n^2$. But $\phi'(0) = 0$ so that $C_n = 0$. The other boundary condition yields $\phi_n(l) = D_n\cos\sigma_n l = 0$ so that

$$\sigma_n l = \frac{(2n-1)\pi}{2}, \quad n = 1,2,...$$

Thus the torsional frequencies are

$$\omega_n = \sqrt{\frac{G}{\rho}}\sigma_n$$

and the longitudinal frequencies are

$$\omega_n = \sqrt{\frac{E}{\rho}}\sigma_n$$

where

$$\sigma_n = \frac{(2n-1)\pi}{2l}$$

From the values given $\sqrt{\dfrac{G}{\rho}} = 3162$ m/s and $\sqrt{\dfrac{E}{\rho}} = 5128$ m/s. Thus the natural frequencies of the longitudinal vibration are *1.6 times larger* than the torsional vibrations.

6.35   Consider the aluminum shaft of Problem 6.32. Add a disk of inertia J_0 to the free end of the shaft. Plot the torsional natural frequencies versus increasing the tip inertia J_0 of a single-degree-of-freedom model and for the first natural frequency of the distributed-parameter model in the same plot. Are there any values of J_0 for which the single-degree–of-freedom model gives the same frequency as the full distributed model?

Solution:

Refer to problem 6.32 of the rod clamped at $x = 0$ with inertia J_0 at $x = l$. The *sdof* model of the frequency is given in example 1.5.1 as

$$\omega = \sqrt{\frac{GJ}{lJ_0}}$$

where G = torsional rigidity, J = polar moment of inertia of the rod of length l and J_0 is the disc inertia. The first natural frequency according to distributed parameter theory is given in problem 6.30 as the solution of

$$\tan \sigma / 2 = -\frac{\rho}{\sigma J_0}, \quad \omega = \sigma \frac{G}{\rho}$$

which will have a solution for a given value of J_0 equivalent to that of the *sdof* system.

6.36 Calculate the mode shapes and natural frequencies of a bar with circular cross section in torsional vibration with free-free boundary conditions. Express your answer in terms of G, l, and ρ.

Solution:

The separated equations are $\ddot{q} + \left(\dfrac{G}{\rho} \sigma^2 \right) q = 0$ and $\phi'' + \sigma^2 \phi = 0$

where $\omega_n = \sqrt{\dfrac{G}{\rho}} \sigma_n$. Thus

$$q_n = A_n \sin \omega_n t + B_n \cos \omega_n t \quad \text{and} \quad \phi_n = C_n \sin \sigma_n x + D_n \cos \sigma_n x$$

The boundary conditions are

$$\phi_n'(0) = 0$$
$$\phi_n'(l) = 0$$

But $\phi_n' = C_n \sigma_n \cos \sigma_n x - D_n \sigma_n \sin \sigma_n x$ so that $\phi_n'(0) = 0 \Rightarrow C_n = 0$ and the frequency equation becomes $\phi_n'(l) = 0 = -D_n \sigma_n \sin \sigma_n 0$. This has the solution

$\sigma_n l = n\pi$ or $\sigma_n = \dfrac{n\pi}{l}$. Hence

$$\omega_n = \sqrt{\dfrac{G}{\rho}} \dfrac{n\pi}{l} \quad \text{and} \quad \phi_n(x) = \cos \dfrac{n\pi x}{l}.$$

6.37   Calculate the mode shapes and natural frequencies of a bar with circular cross section in torsional vibration with fixed boundary conditions. Express you answer in terms of G, l, and ρ,

Solution: From equation 6.66

$$\frac{\partial^2 \theta(x,t)}{\partial t^2} = \left(\frac{G}{\rho}\right)\frac{\partial^2 \theta(x,t)}{\partial x^2}$$

Assume a solution of the form $\theta(x,t) = \Theta(x)T(t)$ so that

$$\Theta(x)\ddot{T}(t) = \frac{G}{\rho}\Theta''(x)T(t)$$

Separate where σ^2 is the separation constant and $c^2 = \dfrac{G}{\rho}$

$$\frac{\Theta''(x)}{\Theta(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = -\sigma^2$$

or $\Theta''(x) + \sigma^2\Theta(x) = 0$ and $\ddot{T}(t) + \sigma^2 c^2 T(t) = 0$ where $\omega = \sqrt{\dfrac{G}{\rho}}\,\sigma$. The boundary conditions for a fixed-fixed rod are $\Theta(0) = 0$ and $\Theta(l) = 0$ from the solution of the spatial equations

$$\Theta(0) = a_2 = 0$$
$$\Theta(l) = a_1 \sin \sigma l = 0.$$

.

For the non-trivial solution

$$\sin \sigma l = 0$$
$$\sigma = \frac{n\pi}{l}, \quad n = 0,1,2,..$$

natural frequency

$$\omega = \sqrt{\frac{G}{\rho}}\frac{n\pi}{l}, \quad n = 1,2,...$$

mode shape

$$\Theta(x) = a_1 \sin\frac{n\pi}{l}x, \quad n = 0,1,2,...$$

6.38   Calculate the eigenfunctions of Example 6.4.1.

Solution:

From example 6.4.1 the eigenfunctions are

$$\Theta_n(x) = a_1 \sin \sigma_n x + a_2 \cos \sigma_n x \quad \text{or} \quad \Theta_n(x) = A_n \left(-\frac{\sigma J_1}{\rho J} \sin \sigma_n x + \cos \sigma_n x \right)$$

where σ_n are determined by equation 6.8.4.

6.39   Show that the eigenfunctions of Problem 6.38 are orthogonal.

Solution:

Orthogonality requires $\int_0^1 \theta_n(x)\theta_m(x)dx = 0, \quad m \neq n.$ From direct calculation

$$\int_0^l \left(-\frac{\sigma J_1}{\rho J} \sin \sigma_n x + \cos \sigma_n x \right)\left(-\frac{\sigma J_1}{\rho J} \sin \sigma_m x + \cos \sigma_m x \right) dx$$

$$= \left(\frac{\sigma J_1}{\rho J} \right)^2 \int_0^l \sin \sigma_m x \sin \sigma_n x dx$$

$$- \frac{\sigma J_1}{\rho J} \int_0^l \sin \sigma_n x \sin \sigma_m x dx - \frac{\sigma J_1}{\rho J} \int_0^l \sin \sigma_m x \sin \sigma_n x dx$$

$$+ \int_0^l \cos \sigma_n x \cos \sigma_m x dx$$

where each integral vanishes. Also one can use the same calculation as problem 6.3 since the natural frequencies have distinct values.

Problems and Solutions Section 6.5 (6.40 through 6.47)

6.40 Calculate the natural frequencies and mode shapes of a clamped-free beam. Express your solution in terms of E, I, ρ, and l. This is called the cantilevered beam problem.

Solution:

Clamped-free boundary conditions are

$$w(0,t) = w_x(0,t) = 0 \quad \text{and} \quad w_{xx}(l,t) = w_{xxx}(l,t) = 0$$

assume E, I, ρ, l constant. The equation of motion is

$$\frac{\partial^2 w}{\partial t^2} + \left(\frac{EI}{\rho A}\right)\frac{\partial^4 w}{\partial x^4} = 0$$

assume separation of variables $\quad w(x,t) = \phi(x)q(t)$ to get

$$\left(\frac{EI}{\rho A}\right)\frac{\phi''''}{\phi} = -\frac{\ddot{q}}{q} = \omega^2$$

The spatial equation becomes

$$\phi'''' - \left(\frac{\rho A}{EI}\right)\omega^2\phi = 0$$

define $\beta^4 = \dfrac{\rho A\omega^2}{EI}$ so that $\phi'''' - \beta^4\phi = 0$ which has the solution:

$$\phi = C_1\sin\beta x + C_2\cos\beta x + C_3\sinh\beta x + C_4\cosh\beta x$$

Applying the boundary conditions
$$w(0,t) = w_x(0,t) = 0 \quad \text{and} \quad w_{xx}(l,t) = w_{xxx}(l,t) = 0 \Rightarrow$$
$$\phi(0) = \phi'(0) = 0 \quad \text{and} \quad \phi''(l) = \phi'''(l) = 0$$
Substitution of the expression for ϕ into these yields:
$$C_2 + C_4 = 0$$
$$C_1 + C_3 = 0$$
$$-C_1\sin\beta l - C_2\cos\beta l + C_3\sinh\beta l + C_4\cosh\beta l = 0$$
$$-C_1\cos\beta l + C_2\sin\beta l + C_3\cosh\beta l + C_4\sinh\beta l = 0$$
Writing these four equations in four unknowns in matrix form yields:

$$
\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
-\sin\beta l & -\cos\beta l & \sinh\beta l & \cosh\beta l \\
-\cos\beta l & \sin\beta l & \cosh\beta l & \sinh\beta l
\end{bmatrix}
\begin{bmatrix}
c_1 \\ c_2 \\ c_3 \\ c_4
\end{bmatrix} = 0
$$

For a nonzero solution, the determinant must be zero to that (after expansion)

$$
\begin{vmatrix}
-\sin\beta l - \sinh\beta l & -\cos\beta l - \cosh\beta \\
-\cos\beta l - \cosh\beta & \sin\beta l - \sinh\beta
\end{vmatrix} =
$$

$$
(-\sin\beta l - \sinh\beta l)(\sin\beta l - \sinh\beta l) -
$$
$$
(-\cos\beta l - \cosh\beta l)(-\cos\beta l - \cosh\beta l) = 0
$$

Thus the frequency equation is $\cos\beta l \cosh\beta l = -1$ or $\cos\beta_n l = -\dfrac{1}{\cosh\beta_n l}$ and

frequencies are $\omega_n = \sqrt{\dfrac{\beta_n^4 EI}{\rho A}}$. The mode shapes are

$$
\phi_n = C_{1n}\sin\beta_n x + C_{2n}\cos\beta_n x + C_{3n}\sinh\beta_n x + C_{4n}\cosh\beta_n x
$$

Using the boundary condition information that $C_4 = -C_2$ and $C_3 = -C_1$ yields

$$
-C_1\sin\beta l - C_2\cos\beta l - C_1\sinh\beta l - C_2\cosh\beta l
$$
$$
-C_1(\sin\beta l + \sinh\beta l) = C_2(\cos\beta l + \cosh\beta l)
$$

so that

$$
C_1 = -C_2\left(\frac{\cos\beta l + \cosh\beta l}{\sin\beta l + \sinh\beta l}\right)
$$

and the mode shapes can be expressed as:

$$
\phi_n = -C_{2n}\left[-\left(\frac{\cos\beta_n l + \cosh\beta_n l}{\sin\beta_n l + \sinh\beta_n l}\right)\sin\beta_n x + \cos\beta_n x\right.
$$

$$
\left. +\left(\frac{\cos\beta_n l + \cosh\beta_n l}{\sin\beta_n l + \sinh\beta_n l}\right)\sinh\beta_n x - \cosh\beta_n x\right]
$$

6.41 Plot the first three mode shapes calculated in Problem 6.40. Next calculate the strain mode shape [i.e., $X'(x)$], and plot these next to the displacement mode shapes $X(x)$. Where is the strain the largest?

Solution: The following Mathcad session yields the plots using the values of β taken from Table 6.4.

$$\beta := \begin{bmatrix} 1.87510407 \\ 4.69409113 \\ 7.85475744 \end{bmatrix}$$

$$\Phi(x,b) := \left[\left(-\frac{\cos(b) + \cosh(b)}{\sin(b) + \sinh(b)} \cdot \sin(b \cdot x) \right) + \cos(b \cdot x) + \left(\frac{\cos(b) + \cosh(b)}{\sin(b) + \sinh(b)} \right) \cdot \sinh(b \cdot x) - \cosh(b \cdot x) \right]$$

$$\left[\left(-\frac{\cos(b) + \cosh(b)}{\sin(b) + \sinh(b)} \cdot \sin(b \cdot x) \right) + \cos(b \cdot x) + \left(\frac{\cos(b) + \cosh(b)}{\sin(b) + \sinh(b)} \right) \cdot \sinh(b \cdot x) - \cosh(b \cdot x) \right]$$

$$\left[\left(\frac{-(\cos(b) + \cosh(b))}{(\sin(b) + \sinh(b))} \cdot \cos(b \cdot x) \cdot b - \sin(b \cdot x) \cdot b + \frac{(\cos(b) + \cosh(b))}{(\sin(b) + \sinh(b))} \cdot \cosh(b \cdot x) \cdot b \right) - \sinh(b \cdot x) \cdot b \right]$$

$$\Phi p(x,b) := \frac{-(-\cos(b) - \cosh(b))}{(\sin(b) + \sinh(b))} \cdot \sin(b \cdot x) \cdot b^2 - \cos(b \cdot x) \cdot b^2 + \frac{(\cos(b) + \cosh(b))}{(\sin(b) + \sinh(b))} \cdot \sinh(b \cdot x) \cdot b^2 - \cosh(b \cdot x) \cdot b^2$$
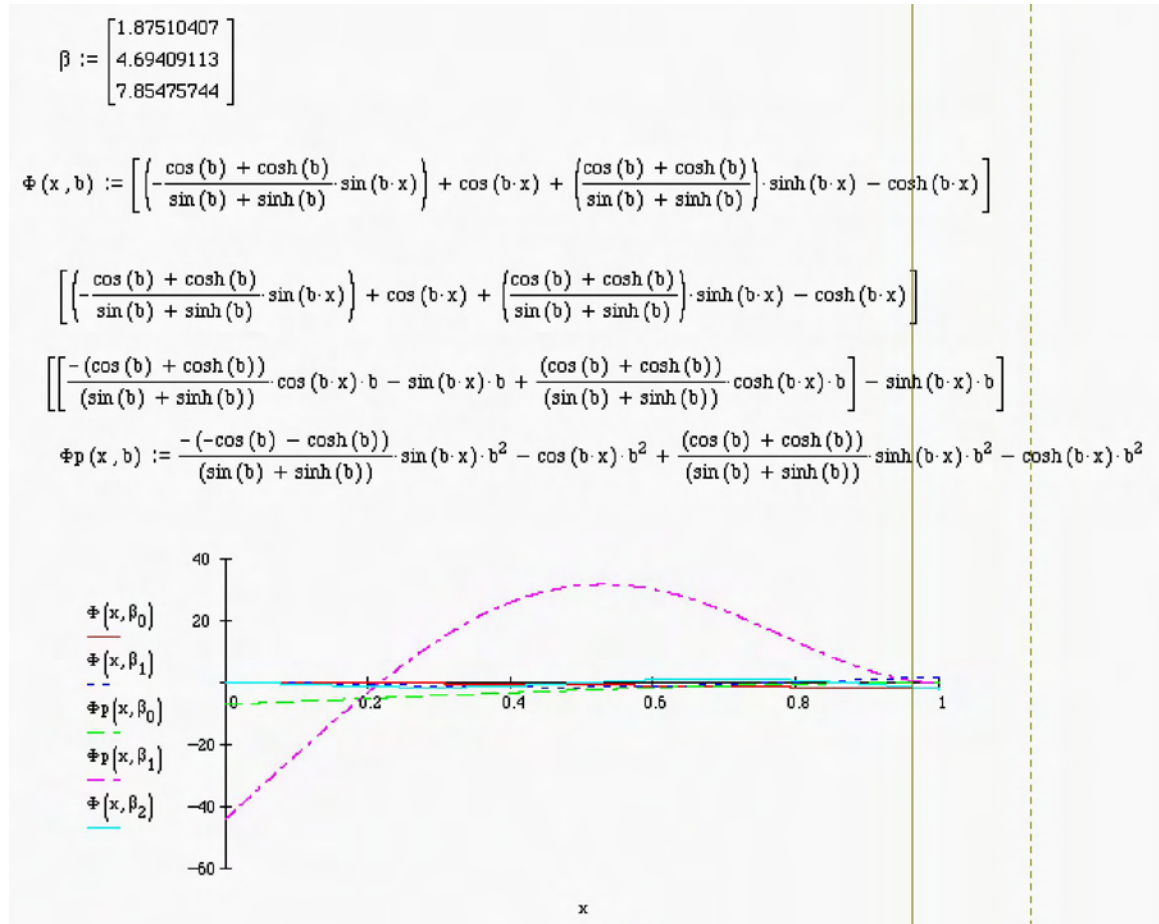
The strain is largest at the free end.

6.42 Derive the general solution to a fourth-order ordinary differential equation with constant coefficients of equation (6.100) given by equation (6.102).

Solution:

From equation (6.100) with $\beta^4 = \rho A \omega^2 / EI$, the problem is to solve $X'''' - \beta^4 X = 0$. Following the procedure for the second order equations suggested in example 6.2.1 let $X(x) = Ae^{\lambda t}$ which yields

$$\left(\lambda^4 - \beta^4 \right) A e^{\lambda x} = 0 \text{ or } \lambda^4 = \beta^4$$

This characteristic equation in λ has 4 roots

$$\lambda = -\beta, \beta, -\beta j, \text{ and } \beta j$$

each of which corresponds to a solution, namely $A_1 e^{-\beta x}$, $A_2 e^{\beta x}$, $A_3 e^{-\beta j x}$ and $A_4 e^{\beta j x}$. The most general solution is the sum of each of these or

$$X(x) = A_1 e^{-\beta x} + A_2 e^{\beta x} + A_3 e^{-\beta j x} + A_4 e^{\beta j x} \tag{a}$$

Now recall equation (A.19), i.e., $e^{\beta x} = \cos \beta x + j \sin \beta x$, and add equations (A.21) to yield $e^{\beta j x} = \sinh \beta x + \cosh \beta x$. Substitution of these two expressions into (a) yields

$$X(x) = A \sin \beta x + B \cos \beta x + C \sinh \beta x + D \cosh \beta x$$

where *A, B, C,* and *D* are combinations of the constants A_1, A_2, A_3 and A_4 and may be complex valued.

6.43 Derive the natural frequencies and mode shapes of a pinned-pinned beam in transverse vibration. Calculate the solution for $w_0(x) = \sin 2\pi x/l$ min and $\dot{w}_0(x) = 0$.

Solution: Use $w(x,t) = \phi(x)q(t)$ in equation (6.29) with $\dot{w}(x,0) = 0$ or $\dot{q}(0) = 0$. Then the temporal solution $q = A \sin \omega t + B \cos \omega t$ with $\dot{q}(0) = 0$ yields $A = 0$. The spatial solution is $\phi = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x$ where $\beta^4 = \dfrac{\rho A \omega^2}{EI}$. The boundary conditions become

$$\phi(0) = \phi''(0) = \phi(l) = \phi''(l) = 0$$

Applied to $\phi(x)$ these yield the matrix equation

$$
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & -1 & 0 & 1 \\
\sin \beta l & \cos \beta l & \sinh \beta l & \cosh \beta l \\
-\sin \beta l & -\cos \beta l & \sinh \beta l & \cosh \beta l
\end{bmatrix}
\begin{bmatrix}
C_1 \\ C_2 \\ C_3 \\ C_4
\end{bmatrix} = 0
$$

But $C_2 + C_4 = 0$ and $-C_2 + C_4 = 0$ so $C_2 = C_4 = 0$ and this reduces to

$$
\begin{bmatrix}
\sin \beta l & \sinh \beta \\
-\sin \beta & \sinh \beta
\end{bmatrix}
\begin{bmatrix}
C_1 \\ C_3
\end{bmatrix} = 0
$$

or $\sin \beta l \sinh \beta l + \sin \beta l \sinh \beta l = 0$,

$C_3 = -\dfrac{C_1 \sin \beta l}{\sinh \beta l}$, and $-C_1 \sin \beta l - C_1 \sin \beta l = 0$ so that the frequency equation

becomes $\sin \beta l = 0$ and thus $\beta_n l = n\pi$, $n = 1,2,3,\ldots$ and $\beta_n = \dfrac{n\pi}{l}$, $n = 1,2,3,\ldots$so

that $C_3 = 0$ and the frequencies are $\omega_n = \left(\dfrac{n\pi}{l}\right)^2 \sqrt{\dfrac{EI}{\rho A}}$ with mode shapes $\phi_n(x) =$

$C_{1n} \sin \beta_n x$. The total solution is the series $w(x,t) = \sum_{n=1}^{\infty} \left\{ \beta_n \cos \omega_n t \sin \beta_n x \right\}$.

Applying the second initial condition yields $w(x,0) = \sin \dfrac{2\pi x}{l} = \sum_{n=1}^{\infty} \beta_n \sin \dfrac{n\pi x}{l}$

and therefore

$$
B_n = \begin{cases}
0 & n = 1 \\
& n = 3,4,\ldots \\
1 & n = 2
\end{cases}
$$

so that

$$w(x,t) = \cos \omega_2 t \sin \frac{2\pi x}{l}$$

6.44 Derive the natural frequencies and mode shapes of a fixed-fixed beam in transverse vibration.

Solution: Follow example 6.5.1 to get the solution in the 5th entry of table 6.4. The spatial equation for the transverse vibration of a beam has solution of the form (6.102)

$$X(x) = a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x$$

where $\beta^4 = \rho A \omega^2 / EI$. The clamped boundary conditions are given by equation (6.94) as $X(0) = X'(0) = X(l) = X'(l) = 0$. Applying these boundary conditions to the solution yields

$$X(0) = 0 = a_1(0) + a_2(1) + a_3(0) + a_4(1) \tag{1}$$

$$X'(0) = 0 = \beta a_1(1) - \beta a_2(0) + \beta a_3(1) + \beta a_4(0) \tag{2}$$

$$X(l) = 0 = a_1 \sin \beta l + a_2 \cos \beta l + a_3 \sinh \beta l + a_4 \cosh \beta l \tag{3}$$

$$X'(l) = 0 = \beta a_1 \cos \beta l - \beta a_2 \sin \beta l + \beta a_3 \cosh \beta l + \beta a_4 \sinh \beta l$$

(4)

dividing (2) and (3) by $\beta \neq 0$ and writing in matrix form yields

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \sin \beta l & \cos \beta l & \sinh \beta l & \cosh \beta l \\ \cos \beta l & -\sin \beta l & \cosh \beta l & \sinh \beta l \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix must have zero determinant for a nonzero solution for the a_n. Taking the determinant yields (expanding by minors across the top row).

$$\sinh^2 \beta l - \cosh^2 \beta l - \sin \beta l \sinh \beta l + \cos \beta l \cosh \beta l +$$

$$\cos \beta l \cosh \beta l \sin \beta l \sinh \beta l - \sin^2 \beta l - \cos^2 \beta l = 0$$

which reduces to

$$-1 + 2 \cos \beta l \cosh \beta l - 1 = 0 \quad \text{or} \quad \cos \beta l \cosh \beta l = 1$$

since $\sinh^2 \beta l - \cosh^2 \beta l = -1$ and $\sin^2 x + \cos^2 x = 1$. The solutions of this characteristic equation are given in table 6.4. Next from equation (1) $a_2 = -a_4$ and from equation (2) $a_1 = -a_3$ so equation (3) can be written as

$$-a_3 \sin \beta l - a_4 \cos \beta l + a_3 \sinh \beta l + a_4 \cosh \beta l = 4$$

Solving this for a_3 yields

$$a_3 = a_4 \left(\frac{\cos \beta l - \cosh \beta l}{\sinh \beta l - \sin \beta l} \right)$$

Recall also that $a_1 = -a_3$. Substitution into the solution $X(x)$ and factoring out a_4 yields

$$X(x) = a_4 \left[\cosh \beta x - \cosh \beta x - \left(\frac{\cos \beta l - \cosh \beta l}{\sin \beta l - \sinh \beta l} \right) \left(\sinh \beta x - \sin \beta x \right) \right]$$

in agreement with table 6.4. Note that a_4 is arbitrary as it should be.

6.45 Show that the eigenfunctions or mode shapes of Example 6.5.1 are orthogonal. Make them normal.

Solution:

The easiest way to show the orthogonality is to use the fact that the eigenvalues are not repeated and follow the solution to problem 6.2. The eigenfunctions are (table 6.4 or example 6.5).

$$X_n(x) = a_n \left\{ \cosh \beta_n x - \cos \beta_n x - \sigma_n \left(\sinh \beta_n x - \sin \beta_n x \right) \right\}$$

Note that the constant a_n is arbitrary (a constant times a mode shape is still a mode shape) and normalizing involves choosing the constant a_n so that $\int X_n X_n dx = 1$. Calculating this integral yields:

$$a_n^2 \int_0^l \left\{ \cosh^2 \beta_n x - 2\cos \beta_n x \cosh \beta_n x + \cos^2 \beta_n x \right.$$
$$- 2\sigma_n \left(\sinh \beta_n x - \sin \beta_n x \right)\left(\cosh \beta_n x - \cos \beta_n x \right)$$
$$\left. + \sigma_n^2 \left(\sinh^2 \beta_n x - 2\sin \beta_n x \sinh \beta_n x + \sin^2 \beta_n x \right) \right\} dx$$

so

$$1 = a_n^2 \left[\frac{1}{\beta_n} \left(\frac{\sinh 2\beta_n l + \sin 2\beta_n l}{4} \right) + \beta_n l \right.$$
$$- \frac{1}{\beta_n} \left(\sinh \beta_n l \sin \beta_n l + \cos \beta_n l \cosh \beta_n l \right) - \frac{\sigma_n}{\beta_n} \cos^2 \beta_n l + \cosh 2\beta_n l$$
$$+ \sinh \beta_n l \left(\sin \beta_n l + \cos \beta_n l \right) - \cosh \beta_n l \left(\cos \beta_n - \sin \beta_n l \right)$$
$$\left. + \frac{\sigma_n^2}{\beta_n} \left[\frac{\sinh^2 \beta_n l - \sin 2\beta_n l}{4} - 1 - \sin \beta_n l \sinh \beta_n l + \cosh \beta_n l \cos \beta_n l \right] \right]$$

So denoting the term in [ ] as γ_n and solving for $a_n = 1/\sqrt{\gamma_n}$ yields the normalization constant.

6.46   Derive equation (6.109) from equations (6.107) and (6.108).

Solution:

Using subscript notation for the partial derivatives, equation (6.108) with $f = 0$ yields an expression for φ_x, i.e.

$$\varphi_x = (\kappa AGW_{xx} - \rho Aw_{tt})/\kappa^2 AG \tag{a}$$

Equation (6.107) can be differentiated once with respect to x to yield a middle term identical to the first term of equation (6.108). Substitution yields

$$EI\varphi_{xx} + \rho Aw_{tt} = \rho I\varphi_{xtt} \tag{b}$$

Equation (a) can be differentiated twice with respect to time to get an expression for $\rho I\varphi_{xx}$ in terms of $w(x,t)$ which when substituted into (b) yields

$$EI\varphi_{xxx} + \rho Aw_{tt} = \rho Iw_{xxtt} - \left(\rho^2 I/\kappa^2 G\right)w_{tttt}$$

The first term $EI\varphi_{xxx}$ can be eliminated by differentiating (a) twice with respect to x to yield

$$EI\left(\kappa^2 AGw_{xxxx} - \rho Aw_{ttxx}\right) + \rho Aw_{tt} = \kappa^2 AGw_{xxtt} - \rho AEIw_{tttt}$$

when substituted into (c). This is an expression in $w(x,t)$ only. Rearranging terms and dividing by $\kappa^2 AG$ yields equation (6.109).

6.47   Show that if shear deformation and rotary inertia are neglected, the Timoshenko equation reduces to the Euler-Bernoulli equation and the boundary conditions for each model become the same.

Solution:

This is a bit of a discussion problem. Since ρI is the inertia of the beam in rotation about φ the term ρIw_{xxtt} represents rotary inertia. The term $(\rho IE/\kappa^2 G)w_{tttt}$ is the shear distortion and the term $(\rho^2 I/\kappa^2 G)w_{xxtt}$ is a combination of shear distortion and rotary inertia. Removing these terms from equation (6.109) results in equation (6.92).

Problems and Solutions Section 6.6 (6.48 through 6.52)

6.48   Calculate the natural frequencies of the membrane of Example 6.6.1 for the case that one edge $x = 1$ is free.

Solution:

The equation for a square membrane is

$$w_{tt} + w_{yy} = \left(\frac{\rho}{\tau} w_{tt} \right)$$

with boundary condition given by $w(0,y) = 0$, $w_x(l,y) = 0$, $w(x,0) = 0$, $w(x,l) = 0$. Assume separation of variables $w = X(x)Y(y)q(t)$ which yields

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2} \frac{\ddot{q}}{q} = -\omega^2 \quad \text{where} \quad c = \sqrt{\rho/\tau}$$

Then

$$\ddot{q} + c^2 \omega^2 q = 0$$

is the temporal equation and

$$\frac{X''}{X} = -\omega^2 - \frac{Y''}{Y} = -\alpha^2$$

yields

$$X'' + \alpha^2 X = 0$$
$$Y'' + \gamma^2 Y = 0$$

as the spatial equation where $\gamma^2 = \omega^2 - \alpha^2$ and $\omega^2 = \alpha^2 + \gamma^2$. The separated boundary conditions are $X(0) = 0$, $X'(l) = 0$ and $Y(0) = Y(l) = 0$. These yield

$$X = A\sin\alpha x + B\cos\alpha x$$
$$B = 0$$
$$A\cos\alpha l = 0$$
$$\alpha_n l = \frac{(2n-1)\pi}{2}$$
$$\alpha_n = \frac{(2n-1)\pi}{2l}$$

Next $Y = C \sin \gamma y + D \cos \gamma y$ with boundary conditions which yield $D = 0$ and $C \sin \gamma l = 0$. Thus

$$\gamma_m = m\pi l$$

and for $l = 1$ we get $a_n = \dfrac{(2n-1)\pi}{2}$, for $\gamma_m = m\pi$ $n,m = 1, 2, 3,\ldots$

$$\omega_{nm}^2 = \alpha_n^2 + \gamma_m^2 = \frac{(2n-1)^2 \pi^2}{4} + m^2\pi^2 = \left[\frac{(2n-1)^2 + 4m^2}{4}\right]\pi^2$$

$$c^2\omega_{nm}^2 = c^2 \left[\frac{(2n-1)^2 + 4m^2}{4}\right]\pi^2$$

So that

$$\omega_{nm} = \sqrt{(2n-1)^2 + 4m^2}\,\frac{c\pi}{2}$$

are the natural frequencies.

6.49 Repeat Example 6.6.1 for a rectangular membrane of size a by b. What is the effect of a and b on the natural frequencies?

Solution:

The solution of the rectangular membrane of size $a \times b$ is the same as given in example 6.6.1 for a unit membrane until equation 6.13.1. The boundary condition along $x = a$ becomes

$$A_1 \sin \alpha a \sin \gamma y + A_2 \sin \alpha a \cos \gamma y = 0$$

or

$$\sin \alpha a (A_1 - \sin \gamma y + A_2 \cos \gamma y) = 0$$

Thus $\sin \alpha a = 0$ and $\alpha a = n\pi$ or $\alpha = n\pi/a$, $n = 1, 2,...$ Similarly, the boundary conditions along $y = b$ yields that

$$\gamma = \frac{n\pi}{b} \quad n = 1,2,3,...$$

Thus the natural frequency becomes

$$\omega_{nm} = \pi \sqrt{a^2 n^2 + b^2 m^2} \quad n, m = 1, 2, 3,...$$

Note that ω_{nm} are no longer repeated, i.e., $\omega_{12} \neq \omega_{21}$, etc.

6.50   Plot the first three mode shapes of Example 6.6.1.

Solution: A three mesh routine from any of the programs can be used. Mathcad results follow for the 11, 12, 21 and 31 modes:

$N := 10$ $i := 0 .. N$ $j := 0 .. N$ $x_i := i \cdot 0.1$ $y_j := j \cdot 0.1$

$$w(x, y) := \left(\sin(\pi \cdot x) \right) \cdot \sin(\pi \cdot y)$$

$$M_{(i, j)} := w(x_i, y_j)$$

M
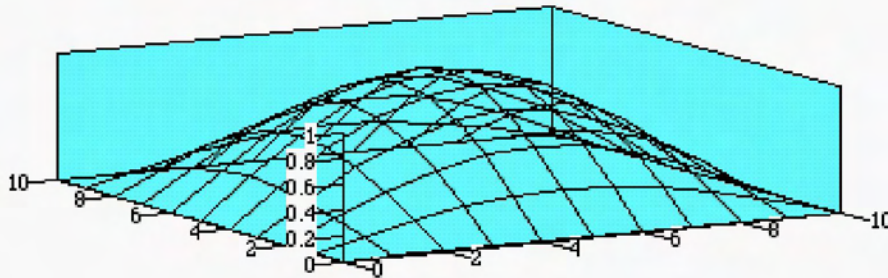
$N := 10$ $i := 0 .. N$ $j := 0 .. N$ $x_i := i \cdot 0.1$ $y_j := j \cdot 0.1$

$$w(x, y) := \left(\sin(\pi \cdot x) \right) \cdot \sin(2 \, \pi \cdot y)$$

$$M_{(i, j)} := w(x_i, y_j)$$

M

$N := 10 \quad i := 0 .. N \qquad j := 0 .. N \qquad x_i := i \cdot 0.1 \qquad y_j := j \cdot 0.1$

$w(x, y) := \left(\sin\left(2 \, \pi \cdot x\right) \right) \cdot \sin\left(\pi \cdot y\right)$

$M_{(i, j)} := w\left(x_i, y_j\right)$

M
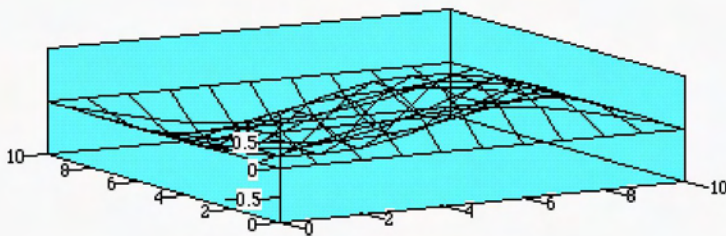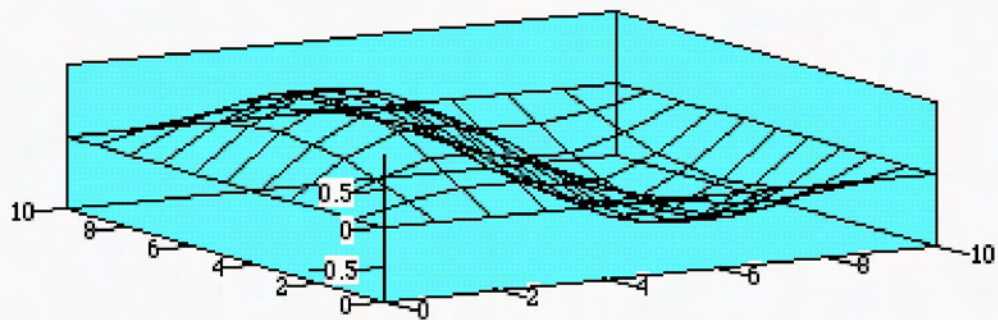
$w(x, y) := \left(\sin\left(3 \, \pi \cdot x\right) \right) \cdot \sin\left(\pi \cdot y\right)$

$M_{(i, j)} := w\left(x_i, y_j\right)$
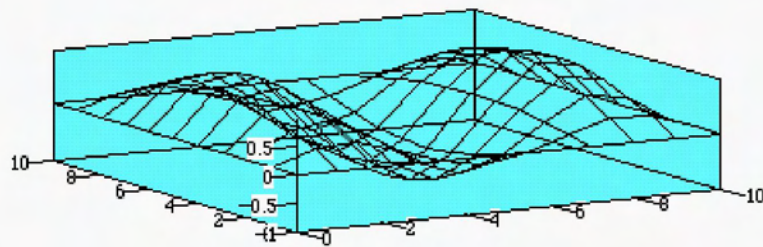
M

6.51   The lateral vibrations of a circular membrane are given by

$$\frac{\partial^2 \omega(r,\phi,t)}{\partial r^2} + \frac{1}{r}\frac{\partial \omega(r,\phi,t)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega(r,\phi,t)}{\partial\phi\partial r} = \frac{\rho}{\tau}\frac{\partial^2 \omega(r,\phi,t)}{\partial t^2}$$

where r is the distance form the center point of the membrane along a radius and ϕ is the angle around the center. Calculate the natural frequencies if the membrane is clamped around its boundary at $r = R$.

Solution:

This is a tough problem. Assign it only if you want to introduce Bessel functions. The differential equation of a circular membrane is:

$$\frac{\partial^2 W(r,\phi)}{\partial r^2} + \frac{1}{r}\frac{\partial W(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W(r,\phi)}{\partial\phi^2} + \beta^2 W(r,\phi) = 0$$

$$\beta^2 = \left(\frac{\omega}{c}\right)^2 \quad c = \frac{T}{\rho}$$

Assume:

$$W(r,\phi) = F(r)G(\phi)$$

The differential equation separates into:

$$\frac{d^2 G}{d\phi^2} + m^2 G = 0$$

$$\frac{d^2 F}{dr^2} + \frac{1}{r}\frac{dF_1}{dr} + \left(\beta^2 - \frac{m^2}{r^2}\right)F = 0$$

Since the solution in ϕ must be continuous, m must be an integer. Therefore

$$G_m(\phi) = B_{1m}\sin m\phi + B_{2m}\cos m\phi$$

The equation in r is a Bessel equation and has the solution

$$F_m(r) = B_{3m}J_m(\beta r) + B_{4m}Y_m(\beta r)$$

Where $J_m(\beta r) + Y_m(\beta r)$ are the mth order Bessel functions of the first and second kind, respectively. Writing the general solution $F(r)G(\phi)$ as

$$W_m(r,\phi) = A_{1m}J_m(\beta r)\sin m\phi + A_{2m}J_m(\beta r)\cos m\phi$$
$$+ A_{3m}Y_m(\beta r)\sin m\phi + A_{4m}Y_m(\beta r)\cos m\phi$$

Enforcing the boundary condition

$$W_m(R,\phi) = 0 \quad m = 0,1,2,...$$

Since every interior point must be finite and $Y_m(\beta r)$ tends to infinity as $r \to 0$, $A_{3m} = A_{4m} = 0$. At $r = R$

$$W_m(R,\phi) = A_{1m}J_m(\beta R)\sin m\phi + A_{2m}J_m(\beta R)\cos m\phi = 0$$

This can only be satisfied if

$$J_m(\beta R) = 0 \quad m = 1,2,...$$

For each m, $J_m(\beta R) = 0$ has an infinite number of solutions. Denote β_{mn} as the nth root of the mth order Bessel function of the first kind, normalized by R. Then the natural frequencies are:

$$\omega_{mn} = c\beta_{mn}$$

6.52 Discuss the orthogonality condition for Example 6.6.1.

Solution:

The eigenfuncitons of example 6.6.1 are given as

$$X_n(x)Y_n(y) = A_{nm}\sin m\pi x \sin n\pi y$$

Orthogonality in this case is generalized to two dimensions and becomes

$$\int_0^1\int_0^1 A_{nm}A_{pq}\sin m\pi x \sin n\pi y \sin p\pi y \sin q\pi y\,dxdy = 0 \quad mn \ne pq$$

Integrating yields

$$A_{nm}A_{pq}\int_0^1 \sin n\pi x \sin p\pi x dx \int_0^1 \sin m\pi g \sin gy\,dy$$

$$= A_{nm}A_{pq}\left[\frac{\sin(n-p)\pi x}{2(n-p)} - \frac{\sin(n+p)\pi x}{2(m+p)}\right]\left[\frac{\sin(m-1)\pi x}{2(m-q)} - \frac{\sin(m+p)\pi x}{2(m+p)}\right]$$

Evaluating at $x = 0$ and $x = 1$ this expression is zero. The expression is also zero provided $n = p$ and $n \neq q$ illustrating that the modes are in fact orthogonal.

Problems and Solutions Section 6.7 (6.53 through 6.63)

6.53 Calculate the response of Example 6.7.1 for $l = 1$ m, $E = 2.6 \times 10^{10}$ N/m^2 and $\rho = 8.5 \times 10^3$ kg/m^3. Plot the response using the first three modes at $x = l/2$, $l/4$, and $3l/4$. How many modes are needed to represent accurately the response at the point $x = l/2$?

Solution:

$$w(x,t) = \sum_{n=1}^{\infty} \left(\frac{0.02}{l^2 \sigma_n^2} (-1)^{n+1} \right) e^{-0.01\omega_n t} \cos \omega_{2n} t \sin \sigma_n x$$

Where

$$\sigma_n = \frac{(2n-1)\pi}{2l}$$

$$\omega_n = \sigma_n \sqrt{\frac{E}{\rho}}$$

$$\omega_{dn} = 0.9999 \omega_n$$

For $l = 1$ m

$$E = 2.6 \times 10^{10} \text{ N/m}^2$$
$$\rho = 8.5 \times 10^3 \text{ kg/m}^3$$

Response using first three modes at $x = \dfrac{l}{2}, \dfrac{l}{4}, \dfrac{3l}{4}$ plotted below.

Three modes accurately represents the response at $x = \dfrac{l}{2}$. The error between a three and higher mode approximation is less than 0.2%.

$$\sigma(n) := \frac{2 \cdot n - 1}{2} \cdot \pi \qquad E := 2.6 \cdot 10^{10} \qquad \rho := 8.5 \cdot 10^{3}$$

$$\omega(n) := \sigma(n) \cdot \sqrt{\frac{E}{\rho}}$$

$$w(x, t) := \sum_{n=1}^{3} \frac{.02}{\sigma(n)^2} (-1)^{n+1} \cdot e^{-0.01 \cdot \omega(n) \cdot t} \cdot \cos(\omega(n) \cdot t) \cdot \sin(\sigma(n) \cdot x)$$
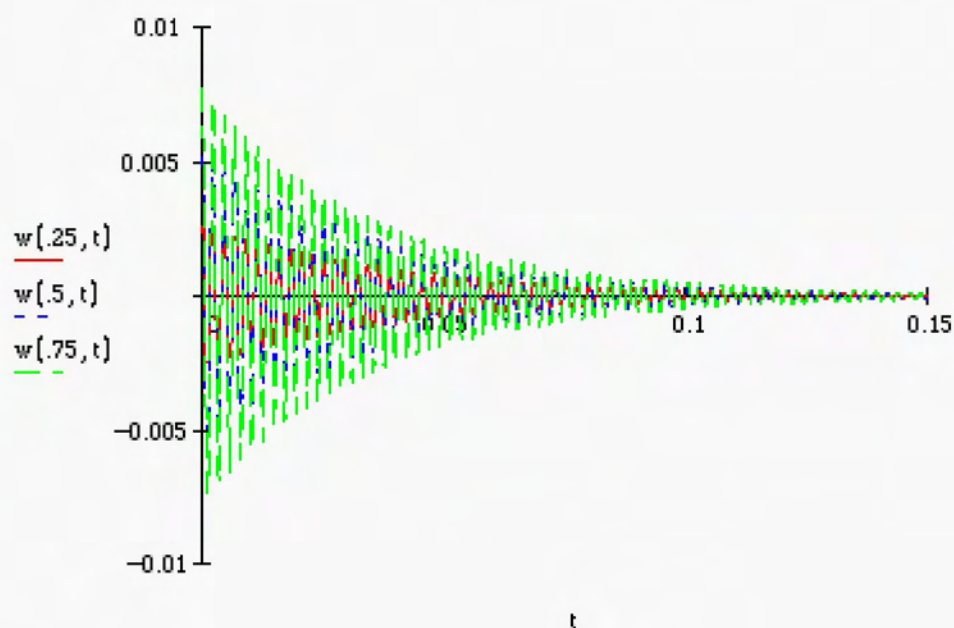
6.54 Repeat Example 6.7.1 for a modal damping ratio of $\zeta_n = 0.01$.

Solution: Using $\zeta_n = 0.01$ and the frequency given in the example

$$\omega_{dn} = \omega_n\sqrt{1-\zeta_n^2} = 0.995\omega_n, \quad \omega_n = \frac{2n-1}{2l}\sqrt{\frac{E}{\rho}} = \sigma_n\sqrt{\frac{E}{\rho}}$$

The time response is then $T_n(t) = A_n e^{-0.1\omega_n t}\sin(\omega_{dn}t + \phi_n)$ and the total solution is:

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-0.1\omega_n t}\sin(\omega_{dn}t + \phi_n)\sin\frac{(2n-1)}{2l}\pi x$$

The initial conditions are:

$$w(x,0) = 0.01\frac{x}{l} \text{ m} \quad \text{and} \quad w_t(x,0) = 0$$

Therefore:

$$0.01\frac{x}{l} = A_n\sin\phi_n\sin\sigma_n x$$

Multiply by $\sin\sigma_m x$ and integrate over the length of the bar to get

$$0.01\frac{(-1)^{m+1}}{l\,\sigma_m^2} = A_m\sin\phi_m\frac{l}{2} \quad m = 1,2,3,...$$

From the velocity initial condition

$$w_t(x,0) = 0 = \sum_{n=1}^{\infty} A_n\left[-0.1\omega_n\sin\phi_n + \omega_{dn}\cos\phi_n\right]\sin\sigma_n x$$

Again, multiply by $\sin\sigma_m x$ and integrate over the length of the bar to get

$$A_m(-0.1\omega_n\sin\phi_n + \omega_{dn}\cos\phi_n)\frac{l}{2} = 0$$

Since A_m is not zero this yields:

$$\tan\phi_n = \frac{\sin\phi_n}{\cos\phi_n} = \frac{\sqrt{1-\zeta_n^3}}{0.1} = 9.9499 \Rightarrow \phi_n = 1.4706 \text{ rad } = 84.3°$$

Substitution into the equation from the displacement initial condition yields:

$$A_m = \frac{0.01}{l^2\sigma_m^2}(-1)^{m+1}\frac{1}{\sin\phi_n} = \frac{0.0201}{l^2\sigma_m^2}(-1)^{m+1}$$

The solution is then

$$w(x,t) = \sum_{n=1}^{\infty}\frac{0.01}{l^2\sigma_m^2}(-1)^{m+1}e^{-0.1\omega_n t}\sin(\omega_{dn}t + \phi_n)\sin\frac{(2n-1)}{2l}\pi x$$

6.55 Repeat Problem 6.53 for the case of Problem 6.54. Does it take more or fewer modes to accurately represent the response at $l/2$?

Solution: Use the result given in 6.54 and

$$l = 1 \text{ m}$$
$$E = 2.6 \times 10^{10} \text{ N/m}^2$$
$$\rho = 8.5 \times 10^3 \text{ kg/m}^3$$

The response is plotted below at $x = \dfrac{l}{4}, \dfrac{l}{2}, \dfrac{3l}{4}$. An accurate representation of the response is obtained with three modes. The error between a three mode and a higher mode representation is always less than 0.2%. The results here are from Mathcad:

$$\sigma(n) := \frac{2 \cdot n - 1}{2} \cdot \pi \qquad E := 2.6 \cdot 10^{10} \qquad \rho := 8.5 \cdot 10^3$$

$$\omega(n) := \sigma(n) \cdot \sqrt{\frac{E}{\rho}} \qquad \omega d(n) := 0.995 \cdot \omega(n)$$

$$w(x,t) := \sum_{n=1}^{3} \frac{.0201}{\sigma(n)^2} (-1)^{n+1} \cdot e^{-0.1 \cdot \omega(n) \cdot t} \cdot \sin\left(\omega d(n) \cdot t + 1.4706\right) \cdot \sin\left(\sigma(n) \cdot x\right)$$
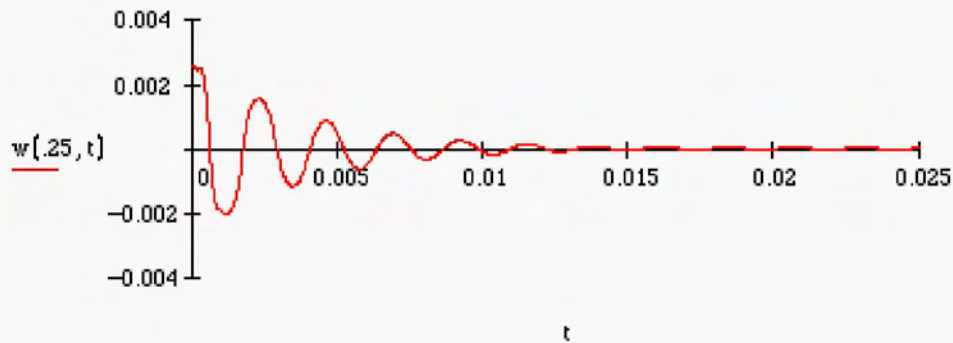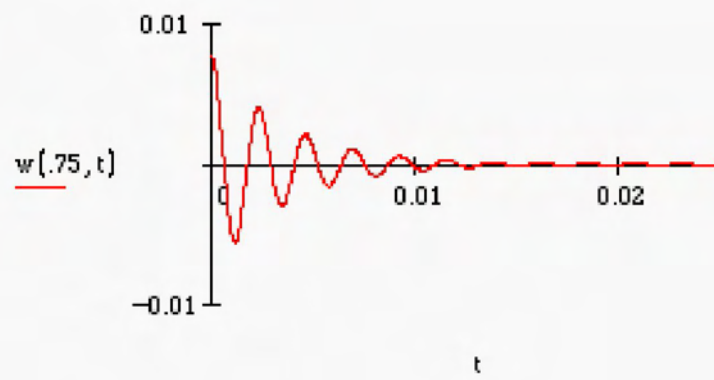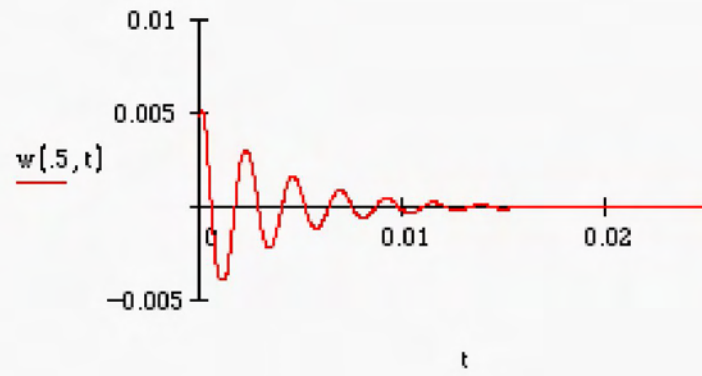
6.56 Calculate the form of modal damping ratios for the clamped string of equation (6.151) and the clamped membrane of equation (6.152).

Solution:

(a) For the string:

$$\rho w_{tt} + \gamma w_t - \tau w_{xx} = 0$$

$$\rho \phi \ddot{q} + \gamma \phi \dot{q} - \tau \phi'' q = 0$$

$$\frac{\rho}{\tau} \frac{\ddot{q}}{q} + \frac{\gamma}{\tau} \frac{\dot{q}}{q} = \frac{\phi''}{\phi} = -\sigma^2$$

$$\ddot{q} + \left(\frac{\gamma}{\rho}\right) \dot{q} + \left(\frac{\tau}{\rho}\right) \sigma^2 q = 0$$

$\phi'' + \sigma^2 \phi = 0$ which has the solution $\phi = A\sin\sigma x + B\cos\sigma x$. The boundary conditions $\phi(0) = \phi(l) = 0$ yield $\sigma_n = \dfrac{n\pi}{l}$, $n = 1,2,3,...$

$$\omega_n^2 = \left(\frac{\tau}{\rho}\right)\sigma_n^2 = \frac{\tau}{\rho}\left(\frac{n\pi}{l}\right)^2$$

$$2\zeta_n \omega_n = \frac{\gamma}{\rho}$$

$$\zeta_n = \frac{\gamma}{2\rho}\sqrt{\frac{\rho}{\tau}}\left(\frac{n\pi}{l}\right)$$

$$\zeta_n = \frac{\gamma}{2\sqrt{\rho\tau}}\left(\frac{n\pi}{l}\right)$$

(b) For the membrane

$$\frac{\rho}{\tau}w_{tt} + \frac{\gamma}{\tau}w_t = w_{xx} + w_{yy}$$

$$\left(\frac{\rho}{\tau}\right)XY\ddot{q} + \left(\frac{\gamma}{\tau}\right)XY\dot{q} = X''Yq + XY''q$$

$$\left(\frac{\rho}{\tau}\right)\frac{\ddot{q}}{q} + \left(\frac{\gamma}{\tau}\right)\frac{\dot{q}}{q} = \frac{X''}{X} + \frac{Y''}{Y} = -\beta^2$$

$$\ddot{q} + \left(\frac{\gamma}{\rho}\right)\dot{q} + \left(\frac{\tau}{\rho}\right)\beta^2 q = 0$$

$\dfrac{X''}{X} = -\dfrac{Y''}{Y} - \beta^2 = -\alpha^2$ .The boundary conditions are $X(0) = X(l) = 0$ and $Y(0) = Y(l) = 0$. The two spatial solutions become

$$X'' + \alpha^2 = 0 \qquad\qquad Y'' + \gamma^2 Y = 0$$
$$X = A\sin\alpha x + B\cos\alpha x \qquad Y = C\sin\gamma x + D\cos\gamma x$$
$$B = 0 \qquad\qquad D = 0$$
$$\alpha_n + \dfrac{n\pi}{l} \quad n = 1,2,3,\dots \qquad \gamma_m = \dfrac{n\pi}{l} \quad m = 1,2,3,\dots$$

Thus

$$\beta_{mn}^2 = \left(n^2 + m^2\right)\left(\dfrac{\pi}{l}\right)^2$$

$$\omega_{mn}^2 = \dfrac{\tau}{\rho}\left(n^2 + m^2\right)\left(\dfrac{\pi}{l}\right)^2$$

$$2\zeta_m n\omega_m n = \dfrac{\gamma}{\rho}$$

$$\zeta_{mn} = \dfrac{\gamma}{2\rho\omega_{mn}} = \dfrac{\gamma}{2\rho}\dfrac{1}{\sqrt{\dfrac{\tau}{\rho}\left(n^2 + m^2\right)}}\dfrac{l}{\pi}$$

$$\zeta_{mn} = \dfrac{\gamma l}{2\sqrt{\rho\tau\left(n^2 + m^2\right)}}$$

6.57 Calculate the units on γ and β in equation (6.153).

   Solution: The units are found from

$$\dfrac{mg}{m^3}m^2\dfrac{m}{s^2} = \gamma\dfrac{m}{s}$$

$$\dfrac{kg}{s^2}\dfrac{s}{m} = \gamma$$

$$\gamma = \dfrac{kg}{m \bullet s}$$

6.58 Assume that E, I, and ρ are constant in equations (6.153) and (6.154) and calculate the form of the modal damping ratio ζ_n.

Solution:

If E, I, and ρ are constant in equation 6.153 and 6.154. Then separation of variables works and the mode shapes become those given in table 6.4, which can be normalized so that $\int_0^1 X_n X_m \, dx = \delta_{nm}$. Substitution of $w(x,t) = a_n(t)X_n(x)$ into equation (6.153) multiplying by $X_m(x)$ and integrating over x yield the mth modal equation:

$$\rho A \ddot{a}_n(t) + \gamma \dot{a}_n(t) + \beta I \left(\frac{\omega_n^2}{c^2} \right) \dot{a}_n(t) + EI \frac{\omega_n^2}{c^2} a_n(t) = 0$$

where equation (6.93) has been used to evaluate X'''' and $c^2 = EI / \rho A$. Dividing by ρA yields

$$\ddot{a}_n(t) + \left(\frac{\gamma}{\rho A} + \frac{\beta}{E} \omega_n^2 \right) \dot{a}_n(t) + \omega_n^2 a_n(t) = 0$$

which is the *sdof* form of windows 6.4. Thus the coefficients of   must be and hence

$$2\zeta_n \omega_n = \frac{\gamma}{\rho A} + \frac{\beta}{E} \omega_n^2$$

and

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho A}}$$

$$\zeta_n = \frac{\gamma}{2\rho A \omega_n} + \frac{\beta}{E} \omega_n$$

where β_n are given in table 6.4.

6.59 Calculate the form of the solution $w(x,t)$ for the system of Problem 6.58.

Solution:

The form of the solution of the m time equation is just

$$A_n e^{-\zeta_n \omega_n t} \sin\left(\omega_{dn} t + \phi_n\right)$$

where ζ_n and ω_n are as given in problem 6.58, $\omega_{dn} = \omega_n \sqrt{1 - \zeta_n^2}$, and A_n and ϕ_n are constants determined by initial conditions. The total solution is of the form

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-\zeta_n \omega_n t} \sin\left(\omega_{dn} t + \phi_n\right) X_n(x)$$

where $X_n(t)$ are the eigenfunctions given in table 6.4.

6.60   For a given cantilevered composite beam, the following values have been
measured for bending vibration:

$$E = 2.71\text{x}10^{10} \text{ N/m}^2 \qquad \rho = 1710 \text{ kg/m}^3$$
$$A = 0.597\text{x}10^{-3} \text{ m}^2 \qquad l = 1 \text{ m}$$
$$I = 1.64\text{x}10^{-9} \text{ m}^4 \qquad \gamma = 1.75 \text{ N s/m}^2$$
$$\beta = 20{,}500 \text{ Ns/m}^2$$

Calculate the solution for the beam to an initial displacement of $w_t(x,0) = 0$ and
$w(x,0) = 3\sin \pi x$.

Solution:

Using the values given and the formulas for $a_n(t)$ from problem 6.58 the temporal
equation becomes

$$\ddot{a}_n + \left(1.714 \times .00000075 G\omega_n^2\right)\dot{a}_n + \omega_n^2 a_n = 0$$

from problem 6.59,

$$w_t(x,t)\Big|_{t=0} = 0 = \sum A_n\left[\left(-\zeta_n\omega_n\right)\sin\phi_n + \omega_{dn}\cos\phi_n\right]X_n(x)$$

and

$$w(x,0) = 3\sin\pi x = \sum A_n \sin\phi_n X_n(x)$$

Multiplying by $X_n(x)$ and integrating yields that

$$\zeta_n\omega_n \sin\phi_n = \omega_{dn}\cos\phi_n \quad \text{or} \quad \tan\phi_n = \frac{\omega_{dn}}{\zeta_n\omega_n}$$

and $3\int_0^l \sin\pi x X_n(x)dx = A_n \sin\phi_n$ so that

$$A_n = \frac{3\int_0^l \sin\pi x X_n(x)dx}{\sin\phi_n} = \frac{3}{\sqrt{1-\zeta_n^2}}\int_0^l \sin\pi x X_n(x)dx$$

where $X_n(x)$ is given in table 6.4.

6.61 Plot the solution of Example 6.7.2 for the case $w_t(x,0) = 0$, $w(x,0)=\sin(nx/\ell)$, $\gamma=10$ Ns/m², $\tau =10^4$ N, $\ell =1$ m and $\rho =0.01$ kg/m³.

Solution: From equation (6.156) and the values given, $\zeta_1 =0.159/n$ or $\zeta_n\omega_n= 500$ and $\omega_{dn} = \sqrt{1= 0.159^2}$, so that:

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-500t} \sin(\omega_{dn}t + \phi_n)\sin n\pi x$$

Applying the initial conditions yields

$$\int_0^1 \sin n\pi x \sin m\pi x dx = \sum_{n=1}^{\infty} A_n \sin(\phi_n)\int_0^1 \sin m\pi x \sin n\pi x dx$$

So that $A_n\sin\phi_n =0$ for all n except $n = 1$, and $A_1\sin\phi_1 = 1$. So either $\phi_n =0$ or $A_n= 0$

for n not zero. The other initial condition yields that $\phi_n = \tan^{-1}(\dfrac{-\sqrt{1-\zeta_n^2}}{\zeta_n})$ so that

$A_n= 0$ for n not zero. Thus the system is only excited in the first mode. Then

$$w(x,t) = A_1 e^{-500t} \sin(\omega_1\sqrt{1-\zeta_n^2}t + \phi_n)\sin \pi x$$

$$= -1.001e^{-500t} \sin(3137.7t - 1.50)\sin \pi x$$

This is plotted in Mathcad below:

$N := 8 \quad i := 0..N \quad j := 0..N \quad x_i := i\cdot\dfrac{1}{8} \quad t_j := j\cdot 0.001$

$\zeta := 0.159 \quad \omega := \pi\cdot 10^3 \quad \phi := -1.52 \quad \omega d := \omega\cdot\sqrt{1 - \zeta^2}$

$w(x,t) := -1.001\, e^{-\zeta\cdot\omega\cdot t}\cdot\left(\sin\left(\pi\cdot x\right)\right)\cdot\sin\left(\omega d\cdot t + \phi\right)$

$M_{i,j} := w(x_i, t_j)$
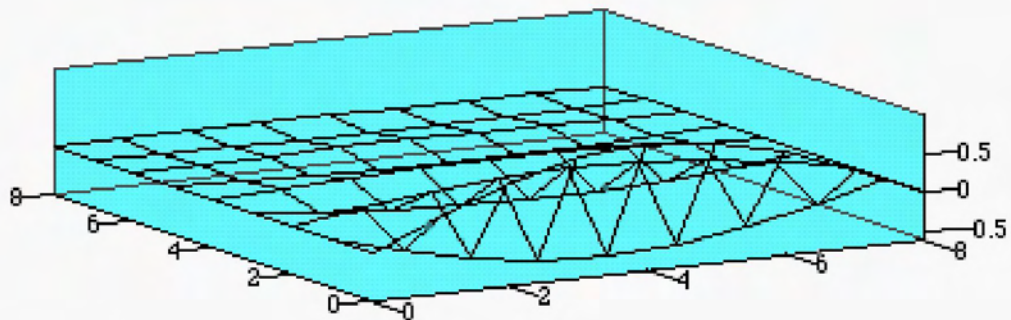
M

6.62 Calculate the orthogonality condition for the system of Example 6.7.2. Then calculate the form of the temporal solution.

Solution: Problem is to fill in the details of example 6.7.2 by checking the coefficients. Equation (6.155) by performing the integration.

6.63 Calculate the form of modal damping for the longitudinal vibration of the beam of Figure 6.14 with boundary conditions specified by equation (6.157).

Solution: This is a discussion problem. The boundary condition given in equation (6.157)

$$AEw_x(0,t) = kw(0,t) + c\frac{\partial w(0,t)}{\partial t}$$

$$AEw_x(l,t) = -kw(l,t) - c\frac{\partial w(l,t)}{\partial t}$$

Do not conform readily to separation of variables and lead to time dependent boundary conditions. However one approach is to treat the damper as applied forces of the bar $cw_t(0,t)$ and $-cw_t(l,t)$. Following this approach the boundary conditions become

$$AEX'(0) = kX(0) \text{ and } AEX'(l) = -kX(l)$$

The general solution of the spatial equation of a bar has the form

$$X(x) = a\sin(\sigma x + b)$$

Where σ is the usual separation constant and a and b are constants. The first boundary condition yields that $\phi = \tan^{-1}(AE/k)$. The second boundary condition yields the characteristic equation

$$-(AE/k)\sigma_n = \tan(\sigma_n l + \phi)$$

Which can be solved for σ_n numerically. Note that σ_n are distinct so that from problem 6.39 the eigenfunctions are orthogonal, i.e. an can be calculated such that

$$X_n(x) = a_n\sin(\sigma_n x + \phi)$$

Are orthonormal. Following the procedure of example 6.8.11, the temporal solution for the forced response is

$$\ddot{r}_n(t) + \omega_n^2 = \int_0^l \left[cw_t(0,t) - cw(l,t)\right]X_r(x)dx$$

$$= \left\{\int_0^l \left[cX_n(0) - cX_n(l)\right]X_n(x)dx\right\}\dot{r}_n(t)$$

Bring the $\dot{r}_n$ term to the left side and comparing its coefficient to $2\zeta_n\omega_n$ yields

$$2\zeta_n\omega_n = \left\{\int_0^l c\left[X_n(l) - X_n(0)\right]X_n(x)\right\}dx$$

The form of the modal damping ratio is thus

$$\zeta_n - \frac{ca_n^2}{2\omega_n\sigma_n}\left[\cos(\sigma_n l + \phi) - \cos\phi\right]$$

Where a_n^2 is the normalization factor, σ_n are the eigenvalues $\omega_n^2 = c^2 \sigma_n^2$ and $\tan^{-1}(AE\,/\,k)$.

Problems and Solutions Section 6.8 (6.64 through 6.68)

6.64   Calculate the response of the damped string of Example 6.8.1 to a disturbance force of $f(x,t) = (\sin \pi x/l) \sin 10t$.

Solution:

$$f(x,t) = \sin\left(\frac{\pi x}{l}\right) \sin 10t. \text{ Assume a solution of the form:}$$

$$w_n(x,t) = T_n(t)X_n(x)$$

where

$$X_n(x) = \sin\frac{n\pi x}{l}$$

Substitute into (6.158)

$$\left\{\rho\ddot{T}_n + \gamma\dot{T}_n - \tau\left[-\left(\frac{n\pi}{l}\right)^2\right]T_n\right\}\sin\frac{n\pi x}{l} = \sin\left(\frac{\pi x}{l}\right)\sin 10t$$

Multiply by $\sin\dfrac{n\pi x}{l}$ and integrate over the length of the string:

$$\left\{\rho\ddot{T}_n + \gamma\dot{T}_n + \tau\left(\frac{n\pi}{l}\right)^2 T_n\right\}\frac{l}{2} = \begin{cases} 0 & \text{for } n = 1 \\ \sin 10t & \text{for } n > 1 \end{cases}$$

Only the particular solution is of interest since we are looking for the response to the disturbance force. Therefore, dropping the subscripts:

$$\rho\ddot{T} + \gamma\dot{T} + \tau\left(\frac{\pi}{l}\right)^2 T = \sin 10t$$

$$\ddot{T} + \left(\frac{\gamma}{\rho}\right)\dot{T} + \left(\frac{c\pi}{l}\right)^2 T = \frac{\sin 10t}{\rho} \text{ where } c = \sqrt{\frac{\tau}{\rho}}$$

Solution is

$$T = A\sin(20t - \phi)$$

where

$$A = \cfrac{1}{\rho\sqrt{\left(\cfrac{c^2\pi^2}{l^2} - 100\right)^2 + 100\cfrac{\gamma^2}{\rho^2}}} = \cfrac{l^2}{\sqrt{\rho^2\left(c^2\pi^2 - 100l^2\right)^2 + 100\gamma^2 l^4}}$$

$$\phi = \tan^{-1}\left[\cfrac{10\cfrac{\gamma}{\rho}}{\cfrac{c^2\pi^2}{l^2} - 100}\right] = \tan^{-1}\left[\cfrac{10\gamma l^2}{\rho c^2\pi^2 - 100\rho l^2}\right]$$

$$w(x,t) = A\sin(10t - \phi)\sin\frac{\pi x}{l}$$

where A and ϕ are given above.

6.65 Consider the clamped-free bar of Example 6.3.2. The bar can be used to model a truck bed frame. If the truck hits an object (at the free end) causing an impulsive force of 100 N, calculate the resulting vibration of the frame. Note here that the truck cab is so massive compared to the bed frame that the end with the cab is modeled as clamped. This is illustrated in Figure P6.65.

Solution: Assume constant area and constant material properties. Equation of motion:

$$\rho A w_{tt} - E A w_{xx} = f(x,t) = -100\delta(x-l)\delta(t)$$

Mode shapes (eigenvalues) of a fixed-free bar are (Table 6.1)

$$X_n(x) = \sin\frac{(2n-1)\pi x}{2l}$$

Assume a solution of the form: $w_n(x,t) = X_n(x)T_n(t)$. Substitute into the equation of motion:

$$\left\{ \ddot{T}_n - \left[-\left(\frac{(2n-1)\pi}{2l}\right)^2 \right] c^2 T_n \right\} \sin\frac{(2n-1)\pi x}{2l} = -\frac{100}{\rho A}\delta(x-l)\delta(t)dx$$

$$\left\{ \ddot{T}_n + \omega_n^2 T_n \right\} \sin\frac{(2n-1)\pi x}{2l} = -\frac{100}{\rho A}\delta(x-l)\delta(t)$$

where $c^2 = \dfrac{E}{\rho}$ and $\omega_n = \dfrac{(2n-1)\pi c}{2l}$. Multiply by $\sin\dfrac{(2n-1)\pi x}{2l}$ and integrate over the length of the rod:

$$\ddot{T}_n + \omega_n^2 T_n = -\frac{2}{l}\int_0^l \frac{100}{\rho A}\sin\left(\frac{(2n-1)\pi x}{2l}\right)\delta(x-l)\delta(t)$$

$$= -\frac{200}{\rho A l}\sin\left(\frac{(2n-1)\pi}{2}\right)\delta(t)$$

which has the solution:

$$T_n(t) = -\frac{200}{\rho A l \omega_n}\sin\left(\frac{(2n-1)\pi}{2}\right)\sin\omega_n t$$

The total solution is:

$$w_n(x,t) = -\sum_{n=1}^{\infty}\left\{\left[\frac{400}{\rho A(2n-1)\pi c}\right]\sin\left(\frac{(2n-1)\pi}{2}\right)\right.$$

$$\left. \sin\left(\frac{(2n-1)\pi c t}{2l}\right)\sin\left(\frac{(2n-1)\pi x}{2l}\right)\right\}$$

6.66   A rotating machine sits on the second floor of a building just above a support column as indicated in Figure P6.66. Calculate the response of the column in terms of E, A, and ρ of the column modeled as a bar.

Solution: Referring to equation (6.55) for the equation of a bar and summing forces to get the effect of the applied force yields

$$\rho A w_{tt} - E A w_{xx} = \delta(x - l) F_0 \sin \omega t$$

subject to the boundary conditions $w(0,t) = w_x(0,t) = 0$. Following the method of example 6.8.1, use separation of variables where the spatial function is the clamped-free mode shapes used in example 6.3.1:

$$w(x,t) = X_n(x) T_n(t) = (a_n \sin \sigma_n x) T_n(t), \quad \sigma_n = \frac{2n-1}{2l} \pi$$

Substitution into the equation of motion yields

$$\left(\rho A \ddot{T}_n(t) + E A \sigma_n^2 T_n(t) \right) a_n \sin \sigma_n x = \delta(x - l) F_0 \sin \omega t$$

(the minus sign in front of EA goes away because of the second derivative of sine being negative). Next, let $a_n = 1$ (recalling that eigenvectors have arbitrary magnitude) and multiply by $\sin \sigma_n x$ and integrate over the length of the beam to get:

$$\left(\rho A \ddot{T}_n(t) + E A \sigma_n^2 T_n(t) \right) \frac{l}{2} = F_0 \sin \omega t \int_0^l \delta(x - l) \sin \sigma_n x\, dx$$

The integral on the right is a bit tricky as the delta function acts at the end of the interval. The details are below, however integrating yields

$$\left(\rho A \ddot{T}_n(t) + E A \sigma_n^2 T_n(t) \right) \frac{l}{2} = \frac{1}{2} F_0 \sin \omega t \frac{\sin \sigma_n l}{2} = (-1)^{n-1} \frac{F_0}{2} \sin \omega t$$

Dividing by the appropriate constants this simplifies to

$$\ddot{T}_n(t) + \frac{E}{\rho} \sigma_n^2 T_n(t) = \frac{(-1)^{n-1} F_0}{\rho A} \sin \omega t$$

This has particular solution

$$T_{np}(t) = \frac{(-1)^{n-1}}{\rho A} \left(\frac{F_0}{\omega_n^2 - \omega^2} \right) \sin \omega t \quad \text{where } \omega_n = \sqrt{\frac{E}{\rho}} \frac{(2n-1)\pi}{2l}$$

Combined with the homogenous solution, the total temporal solution is

$$T_n(t) = C_{1n} \sin \omega_n t + C_{2n} \cos \omega_n t + \left(\frac{(-1)^{n-1}}{\rho A} \frac{F_0}{\omega_n^2 - \omega^2} \right) \sin \omega t$$

So the total solution is

$$w(x,t) = \sum_{n=1}^{\infty} \left\{ C_{1n} \sin \omega_n t + C_{2n} \cos \omega_n t + \frac{(-1)^{n-1}}{\rho A} \left(\frac{F_0}{\omega_n^2 - \omega^2} \right) \sin \omega t \right\} \sin \left(\frac{(2n-1)\pi x}{2l} \right)$$

The following it the evaluation of the Dirac integral used about (courtesy of Jamil Renno)

Start with the integral at hand

$$\int_0^l \delta(x-l)\sin(\sigma_m x)dx = \lim_{\tau \to 0}\left[\int_0^l d_\tau(x-l)\sin(\sigma_m x)dx\right]$$

where $d_\tau(x-l)=\begin{cases}\dfrac{1}{2\tau} & l-\tau < x < l+\tau \\ 0 & x \le l-\tau \text{ or } x \ge l+\tau\end{cases}$ is the pulse over the interval $\left[l-\tau, l+\tau\right]$.

Hence, the integral can be subdivided over two intervals

$$\int_0^l \delta(x-l)\sin(\sigma_m x)dx = \lim_{\tau \to 0}\left[\int_0^{l-\tau} d_\tau(x-l)\sin(\sigma_m x)dx + \int_{l-\tau}^l d_\tau(x-l)\sin(\sigma_m x)dx\right]$$

$$= \lim_{\tau \to 0}\left[\int_0^{l-\tau} 0\sin(\sigma_m x)dx + \int_{l-\tau}^l \frac{1}{2\tau}\sin(\sigma_m x)dx\right] = \lim_{\tau \to 0}\left[\int_{l-\tau}^l \frac{1}{2\tau}\sin(\sigma_m x)dx\right]$$

$$= \lim_{\tau \to 0}\left[\frac{1}{2\tau}\frac{1}{\sigma_m}\left[-\cos(\sigma_m x)\right]_{l-\tau}^l\right] = \lim_{\tau \to 0}\frac{\cos\left[\sigma_m(l-\tau)\right]-\cos\left[\sigma_m l\right]}{2\tau\sigma_m}$$

$$\overset{\text{L'Hopital's Rule}}{=} \lim_{\tau \to 0}\frac{\dfrac{d}{d\tau}\left\{\cos\left[\sigma_m(l-\tau)\right]-\cos\left[\sigma_m l\right]\right\}}{\dfrac{d}{d\tau}\left\{2\tau\sigma_m\right\}}$$

$$= \lim_{\tau \to 0}\frac{\left(\dfrac{-1}{\sigma_m}\right)\left(-\sin\left[\sigma_m(l-\tau)\right]\right)}{2\sigma_m} = \frac{\sin(\sigma_m l)}{2}$$

6.67   Recall Example 6.8.2, which models the vibration of a building due to a rotating machine imbalance on the second floor. Suppose that the floor is constructed so that the beam is clamped at one end and pinned at the other, and recalculate the response (recall Example 6.5.1). Compare your solution and that of Example 6.8.2, and discuss the difference.

Solution:

Clamped-pinned beam conditions yield mode shapes (eigenfunctions) of the form:

$$X_n(x) = a_n \left[\cosh \beta_n x - \cos \beta_n x - \sigma_n (\sinh \beta_n x - \sin \beta_n x) \right]$$

where $\tan \beta_n l = \tanh \beta_n l$ and

$$\sigma_n = \begin{cases} 1.0008 & \text{for } n = 1 \\ 1 & \text{for } n > 1 \end{cases}$$

Normalize the mode shape as follows:

$$\int_0^l X_n^2 dx = 1 \Rightarrow$$

$$a_n^2 \int_0^l [\cosh \beta_n x - \cos \beta_n x - \sigma_n (\sinh \beta_n x - \sin \beta_n x)]^2 dx = 1$$

From Mathematica

$$a_n^2 = 4\beta_n / \left\{ 4\beta_n l + 2\sigma_n \cos(2\beta_n l) - 2\sigma_n \cosh(2\beta_n l) - 4\cosh(\beta_n l)\sin(\beta_n l) \right.$$
$$- 4\sigma_n^2 \cosh(\beta_n l)\sin(\beta_n l) + \sin(2\beta_n l) - \sigma_n^2 \sin(2\beta_n l)$$
$$- 4\cos(\beta_n l)\sinh(\beta_n l) + 4\sigma_n^2 \cos(\beta_n l)\sinh(\beta_n l)$$
$$\left. + 8\sigma_n \sin(\beta_n l)\sinh(\beta_n l) + \sinh(2\beta_n l) + \sigma_n^2 \sinh(2\beta_n l) \right\}$$

The equation of motion for the system is: (constant properties)

$$\rho A w_{tt} + EI w_{xxxx} = f(x,t) = 100 \sin 3t \delta\left(x - \frac{l}{2} \right)$$

Assume a solution of the form: $w_n(x,t) = X_n(x)T_n(t)$

$$\ddot{T}_n X_n + \frac{EI}{\rho A} T_n X_n''' = \frac{100}{\rho A} \sin 3t \delta\left(x - \frac{l}{2} \right)$$

Using the mode shapes given above:

$$X_n''' = \beta_n^4 X_n = \frac{\omega_n^2}{c^2} X_n$$

where

$$\beta_n^4 = \frac{\rho A}{EI}\omega_n^2, \quad c^2 = \frac{EI}{\rho A}$$

The equation of motion reduces to:

$$\left\{ \ddot{T}_n + \omega_n^2 \right\} X_n = \frac{100}{\rho A} \sin 3t \delta\left(x - \frac{l}{2} \right)$$

Multiply by X_n and integrate over the length of the beam:

$$\ddot{T}_n + \omega_n^2 T_n = \frac{100}{\rho A} \sin 3t \int_0^l X_n(x) \delta\left(x - \frac{l}{2} \right) dx$$

$$= \frac{100}{\rho A} \sin 3t X_n\left(\frac{l}{2} \right)$$

$$= \frac{100 a_n}{\rho A} \sin 3t \left[\cosh \frac{\beta_n l}{2} - \cos \frac{\beta_n l}{2} - \sigma_n\left(\sinh \frac{\beta_n l}{2} - \sin \frac{\beta_n l}{2} \right) \right]$$

or:

$$T_n(t) = \left[\frac{100 X_n\left(\frac{l}{2} \right)}{\rho A\left(\omega_n^2 - 9 \right)} \right] \sin 3t$$

The solution is then:

$$w(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \left[\cosh \beta_n x - \cos \beta_n x - \sigma_n\left(\sinh \beta_n x - \sin \beta_n x \right) \right] \right.$$

$$\left. \left[\frac{100}{\rho A\left(\omega_n^2 - 9 \right)} \right] X_n\left(\frac{l}{2} \right) \right\} \sin 3t$$

where a_n, ω_n, and β_n are given above. The free time response is stiffer for the clamped case as the frequencies are higher (See Table 6.4).

The comparison of the solution between the two models (one with a pinned end and one with a fixed or clamped end) had two purposes: design and modeling. From the design point of view it is important to know how to construct the floor for a minimum value of response. From the modeling point of view it is important to know how much the solution is effected by the choice of boundary conditions as part of the modeling.

Here the comparison can be made by calculating the response and then evaluating it and plotting it using a truncated solution (say 3 modes, as given in Equation 6.181) at a given point of interest (i.e. for a particular value of x). This gives an accurate comparison.

Next you can compare the differences in the details. For instance the clamped-pinned natural frequencies are lower then the clamped-clamped frequencies (just look at Table 6.4) because the clamped-clamped system is stiffer. Next, one of these sets of frequencies is going to have a natural frequency that is closer to the driving frequency, and hence produce a larger response. To make such comparisons, pick a value for the physical parameters (let omega = beta squared for instance) and check. In this case the clamped-pinned frequency is about 3.9 rad/s, which is much closer to the driving frequency of 3 rad/s then the clamped-clamped first natural frequency of 4.7 rad/s. Thus the first term in the series solution for the example will be larger then the corresponding term in the series solution for the clamped-clamped case.

6.68 Use the modal analysis procedure suggested at the end of Section 6.8 to calculate the response of a clamped free beam with a sinusoidal loading $F_0 \sin \omega t$ at its free end.

Solution:

The equation of motion is:

$$\rho A w_{tt} + EI w_{xxxx} = f(x,t) = F_0 \delta(x-l) \sin \omega t$$

Assume a solution of the form $w_n(x,t) = X_n(x) T_n(t)$

$$\ddot{T}_n X_n + \frac{EI}{\rho A} T_n X_n''' = \frac{F_0}{\rho A} \delta(x,l) \sin \omega t$$

The mode shapes are given in Table 6.4 for a fixed-free beam:

$$X_n(x) = a_n \left[\cosh \beta_n x - \cos \beta_n x - \sigma_n \left(\sinh \beta_n x - \sin \beta_n x \right) \right]$$

Where

$$\sigma_n = \frac{\sinh \beta_n l - \sin \beta_n l}{\cosh \beta_n l + \cos \beta_n l}$$

$$\beta_n^4 = \frac{\rho A}{EI} \omega_n^2$$

And

$$\cos \beta_n l \cosh \beta_n l = -1$$

From the unforced vibration problem:

$$\ddot{T}_n X_n + \frac{EI}{\rho A} T_n X_n''' = 0$$

$$\frac{\ddot{T}_n}{T_n} = -\left(\frac{EI}{\rho A} \right) \frac{X_n''''}{X_n} = -\omega_n^2$$

Therefore

$$X_n''' \pm \frac{\rho A}{EI}\omega_n^2 X_n = \beta_n^4 X_n$$

Substitute into the equation of motion and rearrange:

$$\{\ddot{T}_n + \omega_n^2 T_n\}X_n = \frac{F_0}{\rho A}\delta(x-l)\sin\omega t$$

Normalize the mode shapes as follows:

$$\int_0^l X_n^2 dx = 1$$

$$a_n^2 \int_0^l \left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right]^2 dx = 1$$

$$a_n^2 = 4\beta_n / \{4\beta_n l + 2\sigma_n \cos(2\beta_n l) - 2\sigma_n \cosh(2\beta_n l) - 4\cosh(\beta_n l)\sin(\beta_n l)$$
$$- 4\sigma_n^2 \cosh(\beta_n l) + \sin(2\beta_n l) - \sigma_n^2 \sin(2\beta_n l)$$
$$- 4\cos(\beta_n l)\sinh(\beta_n l) + 4\sigma_n^2 \cos(\beta_n l)\sinh(\beta_n l)$$
$$+ 8\sigma_n \sin(\beta_n l)\sinh(\beta_n l) + \sinh(2\beta_n l) + \sigma_n^2 \sinh(2\beta_n l)\}$$

Multiply the equation of motion $X_n(x)$ and integrate over the length of the beam:

$$\ddot{T}_n + \omega_n^2 T_n = \frac{F_0}{\rho A}\int_0^l X_n(x)\delta(x-l)dx\sin\omega t$$

$$= \frac{F_0}{\rho A}X_n(l)\sin\omega t$$

Solving:

$$T_n(t) = \left(\frac{F_0}{\rho A}\right)\left(\frac{X_n(l)}{\omega_n^2 - \omega^2}\right)\sin\omega t$$

The total solution is:

$$w(x,t) = \sum_{n=1}^{\infty}\left\{a_n\left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right]\right.$$
$$\left.\left(\frac{F_0}{\rho A}\right)\left[\frac{X_n(l)}{\omega_n^2 - \omega^2}\right]\right\}\sin\omega t$$

Where σ_n, w_n are given above and $\cos\beta_n l \cosh\beta_n l = -1$.

Problems and Solutions Section 7.2 (7.1-7.5)

7.1   A low-frequency signal is to be measured by using an accelerometer. The signal is physically a displacement of the form 5 sin (0.2*t*). The noise floor of the accelerometer (i.e. the smallest magnitude signal it can detect) is 0.4 volt/g. The accelerometer is calibrated at 1 volt/g. Can the accelerometer measure this signal?

Solution:

From the problem statement:

$$x(t) = 0.5\sin(0.2t) \qquad \text{m}$$
$$\dot{x}(t) = 0.1\cos(0.2t) \qquad \text{m} \big/ \text{s}$$
$$\ddot{x}(t) = -0.02\sin(0.2t) \quad \text{m} \big/ \text{s}^2$$

The peak acceleration is:

$$\pm 0.2 \, \text{m} \big/ \text{s}^2 \left[\frac{1g}{9.8 \, \text{m} \big/ \text{s}^2} \right] = \pm 0.0204 \, g$$

Accelerometer calibration is $1\text{V} \big/ g$, therefore the peak output of the accelerometer is:

$$\pm 0.0204 g \left[\frac{1\text{V}}{g} \right] = \pm 0.0204 \, \text{V}$$

Since the noise floor on the accelerometer is 0.4 V, then this acceleration cannot be measured.

7.2 Referring to Chapter 2, calculate the response of a single-degree-of-freedom system to a unit impulse and then to a unit triangle input lasting T second. Compare the two responses. The differences correspond to the differences between a "perfect" hammer hit and a more realistic hammer hit, as indicated in Figure 7.2. Use $\zeta = 0.01$ and $\omega = 4$ rad/s for your model.

Solution:

System: $\quad \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f(t)$ $\qquad\qquad$ (Letting $m = 1$)

(i) $\quad f(t) = \delta(t)$, a unit impulse

$x(t) = e^{-\zeta\omega_n t}\sin(\omega_d t)$ $\qquad\qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$

(ii) $\quad f(t) = \dfrac{t}{T}u(t) - \dfrac{2}{T}(t-T)u(t-T) + \dfrac{1}{T}(t-2T)u(t-2T)$

$u(t\text{-}a) =$ unit step at $t = a$.

$x(t) = \dfrac{1}{T}\{r(t) - 2r(t-T) + r(t-2T)\}$

From table of Laplace transforms:

$$r(t) = \frac{1}{\omega_n^2}\left\{ t - \frac{2\zeta}{\omega_n} + \frac{1}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \theta)\right\}u(t) \qquad \cos\theta = 2\zeta^2 - 1$$

$$x(t) \approx \frac{1}{T\omega_n^3}\left\{ \omega_n t - 2\zeta - e^{-\zeta\omega_n t}\sin(\omega_d t)u(t) \right\}$$

$$-2\left[\omega_n(t-T) - 2\zeta - e^{-\zeta\omega_n(t-T)}\sin(\omega_d(t-T))\right]u(t-T)$$

$$+\left[\omega_n(t-2T) - 2\zeta - e^{-\zeta\omega_n(t-2T)}\sin(\omega_d(t-2T))\right]u(t-2T)$$

since $\omega_n \approx \omega_d$ and $\theta \approx \pi/2$

7.3 Compare the Laplace transform of $\delta(t)$ with the Laplace transform of the triangle input of Figure 7.2 and Problem 7.2.

Solution:

(i) $f(t) = \delta(t)$, unit impulse
$F(s) = 1$

(ii) $f(t) = \dfrac{t}{T}u(t) - \dfrac{2}{T}(t-T)u(t-T) + \dfrac{1}{T}(t-2T)u(t-2T)$, unit triangle with period T.

$$F(s) = \frac{1}{T}\left\{ \int_0^\infty te^{-st}dt - 2\int_T^\infty (t-T)e^{-st}dt + \int_{2T}^\infty (t-2T)e^{-st}dt \right\}$$

$$F(s) = \frac{1}{Ts^2}\left\{ 1 + e^{-sT} + e^{s2T} \right\}$$

7.4 Plot the error in measuring the natural frequency of a single-degree-of-freedom system of mass 10 kg and stiffness 350 N/m if the mass of the excitation device (shaker) is included and varies from 0.5 to 5 kg.

Solution:

$m = 10$ kg
$k = 350$ N/m
$0.5 \leq m_s \leq 5.0$ kg

$$\text{Error} = \sqrt{\frac{k}{m+m_s}} - \sqrt{\frac{k}{m}}$$

$k := 350 \qquad m := 10$

$$E(ms) := \sqrt{\frac{k}{m+ms}} - \sqrt{\frac{k}{m}}$$
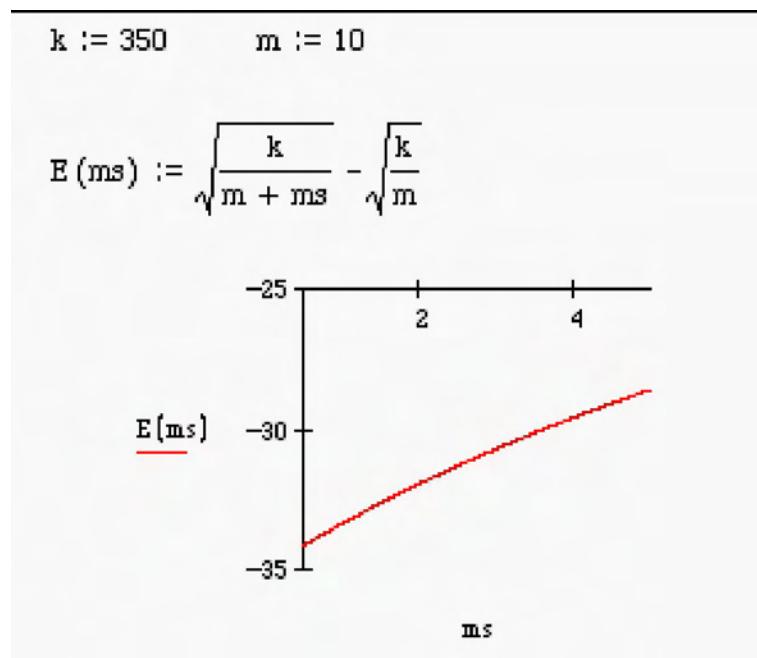
7.5 Calculate the Fourier transform of $f(t) = 3 \sin 2t - 2 \sin t - \cos t$ and plot the spectral coefficients.

Solution:

$F(t) = 3 \sin(2t) - 2\sin(t) - \cos(t)$
$\omega_T = 1$ rad/sec
$a_1 = -1 \qquad b_1 = -2 \qquad b_2 = 3$
$a_n = 0, n = 2, 3, \dots \qquad b_n = 0, n = 3, 4, 5, \dots$
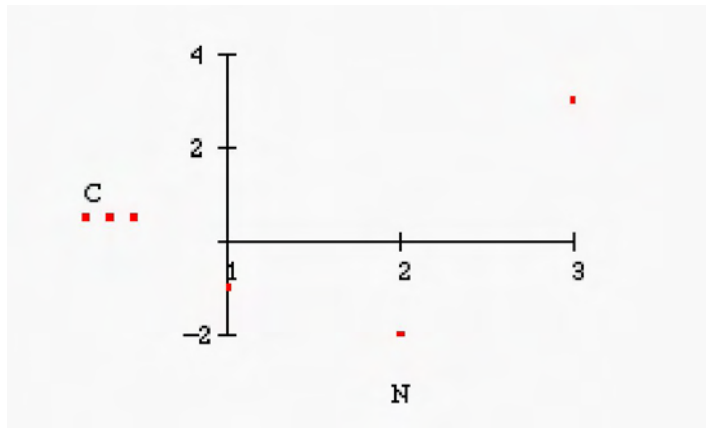
Problems and Solutions Section 7.3 (7.6-7.9)

7.6 Represent 5 sin 3*t* as a digital signal by sampling the signal at π/3, π/6 and π/12 seconds. Compare these three digital representations.

Solution: Four plots are shown. The one at the top far right is the exact wave form. The one on the top left is sampled at π/3 seconds.

$$x(t) := 5 \cdot \sin(3 \cdot t)$$

$$t := 0, \frac{\pi}{3} .. 22$$

$$s1(t) := x(t)$$

The next plot is sampled at π/6 seconds.

$$t := 0, \frac{\pi}{6} .. 22$$

$$s2(t) := x(t)$$
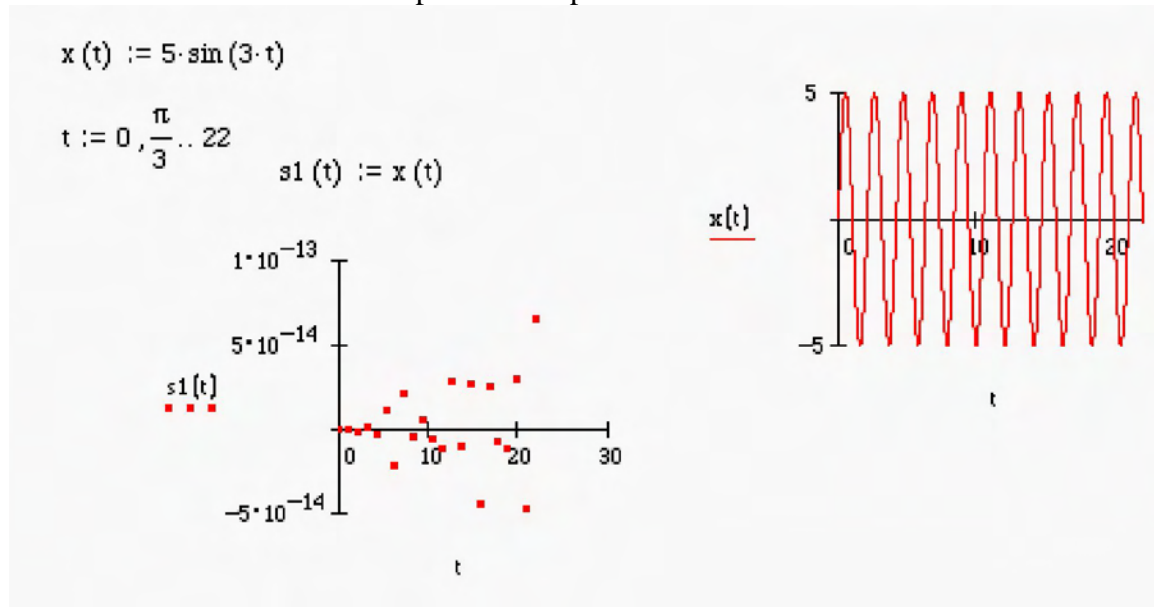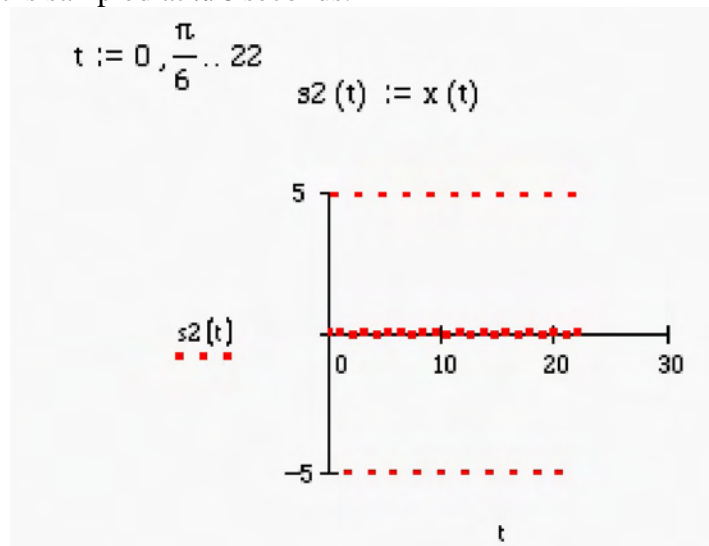
The next plot is sampled at $\pi/12$ seconds.

$$t := 0, \frac{\pi}{12} .. 22$$

$$s3(t) := x(t)$$
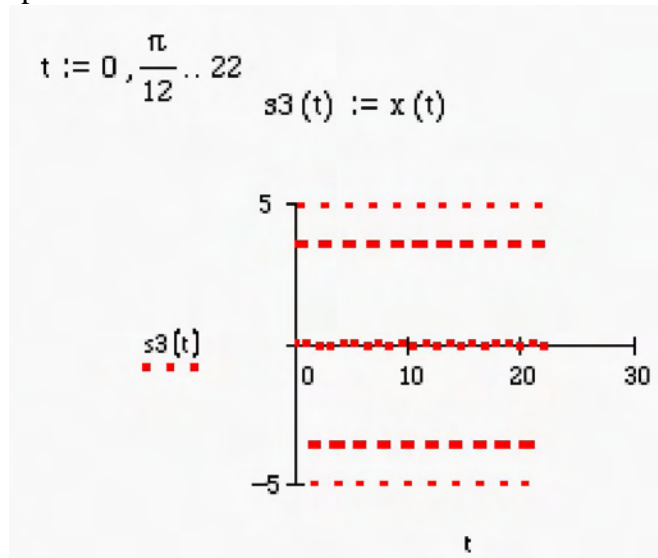
None of the plots give the shape of a sine wave. However if the s3 is connected by lines, the wave shape is close.

7.7   Compute the Fourier coefficient of the signal $|1120 \sin (120 \pi t)|$.

Solution:

$f(t) = |120 sin(120\pi t)|$ (absolute value of the sine wave)

To calculate the Fourier series:

$$T = 1/120 \text{ sec} \quad \omega_T = 240 \ \pi \text{ rad/sec}$$

$$a_o = 240 \int_0^{1/120} 120 \sin(120\pi t)dt$$

$$a_o = 480 \Big/ \pi$$

$$a_n = 240 \int_0^{1/120} 120 \sin(120\pi t)\cos(240\pi nt)dt$$

$$a_n = \frac{480}{\pi(1-4n^2)}$$

$$b_n = 240 \int_0^{1/120} 120 \sin(120\pi t)\sin(240\pi nt)dt$$

$$b_n = 0$$

$$f(t) = \frac{240}{\pi}\left\{1 + \sum_{n=1}^{\infty}\frac{2}{1-4n^2}\cos(240\pi n)t\right\}$$

7.8   Consider the periodic function

$$x(t) = \begin{cases} -5 & 0 < t < \pi \\ 5 & \pi < t < 2\pi \end{cases}$$

and $x(t) = (t + 2\pi)$.   Calculate the Fourier coefficients.   Next plot $x(t)$: $x(t)$ represented by the first term in the Fourier series, $x(t)$ represented by the first two terms of the series, and $x(t)$ represented by the first three terms of the series. Discuss your results.

Solution: For the Fourier Series: $T = 2\pi$ $\omega_T = 1$

$a_0 = 0$

$$a_n = \frac{2}{2\pi} \left\{ \int_0^\pi -5\cos(nt)dt + \int_\pi^{2\pi} 5\cos(nt)dt \right\}$$

$$\Rightarrow a_n = 0$$

$$b_n = \frac{2}{2\pi} \left\{ \int_0^\pi -5\sin(nt)dt \int_\pi^{2\pi} 5\sin(nt)dt \right\} \Rightarrow$$

$$b_n = \frac{-5}{\pi n}[1 - 2\cos(n\pi) + \cos(2n\pi)]$$

$$x(t) = -\frac{5}{\pi}\sum_{n=1}^\infty \frac{1}{n}(1 - 2\cos(n\pi) + \cos(2n\pi))\sin(nt)$$

$$x(t) := -5 + 10 \cdot \Phi(t - \pi) \qquad \text{The funtion for one cycle (0<t<2pi)}$$

$$xf(t, N) := \frac{-5}{\pi} \cdot \sum_{n=1}^N \frac{\left(-2\cos(n\cdot\pi) + 1 + \cos(2\cdot n\cdot\pi)\right)}{n} \cdot \sin(n\cdot t)$$

x(t)

xf(t, 1)

xf(t, 2)

xf(t, 10)
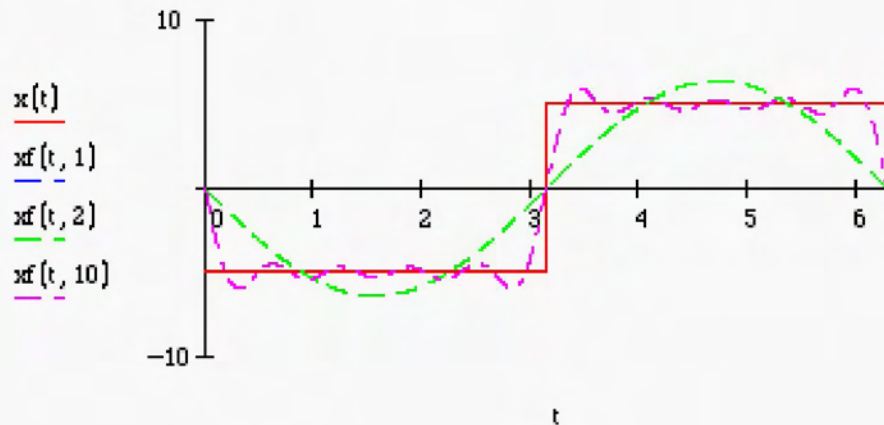
7.9   Consider a signal $x(t)$ with maximum frequency of 500 Hz. Discuss the choice of record length and sampling interval.

Solution:

For a signal with maximum frequency of 500 Hz, the sampling rate, f_s, should be

$$f_s > 2(500) = 1000 \text{ Hz}$$

Due to Shannon's sampling theorem. A better choice would be

$$f_s = 2.5(500) = 1250 \text{ Hz}$$

Thus, the minimum sampling rate is 0.001 sec. and the suggested rate is 0.0008 sec.
Lower sampling rates will produce aliasing.

The record length N is usually a power of 2, such as 512, 1024, 2048, etc.
Windowing is performed to reduce leakage.

Problems and Solutions for Section 7.4 (7.10-7.19)

7.10   Consider the magnitude plot of Figure P7.10. How many natural frequencies does this system have, and what are their approximate values?
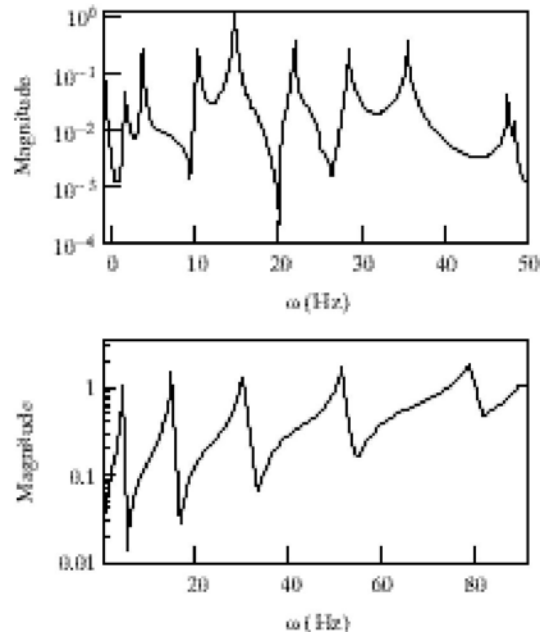
Solution:

The system looks to have 8 modes with approximate natural frequencies of 2, 4, 10, 15, 22, 29, 36, and 47 Hz.

7.11 Consider the experimental transfer function plot of Figure P7.11. Use the methods of Example 7.4.1 to determine ζ_i and ω_i.
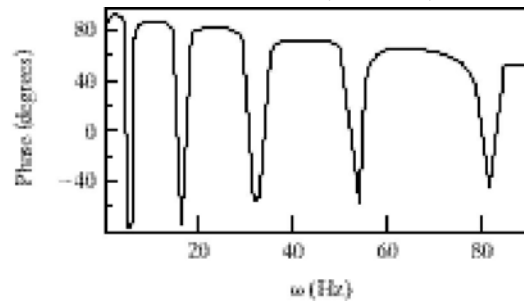
Solution:

For each mode:

$$\zeta_i = \frac{\omega_{bi} - \omega_{ai}}{2\omega_i}$$

where ω_{bi} and ω_{ai} are the frequencies where the magnitude is $\frac{1}{\sqrt{2}}$ of the resonant magnitude. All values given in the following table are approximate.

Mode	ω_i (Hz)	$\left|H(\omega_i)\right|$	$\dfrac{\left|H(\omega_i)\right|}{\sqrt{2}}$	ω_{ai} (Hz)	ω_{bi} (Hz)	ζ_i
1	4.80	0.089	0.063	4.56	5.04	0.049
2	15.20	1.050	0.742	14.76	15.48	0.024
3	30.95	1.800	1.270	30.47	31.19	0.012
4	52.62	2.000	1.414	52.14	52.85	0.007
5	80.00	2.100	1.480	79.05	80.48	0.009

7.12 Consider a two-degree-of-freedom system with frequencies $\omega_1 = 10$ rad/s, $\omega_2 = 15$ rad/s, and damping ratios $\zeta_1 = \zeta_2 = 0.01$. With modal $s = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, calculate the transfer function of this system for an input at x_1 and a response measurement at x_2.

Solution:

Since the natural frequencies, damping ratios and mode shapes are given, the system can be expressed in modal coordinates as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{r}} + \begin{bmatrix} 2(.01)10 & 0 \\ 0 & 2(.01)15 \end{bmatrix}\dot{\mathbf{r}} + \begin{bmatrix} 10^2 & 0 \\ 0 & 15^2 \end{bmatrix}\mathbf{r} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}f(t) = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}f(t)$$

$$y = \frac{1}{\sqrt{2}}\{0 \quad 1\}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{r} = \frac{1}{\sqrt{2}}\{1 \quad 1\}\mathbf{r}$$

This is the representation of the system in modal coordinates, if proportional damping is assumed. The transfer function is:

$$Y(s) = \frac{1}{\sqrt{2}}\{1 \quad 1\}R(s)$$

where

$$R(s) = \frac{1}{\sqrt{2}}\begin{Bmatrix} \dfrac{1}{s^2 + 0.2s + 100} \\ \dfrac{-1}{s^2 + 0.3s + 225} \end{Bmatrix}F(s)$$

Combining the previous two expressions yields

$$\frac{Y(s)}{F(s)} = \frac{(0.1)(s + 1250)}{(s^2 + 0.2s + 100)(s^2 + 0.3s + 225)}$$

7.13 Plot the magnitude and phase of the transfer function of Problem 7.12 and see if you can reconstruct the modal data (ω_1, ω_2, ζ_1, and ζ_2) from your plot.
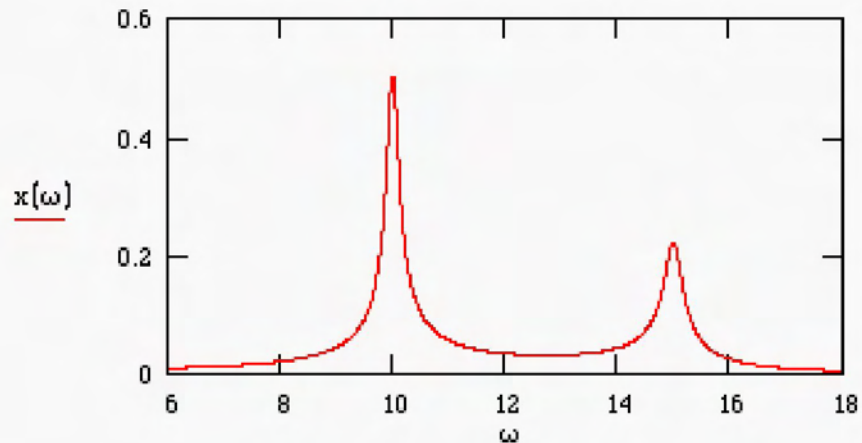
Solution:

For each mode:

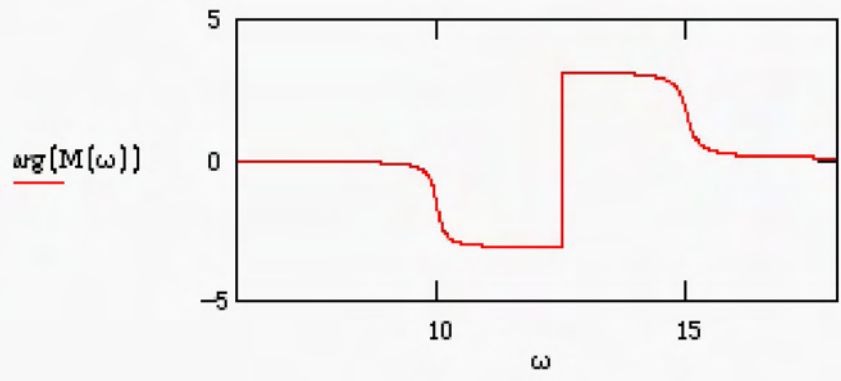$$\zeta_i = \frac{\omega_{bi} - \omega_{ai}}{2\omega_i}$$

where ω_{bi} and ω_{ai} are the frequencies where the magnitude is $1/\sqrt{2}$ of the resonant magnitude. All values in the following table are approximate.

Mode	ω_i (rad/s)	$\left|H(\omega_i)\right|$	$\dfrac{\left|H(\omega_i)\right|}{\sqrt{2}}$	ω_{ai} (rad/s)	ω_{bi} (rad/s)	ζ_i
1	10	0.50	0.354	9.89	10.07	0.009
2	15	0.22	0.156	14.83	15.16	0.011

$$M(\omega) := \frac{(0.1)\cdot(\omega\cdot j + 1250)}{\left(-\omega^2 + 0.2\cdot\omega\cdot j + 100\right)\cdot\left(-\omega^2 + 0.3\cdot\omega\cdot j + 225\right)}$$

$$x(\omega) := \left|M(\omega)\right|$$

7.14 Consider equation (7.14) for determining the damping ratio of a single mode. If the measurement in frequency varies by 1%, how much will the value of ζ change?

Solution:

$$\zeta = \frac{\omega_b - \omega_a}{2\omega_d}$$

If $\omega_d = \omega_{do}(1 \pm 0.01)$ where ω_{do} is the measured natural frequency, then the damping ratio is

$$\zeta = \frac{\omega_b - \omega_a}{2\omega_{do}}\left\{\frac{1}{1 \pm 0.01}\right\} = \zeta_o\left\{\frac{1}{1 \pm 0.01}\right\}$$

If ω_d is $0.99\,\omega_{do}$, then $\zeta = 1.01\,\zeta_o$

If ω_d is $1.01\,\omega_{do}$, then $\zeta = 0.99\,\zeta_o$

Thus, 1 percent changes in the measured natural frequency produce similar changes in the measured damping ratio.

7.15 Discuss the problems of using equation (7.14) if the natural frequencies of the structure are very close together.

Solution:

Equation (7.14) assumes that the response at resonance is due to a single degree of freedom system. If the natural frequencies are very close together, this assumption is not valid. This will introduce error into the damping ratio calculation since the peak response at each resonant frequency will be due to a combination of responses from each of the closely spaced modes.

7.16 Discuss the limitation of using equation (7.15) if ζ is very small. What happens if ζ is very large?

Solution: When ζ is very small (<0.01), it is difficult to determine where $R(\alpha)$ is the largest since equation (7.15) is changing very rapidly in the vicinity of resonance. When ζ is very large (>0.707), the frequency response near resonance is very flat, again making it difficult to determine the damped natural frequency. In either case, experimentally determined damping ratios will contain error since they depend on an accurate determination of the resonant frequency. Problem 7.18 contains plots that illustrate these ideas.

7.17 Consider the two-degree-of-freedom system described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_0 \sin \omega t \\ 0 \end{bmatrix}$$

and calculate the transfer function $|X/F|$ as a function of the damping parameter c.

Solution:

The equations of motion for the system are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\mathbf{x} = \begin{Bmatrix} f_o \\ 0 \end{Bmatrix} f(t)$$

Taking the Laplace transform yields

$$\begin{bmatrix} s^2 + 2 & -1 \\ -1 & s^2 + cs + 2 \end{bmatrix} X(s) = \begin{Bmatrix} f_o \\ 0 \end{Bmatrix} F(s)$$

Inverting the matrix on the left hand side leads to an expression for $X(s)$:

$$X(s) = \frac{1}{(s^2 + 2)(s^2 + cs + 2) - 1}\begin{bmatrix} s^2 + cs + 2 & 1 \\ 1 & s^2 + 2 \end{bmatrix}\begin{Bmatrix} f_o \\ 0 \end{Bmatrix} F(s)$$

Performing the multiplication leads to

$$\frac{X_1(s)}{f_o F(s)} = \frac{s^2 + cs + 2}{s^4 + cs^3 + 4s^2 + 2cs + 3}$$

$$\frac{X_2(s)}{f_o F(s)} = \frac{1}{s^4 + cs^3 + 4s^2 + 2cs + 3}$$

7.18   Plot the transfer function of Problem 7.17 for the four cases: $c = 0.01$, $c = 0.2$, $c = 1$, and $c = 10$. Discuss the difficulty in using these plots to measure ζ_i and ω_i for each value of c.

Solution:

For $c = 0.01$, the resonant peaks are very sharp, making an accurate determination of ζ_i difficult. In the case $c = 0.2$, ζ_i and ω_i could be determined fairly easily using the techniques of section 7.4. Increasing c to 1.0 makes the frequency response very flat, which again makes finding ζ_i and ω_i difficult. Finally, when $c = 10$, it almost looks as if there is one resonant peak, which would lead to a completely erroneous result.

$$M1\left(\omega, c\right) := \frac{-\omega^2 + c \cdot \omega \cdot j + 2}{\omega^4 - c \cdot \omega^3 \cdot j - 4 \cdot \omega^2 + 2 \cdot c \cdot \omega \cdot j + 3}$$
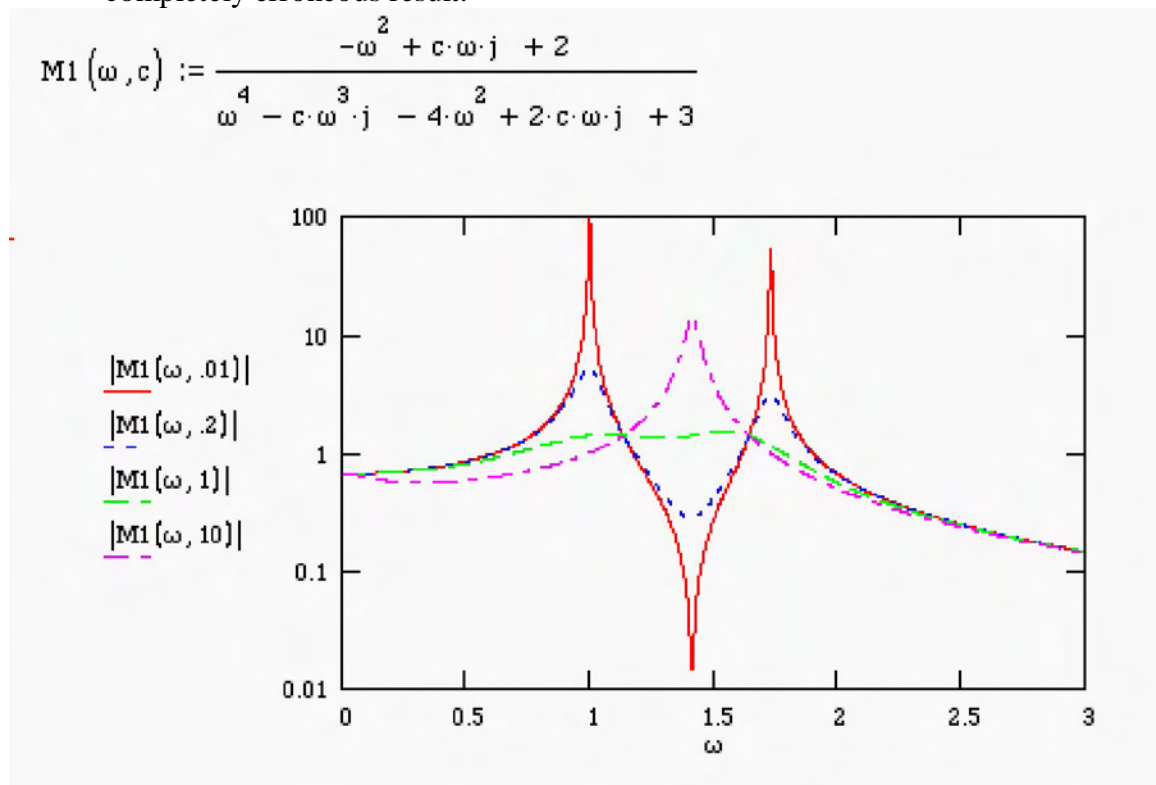
7.19 Use a numerical procedure to calculate the natural frequencies and damping ratios of the system of Problem 7.18. Label these on your plots from Problem 7.18 and discuss the possibility of measuring these values using the methods of Section 7.4

Solution:

For the case where $c = 0.01$

Mode	ω_i (rad/s)	$|H(\omega_i)|$	$\dfrac{|H(\omega_i)|}{\sqrt{2}}$	ω_{ai} (rad/s)	ω_{bi} (rad/s)	ζ_i
1	1.0	59	41.72	0.99	1.02	0.015
2	1.7	48	33.94	1.71	1.69	0.006

Actual values: $\quad \omega_1 = 1.00 \qquad \zeta_1 = 0.003$

$\qquad\qquad\qquad\quad\; \omega_2 = 1.73 \qquad \zeta_2 = 0.001$

The actual values are calculated directly from the equations.

For the case where $c = 0.2$

Mode	ω_i (rad/s)	$|H(\omega_i)|$	$\dfrac{|H(\omega_i)|}{\sqrt{2}}$	ω_{ai} (rad/s)	ω_{bi} (rad/s)	ζ_i
1	1.0	5.1	3.61	0.93	1.06	0.064
2	1.7	2.9	2.05	1.69	1.79	0.030

Actual values: $\quad \omega_1 = 1.00 \qquad \zeta_1 = 0.050$

$\qquad\qquad\qquad\quad\; \omega_2 = 1.73 \qquad \zeta_2 = 0.029$

For the case $c = 0.01$, there is more error in the measured parameters than for the case $c = 0.2$ due to the sharpness of the resonant peak.

$$M1\left(\omega, c\right) := \frac{-\omega^2 + c \cdot \omega \cdot j + 2}{\omega^4 - c \cdot \omega^3 \cdot j - 4 \cdot \omega^2 + 2 \cdot c \cdot \omega \cdot j + 3}$$
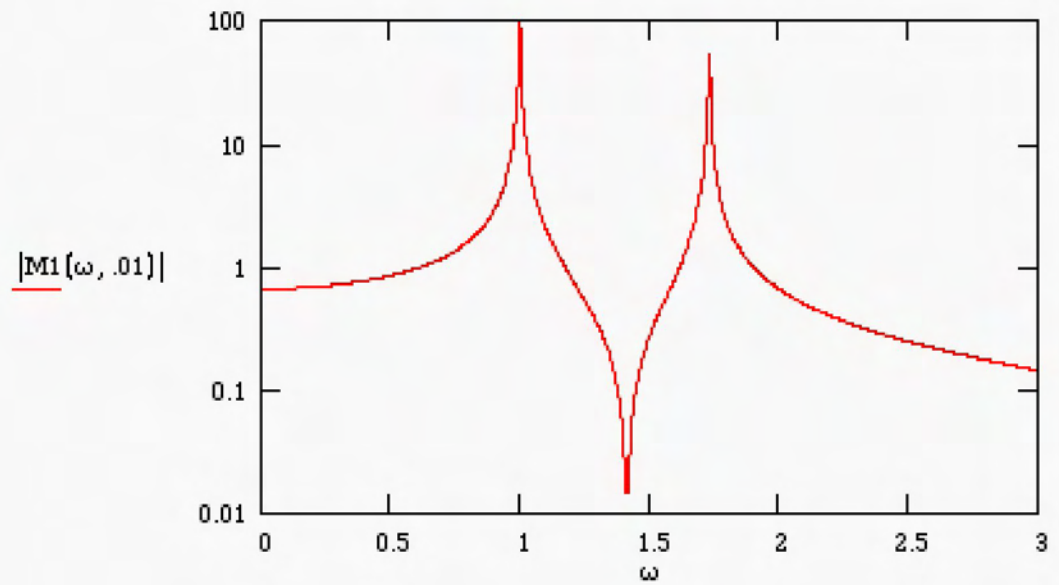
Problems and Solutions Section 7.5 (7.20-7.24)

7.20   Using the definition of the mobility transfer function of Window 7.4, calculate the Re and Im parts of the frequency response function and hence verify equations (7.15) and (7.16).

Solution:

From Window 7.4:

$$\frac{sX(s)}{F(s)} = \frac{s}{ms^2 + cs + k}$$

$$\alpha(\omega) = \frac{j\omega}{(k - m\omega^2) + j\omega c}$$

$$\alpha(\omega) = \frac{j\omega\left[(k - m\omega^2) - j\omega c\right]}{(k - m\omega^2)^2 + (\omega c)^2}$$

$$\alpha(\omega) = \frac{\omega c^2 + j\omega(k - m\omega^2)}{(k - m\omega^2)^2 + (\omega c)^2}$$

The previous expression can be separated into real and imaginary parts:

$$Re[\alpha(\omega)] = \frac{\omega^2 c}{(k - \omega^2 m)^2 + (\omega c)^2} \qquad\qquad Im[\alpha(\omega)] = \frac{\omega(k - \omega^2 m)}{(k - \omega^2 m)^2 + (\omega c)^2}$$

7.21 Using equations (7.15) and (7.16), verify that the Nyquist plot of the mobility frequency response function does in fact form a circle.

Solution:

Define
$$A = \frac{\omega^2 c}{(k - \omega^2 m)^2 + (\omega c)^2} - \frac{1}{2c} = \text{Re}(\alpha) - \frac{1}{2c}$$

$$B = \frac{\omega(k - \omega^2 m)}{(k - \omega^2 m)^2 + (\omega c)^2} = \text{Im}(\alpha)$$

Show that
$$A^2 + B^2 = \left(\frac{1}{2c}\right)^2$$

which is a circle of radius $\frac{1}{2c}$ with center at $Re(\alpha) = \frac{1}{2c}$, $Im(\alpha) = 0$.

$$A^2 + B^2 = \left[\frac{\omega^2 c}{(k - \omega^2 m)^2 (\omega c)^2} - \frac{1}{2c}\right]^2 + \left[\frac{\omega(k - \omega^2 m)}{(k - \omega^2 m)^2 + (\omega c)^2}\right]^2$$

$$A^2 + B^2 = \frac{(\omega^2 c)^2}{\left[(k - \omega^2 m)^2 + (\omega c)^2\right]^2} + \frac{\omega^2 (k - \omega^2 m)^2}{\left[(k - \omega^2 m)^2 + (\omega c)^2\right]^2} - \frac{\omega^2}{(k - \omega^2 m)^2 + (\omega c)^2} + \left(\frac{1}{2c}\right)$$

$$A^2 + B^2 = \frac{\omega^2}{(k - \omega^2 m)^2 + (\omega c)^2}\left[\frac{(k - \omega^2 m)^2 + (\omega c)^2}{(k - \omega^2 m)^2 + (\omega c)^2}\right] - \frac{\omega^2}{(k - \omega^2 m)^2 + (\omega c)^2} + \left(\frac{1}{2c}\right)^2$$

$$A^2 + B^2 = \left(\frac{1}{2c}\right)^2$$

Which is the equation of a circle.

7.22 Consider a single-degree-of-freedom system of mass 10 kg, stiffness 1000 N/m, and damping ratio of 0.01. Pick five values of ω between 0 and 20 rad/s and plot five points of the Nyquist circle using equations (7.15) and (7.16). Do these form a circle?

Solution:

SDOF oscillator:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$m = 10 \text{ kg} \qquad k = 1000 \text{ N/m} \quad \zeta = 0.01$$

First, calculate the damping constant c.

$$\omega_n^2 = \frac{k}{m} = 100$$

$$c = 2\zeta\omega_n m = 2(0.01)(10)(10) = 2 \text{ Ns/m}$$

$$Re[\alpha] = \frac{2\omega^2}{(1000 - 10\omega^2)^2 + (2\omega)^2}$$

$$Im[\alpha] = \frac{\omega(1000 - 10\omega^2)}{(1000 - 10\omega^2)^2 + (2\omega)^2}$$

ω	$Re(\alpha)$	$Im(\alpha)$
9.90	0.2487	0.2500
9.95	0.3996	0.2003
10.00	0.5000	0.0000
10.05	0.4004	-0.1997
10.10	0.2512	0.2500

The following plot displays the 5 points listed in the table, as well as the same plot with a fine discretization of the driving frequency ω.

$$R := \begin{bmatrix} 0.2487 \\ 0.3996 \\ 0.5 \\ 0.4004 \\ 0.2512 \end{bmatrix} \qquad I := \begin{bmatrix} 0.25 \\ 0.2003 \\ 0 \\ -0.1997 \\ 0.25 \end{bmatrix}$$
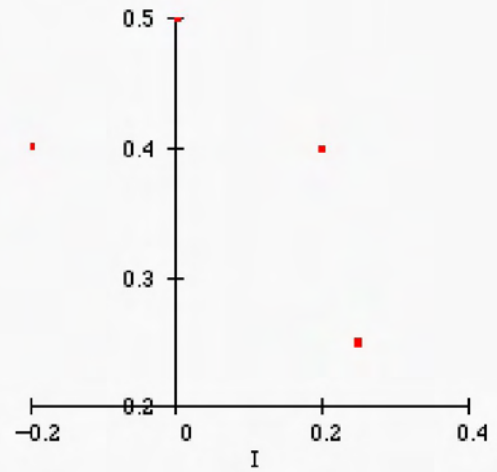
$\omega := 0 , 0.01 .. 10.1$

$$R(\omega) := \frac{2 \cdot \omega^2}{\left(1000 - 10 \cdot \omega^2\right)^2 + \left(2 \, \omega\right)^2}$$

$$I(\omega) := \frac{\omega \cdot \left(1000 - 10 \cdot \omega^2\right)}{\left(1000 - 10 \cdot \omega^2\right)^2 + \left(2 \, \omega\right)^2}$$
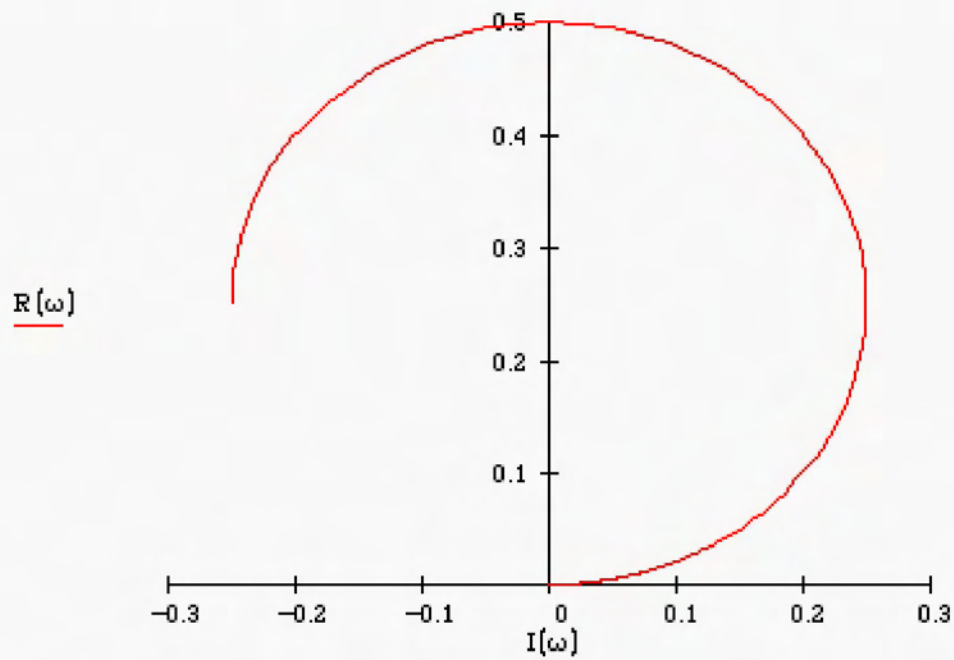
7.23 Derive equation (7.20) for the damping ratio from equations (7.18) and (7.19). Then verify that equation (7.20) reduces to equation (7.21) at the half-power points.

Solution: Begin with equations (7.18) and (7.19)

$$\tan\left(\alpha/2\right)=\frac{\left(\omega_a/\omega_3\right)^2-1}{2\zeta_3\omega_a/\omega_3}$$

$$\tan\left(\alpha/2\right)=\frac{\left(\omega_b/\omega_3\right)^2-1}{2\zeta_3\omega_b/\omega_3}$$

Multiplying the right hand side of each expression by $\dfrac{\omega_3^2}{\omega_3^2}$ yields

$$\tan\left(\alpha/2\right)=\frac{\omega_a^2-\omega_3^2}{2\zeta_3\omega_a\omega_3}$$

$$\tan\left(\alpha/2\right)=\frac{\omega_3^2-\omega_b^2}{2\zeta_3\omega_b\omega_3}$$

After a suitable multiplication, these expressions are:

$$(2\zeta_3\omega_a\omega_3)\tan\left(\alpha/2\right)=\omega_a^2-\omega_3^2$$

$$(2\zeta_3\omega_b\omega_3)\tan\left(\alpha/2\right)=\omega_3^2-\omega_b^2$$

Adding the previous two equations results in:

$$2\zeta_3(\omega_a+\omega_b)\tan\left(\alpha/2\right)=\omega_a^2-\omega_b^2$$

Which can be manipulated to yield equation (7.20)

$$\zeta_3=\frac{\omega_a^2-\omega_b^2}{2\omega_3\left[\omega_a\tan\left(\alpha/2\right)+\omega_b\tan\left(\alpha/2\right)\right]}$$

At the half-power points, $\alpha=90°$ and $\tan\left(\alpha/2\right)=1$, so (7.20) reduces to:

$$\zeta_3=\frac{\omega_a-\omega_b}{2\omega_3}$$

7.24 Consider the experimental curve fit Nyquist circle of Figure P7.24. Determine the modal damping ratio for this mode
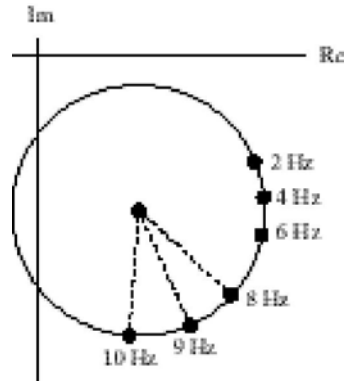
Solution:

From Figure 7.18,

$$\alpha \approx 45°$$
$$\omega_3 = 9\,\text{Hz}$$
$$\omega_b = 10\,\text{Hz}$$
$$\omega_a = 8\,\text{Hz}$$

Using (7.20)

$$\zeta_3 = \frac{10^2 - 8^2}{2(9)\left[8\tan\left(\frac{45°}{2}\right) + 10\tan\left(\frac{45°}{2}\right)\right]}$$

$$\zeta_3 = 0.27$$

Chapter 8
Problems and Solutions Section 8.1 (8.1 through 8.7)

8.1   Consider the one-element model of a bar discussed in Section 8.1. Calculate the finite element of the bar for the case that it is free at both ends rather than clamped.

Solution: The finite element for a rod is derived in section 8.1. Since u_1 is not restrained equations (8.7) and (8.11) are the finite element matrices.

8.2   Calculate the natural frequencies of the free-free bar of Problem 8.1. To what does the first natural frequency correspond? How do these values compare with the exact values obtained from methods of Chapter 6?

Solution:

$$K = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad M = \frac{\rho Al}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$M^{-1}K = \frac{6E}{\rho l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\lambda_{1,2} = 0, \dfrac{12E}{\rho l^2}$ and the corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T / \sqrt{2} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T / \sqrt{2}$$

Therefore, $\omega_1 = 0$, $\omega_2 = \sqrt{\dfrac{12E}{\rho l^2}}$

The first natural frequency corresponds to the rigid body mode, or pure translation.

From the solution to problem 6.8,

$$\omega_1 = 0, \ \omega_2 = \sqrt{\frac{\pi^2 E}{\rho l^2}}$$

The first natural frequency is predicted exactly while the second is 10.2% high. A point of interest is that, due to symmetry, the first mode of a clamped-free rod of length $l/2$ has the same natural frequency as the second mode of a free-free rod of length l.

8.3 Consider the system of Figure P8.3, consisting of a spring connected to a clamped-free bar. Calculate the finite element model and discuss the accuracy of the frequency prediction of this model by comparing it with the method of Chapter 6.
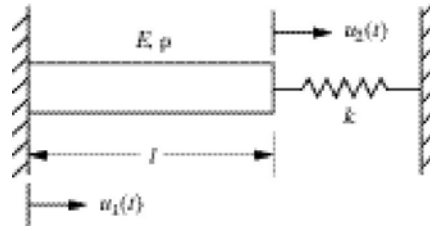
Solution:

The finite element for the clamped-free rod is given by (8.14) as

$$\frac{\rho A l}{3}\ddot{u}_2(t) + \frac{EA}{l}u_2(t) = 0$$

The spring has the effect of adding stiffness K at u_2. Thus,

$$\frac{\rho A l}{3}\ddot{u}_2(t) + \left(\frac{EA}{l} + K\right)u_2(t) = 0$$

From (1.16)

$$\omega = \sqrt{\frac{3(Kl + EA)}{\rho A l}}$$

Next consider the first natural frequency as predicted from the distributed parameter approach of chapter 6. In particular Table 6.1 gives the frequency equation for this system as $\lambda_n \cot \lambda_n = -(Kl/EA)$ where $\lambda_n = \omega_n l/c$, $c^2 = E/\rho$. Approximating $\cot x = 1/x - x/3$ the frequency equation of Table 6.1 becomes

$$\lambda_n (1/\lambda_n - \lambda_n/3) = -(kl/EA) \quad \text{or for } n=1 \quad \omega^2 l^2/c^2 = 3(1+kl/EA)$$

which upon solving for ω is identical to the one element FEM frequency derived above.

8.4   Consider a clamped-free bar with a force $f(t)$ applied in the axial direction at the free end as illustrated in Figure P8.4. Calculate the equations of motion using a single-element finite element model.
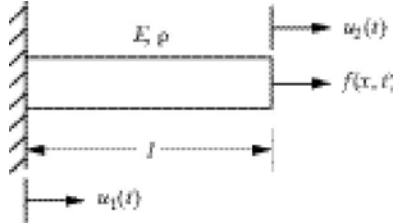
Solution:

The finite element equation of motion for an unforced clamped-free bar is given by equation (8.14). Using (8.13) it can be seen that the forced equation is

$$\frac{\rho A l}{3} \ddot{u}_2(t) + \frac{EA}{l} u_2(t) = f(t)$$

8.5 Compare the solution of a cantilevered bar modeled as a single finite element with that of the distributed-parameter method summarized in Figure 8.1 truncated at three modes by calculating (a) $u(x,t)$ and (b) $u(l/2,t)$ for a 1-m aluminum beam at $t = 0.1$, 1, and 10s using both methods. Use the initial condition $u(x,0) = 0.1x$ m and $u_t(x,0) = 0$.

Solution: (8.5, 8.6)

For the finite element of the bar

$$\rho = 2700 \text{ kg/m}^3, \quad E = 7 \times 10^{10} \text{ N/m}^2$$

The unforced equation of motion is then

$$\ddot{u}_2(t) + 7.78 \times 10^7 u_2(t) = 0$$

From window 8.2

$$u_2(t) = .1\cos(8.819 \times 10^3 t)$$

Using the shape functions for the bar

$$u(x,t) = u_2(t)x = .1x\cos(8.819 \times 10^3 t)$$

For the continuous model truncated at 3 modes, (see example6.3.1)

$\omega_{1,2,3} = 8000$ rad/s, 24000 rad/s, 40000 rad/s and the mode shapes are

$$X_1(x) = \sin\left(\frac{\pi}{2}\frac{x}{l}\right) \quad , \quad X_4(x) = \sin\left(\frac{7\pi}{2}\frac{x}{l}\right)$$

$$X_2(x) = \sin\left(\frac{3\pi}{2}\frac{x}{l}\right) \quad , \quad X_5(x) = \sin\left(\frac{9\pi}{2}\frac{x}{l}\right)$$

$$X_3(x) = \sin\left(\frac{5\pi}{2}\frac{x}{l}\right)$$

The solution is given by (6.27) as

$$u(x,t) = \sum_{n=1}^{\infty}(c_n \sin\omega_n t + d_n \cos\omega_n t)X_n(x)$$

Since we are given $\dot{u}(x,t) = 0$, $c_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty}a_n \cos(\omega_n t)X_n(x)$$

Considering the initial condition $u(x,0) = .1x$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{(2n-1)\pi}{2}\frac{x}{l} = .1x$$

Multiplying by $\sin\frac{(2m-1)\pi}{2}\frac{x}{l}$ and integrating from $x = 0$ to $x = l$,

$$\int_0^l .1x \sin\left(\frac{(2m-1)\pi}{2}\frac{x}{l}\right)dx = a_n \int_0^l \sin\left(\frac{(2n-1)\pi}{2}\frac{x}{l}\right)\sin\left(\frac{(2m-1)\pi}{2}\frac{x}{l}\right)dx$$

$$\int_0^l .1x \sin\left(\frac{(2m-1)\pi}{2}\frac{x}{l}\right)dx = a_n\left(\frac{l^2}{2}\right)$$

$$a_n = \frac{2}{l^2}\int_0^l .1x \sin\left(\frac{(2n-1)\pi}{2}\frac{x}{l}\right)dx$$

$a_1=.08106$, $a_2=-.009006$, $a_3=.003242$, $a_4=0.001654$, $a_5=.001001$

from (6.63)

$$\omega_n = \frac{(2n-1)\pi}{2}\sqrt{\frac{E}{\rho}}, \quad \omega_1 = \frac{\pi}{2}\sqrt{\frac{E}{\rho}} = 7998\text{rad/s}, \quad \omega_2 = \frac{3\pi}{2}\sqrt{\frac{E}{\rho}} = 23994\text{rad/s},$$

$$\omega_3 = \frac{5\pi}{2}\sqrt{\frac{E}{\rho}} = 39990\text{rad/s}, \quad \omega_4 = \frac{7\pi}{2}\sqrt{\frac{E}{\rho}} = 55987\text{rad/s}$$

$$\omega_5 = \frac{9\pi}{2}\sqrt{\frac{E}{\rho}} = 71982\text{rad/s}$$

Substitution into 6.27 yields

$$\omega(x,t) = .08106\cos(7998t)\sin\left(\frac{\pi}{2}\frac{x}{l}\right) - .00901\cos(23994t)\sin\left(\frac{3\pi}{2}\frac{x}{l}\right)$$

$$+ .00324\cos(39990t)\sin\left(\frac{5\pi}{2}\frac{x}{l}\right) - .00165\cos(55987t)\sin\left(\frac{7\pi}{2}\frac{x}{l}\right)$$

$$+ .001001\cos(71982t)\sin\left(\frac{9\pi}{2}\frac{x}{l}\right)$$

Note that for problem 8.5 the last two terms are neglected.

8.5

$$u(x,t)\big|_{t=.1} = -.021205\sin\left(\frac{\pi x}{2}\right) - .00643\sin\left(\frac{3\pi x}{2}\right) - .00314\left(\frac{5\pi x}{2}\right)$$

$$u(x,t)\big|_{t=1} = .07133\sin\left(\frac{\pi x}{2}\right) - .00077\sin\left(\frac{3\pi x}{2}\right) - .00255\left(\frac{5\pi x}{2}\right)$$

$$u(x,t)\big|_{t=10} = .01900\sin\left(\frac{\pi x}{2}\right) - .00587\sin\left(\frac{3\pi x}{2}\right) - .003\left(\frac{5\pi x}{2}\right)$$

$$u(x,t)\big|_{t=.1,\,x=.5} = -.01732$$

$$u(x,t)\big|_{t=1,\,x=.5} = .05169$$

$$u(x,t)\big|_{t=10,\,x=.5} = .01546$$

8.6   Repeat Problem 8.5 using a five-mode model. Can you draw any conclusions?

Solution:

$$u(x,t)\big|_{t=.1} = -.0212\sin\left(\frac{\pi x}{2}\right) - .00643\sin\left(\frac{3\pi x}{2}\right) - .00314\left(\frac{5\pi x}{2}\right)$$

$$- .00153\sin\left(\frac{7\pi x}{2}\right) - .00069\sin\left(\frac{9\pi x}{2}\right)$$

$$u(x,t)\big|_{t=1} = .07133\sin\left(\frac{\pi x}{2}\right) - .0077\sin\left(\frac{3\pi x}{2}\right) - .00255\left(\frac{5\pi x}{2}\right)$$

$$+ .00129\sin\left(\frac{7\pi x}{2}\right) - .00026\sin\left(\frac{9\pi x}{2}\right)$$

$$u(x,t)\big|_{t=10} = .01900\sin\left(\frac{\pi x}{2}\right) + .00587\sin\left(\frac{3\pi x}{2}\right) - .00300\left(\frac{5\pi x}{2}\right)$$

$$- .00146\sin\left(\frac{7\pi x}{2}\right) + .00085\sin\left(\frac{9\pi x}{2}\right)$$

$$u(x,t)\big|_{t=.1,x=.5} = -.01672$$

$$u(x,t)\big|_{t=1,x=.5} = .05060$$

$$u(x,t)\big|_{t=10,x=.5} = .01709$$

For the finite element solution from (8.17)

$$u(x,t) = .1x\cos(8819.2t)$$

$$u(x,t)\big|_{t=.1} = -.06445x \qquad u(x,t)\big|_{t=.1,x=.5} = -.03222$$

$$u(x,t)\big|_{t=1} = -.07515x \qquad u(x,t)\big|_{t=1,x=.5} = -.03758$$

$$u(x,t)\big|_{t=10} = .06047x \qquad u(x,t)\big|_{t=10,x=.5} = .03024$$

Conclusion: Not nearly enough elements were used to accurately determine the 1st natural frequency. Since the 1st mode dominates the response (this can be seen by comparing the coefficients, a_n), it must be determined well in order to predict the rod's response.

8.7 Repeat Problem 8.5 using only the first mode in the series solution and the initial condition $u(x,0) = 0.1\sin(\pi x/2l)$, $u_t(x,0) = 0$. For this initial condition, the first mode is exact. Why?

Solution:

Using the same procedure as in problem 8.5, the solution is

$$u(x,t) = .1\sin\left(\frac{\pi x}{2}\right)\cos(7998t)$$

$$u(x,t)\big|_{t=.1} = -.02616\sin\left(\frac{\pi x}{2}\right) \qquad u(x,t)\big|_{t=.1, x=.5} = -.01850$$

$$u(x,t)\big|_{t=1} = .08800\sin\left(\frac{\pi x}{2}\right) \qquad u(x,t)\big|_{t=1, x=.5} = .06223$$

$$u(x,t)\big|_{t=10} = .02344\sin\left(\frac{\pi x}{2}\right) \qquad u(x,t)\big|_{t=10, x=.5} = .01657$$

The finite element solution is unchanged. Again there is horrible agreement between the finite element model and the distributed parameter model.

The fist mode is exact because the initial condition is in the first mode. All coefficients, a_n, for modes other than the first mode are zero.

Problems and Solutions Section 8.2 (8.8 through 8.20)

8.8   Consider the bar of Figure P8.3 and model the bar with two elements. Calculate the frequencies and compare them with the solution obtained in Problem 8.3. Assume material properties of aluminum, a cross-sectional area of 1 m, and a spring stiffness of 1×10^6 N/m.

Solution: The finite element model for the two-element bar is
$$M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{0}$$
where $\mathbf{u}(t) = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$

$$u_1 \qquad u_2$$
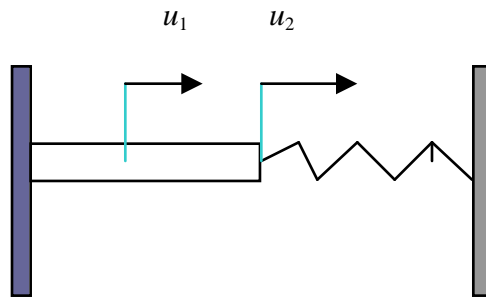
$$M = \frac{\rho A l}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \quad K = \frac{2EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

As in problem 8.3, the spring adds a stiffness K to degree of freedom 2. The equation of motion is then

$$\frac{\rho A l}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\ddot{\mathbf{u}}(t) + \frac{2EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 1+\dfrac{Kl}{2EA} \end{bmatrix}\mathbf{u}(t) = \mathbf{0}$$

The natural frequencies can be found by eigenanalysis. Using the material properties of aluminum

$$\rho = 2700\text{kg/m}^3 \,, \ E = 7 \times 10^{10}\text{Pa}$$

$\omega_1 = 129.0$ rad/s

$\omega_2 = 368.4$ rad/s

The solution obtained in problem 8.4 is $\omega_1 = 149.1$ rad/s.

8.9 Repeat Problem 8.8 with a three-element model. Calculate the frequencies and compare them with those of Problem 8.8.

Solution:

The finite element model of the 3 element rod for equal length elements is (from equation (8.25))

$$\frac{\rho Al}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\ddot{u}+\frac{3EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}u = 0$$

With the spring stiffness included, the global stiffness becomes

$$K = \frac{3EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1+\dfrac{Kl}{3EA} \end{bmatrix}$$

Solving for the natural frequencies gives $\omega_1 = 125.85$ rad/s, $\omega_2 = 333.1$ rad/s, and $\omega_3 = 591.7$ rad/s

The natural frequencies predicted in 8.9 should be better than those predicted in 8.8. You can compare them to the results of 2 element model by using VTB8_2 and loading the file p8_3_10.con.

8.10   Consider Example 8.2.2. Repeat this example with node 2 moved to $\ell/2$ so that
the mesh is uniform. Calculate the natural frequencies and compare them to those
obtained in the example. What happens to the mass matrix?

Solution: (8.10, 8.11)

The equation of motion can be shown to be

$$\frac{\rho A l}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \frac{2EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{u} = \mathbf{0}$$

$$\omega_1 = \frac{1.16114}{l}\sqrt{\frac{E}{\rho}} = 8204.8 \text{ rad/s}$$

$$\omega_2 = \frac{5.6293}{l}\sqrt{\frac{E}{\rho}} = 28663 \text{ rad/s}$$

The first natural frequency is slightly improved (closer to the distributed parameter
'true' value) while the second natural frequency has become worse.

	Truth	Example 8.22	Problem 8.10	Example 8.2.1
ω_1	$1.571\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$1.643\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$1.611\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$1.579\frac{1}{l}\sqrt{\frac{E}{\rho}}$
ω_2	$4.712\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$5.196\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$5.629\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$5.167\frac{1}{l}\sqrt{\frac{E}{\rho}}$

The natural frequencies found using the 3 element model are much better than the
2 element model.

8.11   Compare the frequencies obtained in Problem 8.10 with those obtained in Section
8.2 using three elements.

Solution:

See the solution for problem 8.10.

8.12   As mentioned in the text, the usefulness of the finite element method rests in problems that cannot readily be solved in closed form. To this end, consider a section of an air frame sketched in Figure P8.13 and calculate a two-element finite model of this structure (i.e., find M and K) for a bar with

Solution:

$$A(x) = \frac{\pi}{4}\left[h_1^2 + \left(\frac{h_2 - h_1}{l} \right)^2 x^2 + 2h_1\left(\frac{h_2 - h_1}{l} \right)x \right]$$

Two methods exist for creating a finite element model for this wing. The first is to assume each element has a constant cross section. The second is to derive elements based on the variable cross section. If enough elements are used, constant cross section elements can yield acceptable results. However, since in this example only two elements are used, it is better to use a variable cross section element. Both solutions are given.

A: Variable cross section elements

Following the procedure of section 8.1, the shape function of the first element is given by

$$u(x,t) = \left(1 - \frac{2x}{l} \right)u_1(t) + \frac{2x}{l}u_2(t)$$

The strain energy for element 1 is given by

$$V_1(t) = \int_0^{l/2} EA(x)\left[\frac{\partial u_1(x,t)}{\partial x} \right]^2 dx$$

$$= \frac{E\pi}{48l}[(7h_1^2 + 4h_1h_2 + h_2^2)u_1^2(t) - (14h_1^2 + 8h_1h_2 + 2h_2^2)u_1(t)u_2(t)$$

$$+ (7h_1^2 + 4h_1h_2 + h_2^2)u_2^2(t)]$$

However, since $u_1(t) = 0$,

$$V_1(t) = \frac{E\pi}{48l}(7h_1^2 + 4h_1h_2 + h_2^2)u_2^2(t)$$

For element 2, the shape function is

$$u_2(x,t) = 2\left(1 - \frac{x}{l} \right)u_2(t) + \left(\frac{2x}{l} - 1 \right)u_3(t)$$

The strain energy for element 2 is then given by

$$V_2(t) = \frac{1}{2} \int_{l/2}^{l} EA(x) \left[\frac{\partial u_2(x,t)}{\partial x} \right]^2 dx$$

$$= \frac{E\pi}{48l} \left(h_1^2 + 4h_1 h_2 + 7h_2^2 \right) \left(u_2^2(t) + 2u_2(t)u_3(t) + u_3^2(t) \right)$$

The total strain energy is then

$$V(t) = \frac{E\pi}{48l} \left((f_1 + f_2)u_2^2(t) - 2f_2 u_2(t)u_3(t) + f_2 u_3^2(t) \right)$$

where $f_1 = 7h_1^2 + 4h_1 h_2 + h_2^2$ and $f_2 = h_1^2 + 4h_1 h_2 + 7h_2^2$

In matrix form this is

$$V(t) = \frac{1}{2} \begin{bmatrix} u_2(t) & u_3(t) \end{bmatrix} K \begin{bmatrix} u_2(t) & u_3(t) \end{bmatrix}^T$$

where

$$K = \frac{E\pi}{24l} \begin{bmatrix} f_1 \times f_2 & -f_2 \\ -f_2 & f_2 \end{bmatrix}$$

The kinetic energy of element 1 is given by

$$T_1(t) = \int_{0}^{l/2} A(x) \rho \left[\frac{\partial u_1(x,t)}{\partial x} \right]^2 dx$$

$$= \frac{l\pi\rho}{1920} (16h_1^2 + 18h_1 h_2 + 6h_2^2) \dot{u}_2^2(t)$$

(since $\dot{u}_1(t) = 0$, terms including $\dot{u}_1(t)$ have been dropped)

Similarly, the kinetic energy of element 2 is

$$T_2(t) = \int_{l/2}^{l} A(x)\rho \left[\frac{\partial u_2(x,t)}{\partial x} \right]^2 dx = \frac{l\pi\rho}{1920} [(6h_1^2 + 18h_1 h_2 + 16h_2^2)\dot{u}_2^2$$

$$+ (3h_1^2 + 8h_1 h_2 + 31h_2^2)\dot{u}_2 \dot{u}_3 + (h_1^2 + 8h_1 h_2 + 31h_2^2)\dot{u}_3^2]$$

The total kinetic energy can be written

$$T(t) = \frac{l\pi\rho}{1920}[(22h_1^2 + 36h_1h_2 + 22h_2^2)\dot{u}_2^2 + (3h_1^2 + 14h_1h_2 + 23h_2^2)\dot{u}_2\dot{u}_3$$

$$+ (h_1^2 + 8h_1h_2 + 31h_2^2)\dot{u}_3^2] = \frac{1}{2}[\dot{u}_2\,\dot{u}_3]M[\dot{u}_2\,\dot{u}_3]^T$$

where

$$M = \frac{l\pi\rho}{1920}\begin{bmatrix} 44h_1^2 + 72h_1h_2 + 44h_2^2 & 3h_1^2 + 14h_1h_2 + 23h_2^2 \\ 3h_1^2 + 14h_1h_2 + 23h_2^2 & 2h_1^2 + 16h_1h_2 + 62h_2^2 \end{bmatrix}$$

B: Constant cross section elements

The average cross section area of element 1 is

$$A_1 = \frac{\pi}{48}(7h_1^2 + 4h_1h_2 + h_2^2)$$

and the average cross section area of element 2 is

$$A_2 = \frac{\pi}{48}(h_1^2 + 4h_1h_2 + 7h_2^2)$$

Finding the potential energy again yields the same global stiffness matrix as for the variable cross section model.

The kinetic energy can then be found by

$$T(t) = \frac{1}{2}\int_0^{l/2} A_1\rho\left[\frac{\partial u_1(x,t)}{\partial x}\right]^2 dx + \frac{1}{2}\int_{l/2}^{l} A_2\rho\left[\frac{\partial u_2(x,t)}{\partial x}\right]^2 dx$$

$$= \frac{1}{2}[\dot{u}_2\,\dot{u}_3]M[\dot{u}_2\,\dot{u}_3]^T$$

where

$$M = \frac{\rho l}{12}\begin{bmatrix} 2(A_1 + A_2) & A_2 \\ A_2 & 2A_2 \end{bmatrix}$$

which is not identical to the mass matrix derived using variable cross section elements.

8.13 Let the bar in Figure P8.13 be made of aluminum 1 m in length with $h_1 = 20$ cm and $h_2 = 10$ cm. Calculate the natural frequencies using the finite element model of Problem 8.12.
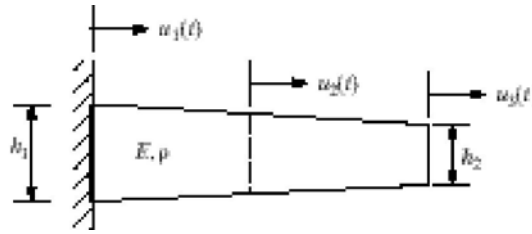
Solution:

$E = 7 \times 10^{10}$ Pa, $\rho = 2700$ kg/m^3

$h_1 = .2$m, $h_2 = .1$m, $l = 1$m

Using the variable cross section elements

$$K = \begin{bmatrix} 2.566 \times 10^9 & -8.705 \times 10^8 \\ -8.705 \times 10^8 & 8.705 \times 10^8 \end{bmatrix}$$

and

$$M = \begin{bmatrix} 16.081 & 2.783 \\ 2.783 & 4.506 \end{bmatrix}$$

The natural frequencies are then $\omega_1 = 7414$ rad/s and $\omega_2 = 20368$ rad/s

The constant cross sectional area mass matrix is

$$M = \begin{bmatrix} 16.493 & 2.798 \\ 2.798 & 5.596 \end{bmatrix}$$

which give $\omega_1 = 7092$ rad/s, $\omega_2 = 18636$ rad/s

8.14 Repeat Problems 8.12 and 8.13 using a three-element four-node finite element model.

Solution:

The shape functions for 3 evenly spaced elements are

$$u_1(x,t) = \left(1 - \frac{3x}{2l}\right)u_1(t) + \frac{3x}{l}u_2(t)$$

$$u_2(x,t) = 2\left(1 - \frac{3x}{2l}\right)u_2(t) + \left(\frac{3x}{l} - 1\right)u_3(t)$$

$$u_3(x,t) = 3\left(1 - \frac{x}{l}\right)u_3(t) + 2\left(\frac{3x}{2l} - 1\right)u_4(t)$$

Integrating to find the strain energy, the strain energies in matrix notation are

$$V_1(t) = \frac{1}{2}\begin{bmatrix} u_1 & u_2 \end{bmatrix} K_1 \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$$

$$V_2(t) = \frac{1}{2}\begin{bmatrix} u_2 & u_3 \end{bmatrix} K_2 \begin{bmatrix} u_2 & u_3 \end{bmatrix}^T$$

$$V_3(t) = \frac{1}{2}\begin{bmatrix} u_3 & u_4 \end{bmatrix} K_3 \begin{bmatrix} u_3 & u_4 \end{bmatrix}^T$$

where

$$K_1 = \frac{E\pi}{36l}(19h_1^2 + 7h_1h_2 + h_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = \frac{E\pi}{36l}(7h_1^2 + 13h_1h_2 + 7h_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_3 = \frac{E\pi}{36l}(h_1^2 + 7h_1h_2 + 19h_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Writing the total strain energy in matrix form, the global stiffness matrix is

$$K = \frac{E\pi}{36l}\begin{bmatrix} f_1 + f_2 & -f_2 & 0 \\ -f_2 & f_2 + f_3 & -f_3 \\ 0 & -f_3 & f_3 \end{bmatrix}$$

where

$$f_1 = 19h_1^2 + 7h_1h_2 + h_2^2, \quad f_2 = 7h_1^2 + 13h_1h_2 + 7h_2^2 \quad \text{and} \quad f_3 = h_1^2 + 7h_1h_2 + 19h_2^2$$

The kinetic energy of each element in matrix form is

$$T_1(t) = \frac{1}{2}\begin{bmatrix} \dot{u}_1 & \dot{u}_2 \end{bmatrix} M_1 \begin{bmatrix} \dot{u}_1 & \dot{u}_2 \end{bmatrix}^T, \quad T_2(t) = \frac{1}{2}\begin{bmatrix} \dot{u}_2 & \dot{u}_3 \end{bmatrix} M_2 \begin{bmatrix} \dot{u}_2 & \dot{u}_3 \end{bmatrix}^T,$$

$$T_3(t) = \frac{1}{2}\begin{bmatrix} \dot{u}_3 & \dot{u}_4 \end{bmatrix} M_3 \begin{bmatrix} \dot{u}_3 & \dot{u}_4 \end{bmatrix}^T$$

where

$$M_1 = \frac{l\pi\rho}{3240}\begin{bmatrix} 76h_1^2 + 13h_1h_2 + h_2^2 & \frac{1}{2}\left(63h_1^2 + 24h_1h_2 + 3h_2^2\right) \\ \frac{1}{2}\left(63h_1^2 + 24h_1h_2 + 3h_2^2\right) & 51h_1^2 + 33h_1h_2 + 6h_2^2 \end{bmatrix}$$

$$M_2 = \frac{l\pi\rho}{3240}\begin{bmatrix} 31h_1^2 + 43h_1h_2 + 16h_2^2 & \frac{1}{2}\left(23h_1^2 + 44h_1h_2 + 23h_2^2\right) \\ \frac{1}{2}\left(23h_1^2 + 44h_1h_2 + 23h_2^2\right) & 16h_1^2 + 43h_1h_2 + 31h_2^2 \end{bmatrix}$$

$$M_3 = \frac{l\pi\rho}{3240}\begin{bmatrix} 6h_1^2 + 33h_1h_2 + 51h_2^2 & \frac{1}{2}\left(3h_1^2 + 24h_1h_2 + 63h_2^2\right) \\ \frac{1}{2}\left(3h_1^2 + 24h_1h_2 + 63h_2^2\right) & h_1^2 + 13h_1h_2 + 76h_2^2 \end{bmatrix}$$

Evaluating and assembling the mass and stiffness matrices gives:

$$K = \begin{bmatrix} 9.285 & -3.726 & 0 \\ -3.729 & 5.987 & -2.2602 \\ 0 & -2.2602 & 2.2602 \end{bmatrix} \times 10^9$$

$$M = \begin{bmatrix} 13.1423 & 2.6573 & 0 \\ 2.6573 & 8.4299 & 1.6101 \\ 0 & 1.6101 & 2.7751 \end{bmatrix}$$

$\omega_1 = 10406$ rad/s, $\omega_2 = 27309$ rad/s, $\omega_3 = 47797$ rad/s

Note that a ten element model yields

$\omega_1 = 10316$ rad/s, $\omega_2 = 25183$ rad/s

8.15 Consider the machine punch of Figure P8.15. This punch is made of two materials and is subject to an impact in the axial direction. Use the finite element method with two elements to model this system and estimate (calculate) the first two natural frequencies. Assume $E_1 = 8 \times 10^{10}$ Pa, $E_2 = 2.0 \times 10^{11}$ Pa, $\rho_1 = 7200$ kg/m^3, $\rho_2 = 7800$ kg/m^3, $l = 0.2$ m, $A_1 = 0.009$ m^2, and $A_2 = 0.0009$ m^2.
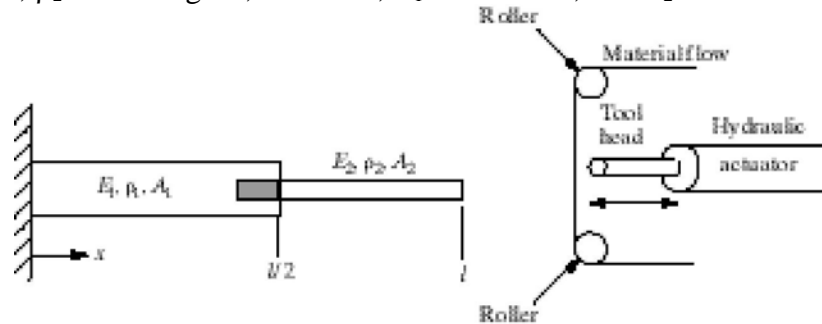
Solution: The total strain energy of the system is

$$V(t) = \frac{1}{2}\left\{ u_1^2 \frac{2E_1 A_1}{l} + \frac{2E_1 A_1}{l}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right\}$$
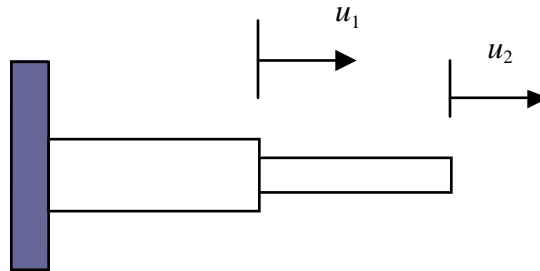
The vector of derivatives of the potential energy gives

$$\begin{bmatrix} \dfrac{\partial V}{\partial u_1} \\ \dfrac{\partial V}{\partial u_2} \end{bmatrix} = \frac{2}{l}\begin{bmatrix} E_1 A_1 + E_2 A_2 & -E_2 A_2 \\ -E_2 A_2 & E_2 A_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The stiffness matrix is then

$$K = \frac{2}{l}\begin{bmatrix} E_1 A_1 + E_2 A_2 & -E_2 A_2 \\ -E_2 A_2 & E_2 A_2 \end{bmatrix}$$

In similar fashion, the total kinetic energy is

$$T(t) = \frac{1}{2}\left\{ \dot{u}_1^2 \frac{\rho_1 A_1 l}{6} + \frac{l}{12}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}^T \begin{bmatrix} 2\rho_2 A_2 & \rho_2 A_2 \\ \rho_2 A_2 & 2\rho_2 A_2 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \right\}$$

The mass matrix is then

$$M = \frac{l}{12}\begin{bmatrix} 2(\rho_2 A_2 + \rho_1 A_1) & \rho_2 A_2 \\ \rho_2 A_2 & 2\rho_2 A_2 \end{bmatrix}$$

$E_1 = 8 \times 10^{10}\,\text{Pa},\ \rho_1 = 7200\text{kg/m}^3,\ E_2 = 2.0 \times 10^{11}\,\text{Pa},\ \rho_2 = 7800\text{kg/m}^3,$

$l = .2 A_1 = .0009,\ A_2 = .0001$

$$K = \begin{bmatrix} 9.2 & -2 \\ -2 & 2 \end{bmatrix} \times 10^8 \quad M = \begin{bmatrix} .242 & .013 \\ .013 & .026 \end{bmatrix}$$

$\omega_1 = 47556.1$ rad/s, $\omega_2 = 101975$ rad/s

8.16 Recalculate the frequencies of Problem 8.15 assuming that it is made entirely of one material and size (i.e., $E_1 = E_2$, $\rho_1 = \rho_2$, and $A_1 = A_2$), say steel, and compare your results to those of Problem 8.15.

Solution:

Assume $A_1 = A_2$, $E_1 = E_2$, $\rho_1 = \rho_2$

$$K = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \times 10^8 \quad M = \begin{bmatrix} .052 & .013 \\ .013 & .026 \end{bmatrix}$$

$\omega_1 = 40798.6$ rad/s, $\omega_2 = 142525$ rad/s

The first natural frequency decreased. This example illustrates how a punch can be modified to raise the first natural frequency by changing the base material.

8.17 A bridge support column is illustrated in Figure P8.17. The column is made of concrete with a cross-sectioned area defined by $A(x) = A_0 e^{-x/l}$, where A_0 is the area of the column at ground. Consider this pillar to be cantilevered (i.e., fixed) at ground level and to be excited sinusoidally at its tip in the longitudinal direction due to traffic over the bridge. Calculate a single-element finite element model of this system and compute its approximate natural frequency.

(a)   (b)
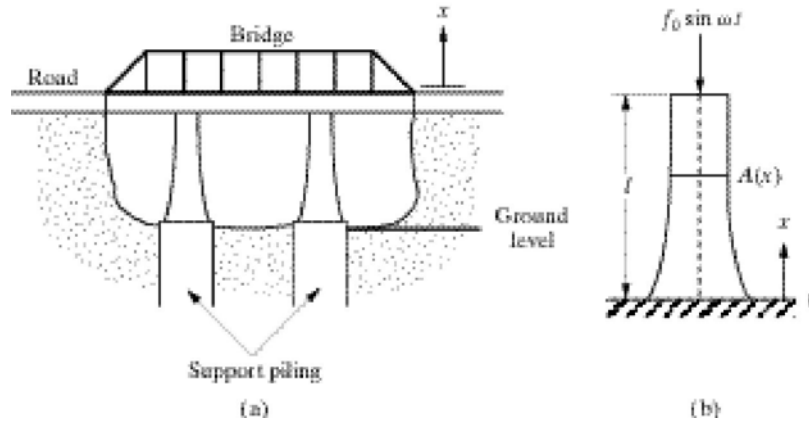
Solution:

$$A(x) = A_0 e^{-x/l}$$

The potential energy is

$$V(t) = \frac{E}{2} \int_0^l A(x) \left[\frac{\partial u(x,t)}{\partial x} \right]^2 dx$$

where $u(x,t) = \left(1 - \frac{x}{l}\right) u_1(t) + \frac{x}{l} u_2(t)$

$$V(t) = \frac{EA_0}{2l} \frac{e-1}{e} \left(u_1(t) - u_2(t)\right)^2$$

$$= \frac{EA}{2l} \frac{e-1}{e} u_2^2(t)$$

The stiffness is then

$$K = \frac{EA}{l} \frac{(e-1)}{e}$$

Likewise, the kinetic energy is

$$T(t) = \frac{1}{2} \int_0^l \rho A \left[\frac{\partial u(x,t)}{\partial x} \right]^2 dx = \frac{Al\rho}{2e} (2e-5) \dot{u}_2^2(t)$$

The mass is then

$$M = \frac{Al\rho}{e}(2e - 5)$$

The first natural frequency is then approximately

$$\omega_1 = \sqrt{\frac{K}{M}} = \sqrt{\frac{E(e-1)}{(2e-5)l^2\rho}} = \frac{1.984}{l}\sqrt{\frac{E}{\rho}}$$

8.18   Redo Problem 8.17 using two elements. What would happen if the "traffic" frequency corresponds with one of the natural frequencies of the support column?

Solution: The shape functions for a 2 element model are

$$u_1(x,t) = \left(1 - \frac{2x}{l}\right)u_1(t) + \frac{2x}{l}u_2(t)$$

$$u_2(x,t) = 2\left(1 - \frac{x}{l}\right)u_2(t) + \left(\frac{2x}{l} - 1\right)u_3(t)$$

The total stain energy in matrix form is

$$V(t) = \frac{1}{2}\begin{bmatrix} u_2 & u_3 \end{bmatrix} K \begin{bmatrix} u_2 & u_3 \end{bmatrix}^T$$

where

$$K = \frac{4A\left(\sqrt{e} - 1\right)E_0}{el}\begin{bmatrix} 1 + \sqrt{e} & -1 \\ -1 & 1 \end{bmatrix}$$

Likewise the mass matrix can be found from the total potential energy to be

$$M = \frac{Al\rho}{e}\begin{bmatrix} 8\left(e - 1 - \sqrt{e}\right) & 10 - 6\sqrt{e} \\ 10 - 6\sqrt{e} & 8 - 13\sqrt{e} \end{bmatrix}$$

and the natural frequencies are then

$$\omega_1 = \frac{1.939}{l}\sqrt{\frac{E}{\rho}} \text{ rad/s}, \quad \omega_2 = \frac{5.605}{l}\sqrt{\frac{E}{\rho}} \text{ rad/s}$$

If the traffic frequency corresponds to a natural frequency of a pillar, the bridge might fail.

8.19   Problems 8.17 and 8.18 represent approximations. As pointed out in Problem 8.18, it is important to know the natural frequencies of this column as precisely as possible. Hence consider modeling this column as a uniform bar of average cross section, calculate the first few natural frequencies, and compare them to the results in Problem 8.17 and 8.18. Which model do you think is closest to reality?
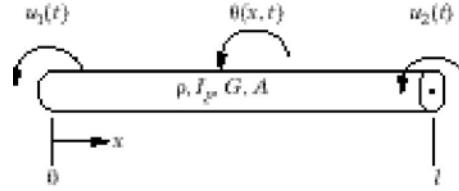
Solution:

The natural frequencies of a rod with constant cross sectional area are independent of the area. Therefore the first 2 natural frequencies are

$$\omega_1 \frac{1.571}{l}\sqrt{\frac{E}{\rho}} \text{ rad/s}, \quad \omega_2 = \frac{4.712}{l}\sqrt{\frac{E}{\rho}} \text{ rad/s}$$

It is doubtful that these results are better since we know from the finite element model that the varying cross sectional area does have an effect.

8.20 Torsional vibration can also be modeled by finite elements. Referring to Figure P8.20, calculate a single-element mass and stiffness matrix for the torsional vibration following the steps of Section 8.1. (*Hint*: $\theta(x,t) = c_1(t)\theta + c_2(t)$,

$T(t) = \dfrac{1}{2}\int_0^l \rho I_\rho \left[\theta_t(x,t)\right]^2 dx$ and $V(t) = \dfrac{1}{2}\int_0^l GI_\rho \left[\theta_t(x,t)\right]^2 dx$.)

Solution:

From equation (6.64), The static (time independent) displacement of the torsional rod element must satisfy

$$\frac{\partial \tau}{\partial x} = 0 = GJ\frac{\partial^2 \theta(x,t)}{\partial x^2}$$

which has the same form as equation (8.1). This can be integrated to yield

$\theta(x) = C_1\theta + C_2$

At $x = 0$

$\theta(0) = \theta_1(t) = C_2$

Likewise, at $x = l$

$\theta(l) = \theta_2(t) = C_1 l + C_2$

$C_1 = \dfrac{\theta_2(t) - C_2}{l} = \dfrac{\theta_2(t) - \theta_1(t)}{l}$

Substituting the values of C_1 and C_2 into the shape function yields

$$\theta(x,t) = \left(1 - \frac{x}{l}\right)\theta_1(t) + \left(\frac{x}{l}\right)\theta_2(t)$$

Evaluating the strain energy yields

$$V(t) = \frac{GJ}{2l}\left(\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2\right)$$

$$= \frac{1}{2}\begin{bmatrix}\theta_1(t) & \theta_2(t)\end{bmatrix}K\begin{bmatrix}\theta_1(t) & \theta_2(t)\end{bmatrix}^T$$

where the stiffness matrix is defined by

$$K = \frac{GJ}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Likewise, evaluating the kinetic energy yields

$$T(t) = \frac{1}{2}\frac{\rho A l}{3}\left(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)$$

$$= \frac{1}{2}\begin{bmatrix} \dot{\theta}_1(t) & \dot{\theta}_2(t) \end{bmatrix} M \begin{bmatrix} \dot{\theta}_1(t) & \dot{\theta}_2(t) \end{bmatrix}^T$$

where the mass matrix is defined by

$$M = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Problems and Solutions Section 8.3 (8.21 through 8.33)

8.21 Use equations (8.47) and (8.46) to derive equation (8.48) and hence make sure that the author and reviewer have not cheated you.

Solution:

$$u(x,t) = C_1(t)x^3 + C_2(t)x^2 + C_3(t)x + C_4(t) \qquad (8.46)$$

$$u(0,t) = u_1(t) \qquad\qquad u_x(0,t) = u_2(t)$$
$$u(l,t) = u_3(t) \qquad\qquad u_x(l,t) = u_4(t) \qquad (8.47)$$

Substituting(8.46) into (8.47)

$$u(0,t) = C_4(t) = u_1(t)$$
$$u_x(0,t) = C_3(t) = u_2(t)$$
$$u(l,t) = C_1(t)l^3 + C_2(t)l^2 + C_3(t)l + C_4(t) = u_3(t)$$
$$u_x(l,t) = 3C_1(t)l + 2C_2(t)l + C_3(t) = u_4(t)$$

This gives

$$C_1 = \frac{1}{l^3}\left(2(u_1 - u_3) + l(u_2 + u_4)\right)$$

$$C_2 = \frac{1}{l^2}\left(3(u_3 - u_1) - l(u_4 + 2u_2)\right)$$

$$C_3 = u_2$$
$$C_4 = u_1$$

8.22 It is instructive, though tedious, to derive the beam element deflection given by equation (8.49). Hence derive the beam shape functions.

Solution:

Substituting (8.48) into (8.46) gives

$$u(x,t) = \left[1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}\right]u_1(t) + l\left[\frac{x}{l} - 2\frac{x^2}{l^2} + \frac{x^3}{l^3}\right]u_2(t)$$

$$+ \left[3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right]u_3(t) + l\left[\frac{-x^2}{l^2} + \frac{x^3}{l^3}\right]u_4(t)$$

8.23 Using the shape functions of Problem 8.22, calculate the mass and stiffness matrices given by equations (8.53) and (8.56). Although tedious, this involves only simple integration of polynomials in x.

Solution:

$$T(t) = \frac{1}{2} \int_0^l \rho A \big(u_t(x,t) \big)^2 dx$$

$$= \frac{1}{2} \dot{\mathbf{u}}^T M \dot{\mathbf{u}}$$

where

$$\mathbf{u} = \begin{bmatrix} u_1(t) & u_2(t) & u_3(t) & u_4(t) \end{bmatrix}^T$$

And M is given by equation (8.35).

Similarly

$$V(t) = \frac{1}{2} \int_0^l EI \big[u_{xx}(x,t) \big]^2 dx$$

$$= \frac{1}{2} \mathbf{u}^T K \mathbf{u}$$

where K is given by (8.56)

8.24 Calculate the natural frequencies of the cantilevered beam given in equation (8.69) using $l = 1$ m and compare your results with those listed in Table 6.1.

Solution:

$$M = \frac{\rho A}{840} \begin{bmatrix} 312 & 0 & 54 & -6.5 \\ 0 & 2 & 6.5 & -.75 \\ 54 & 6.5 & 156 & -11 \\ -6.5 & -.75 & -11 & 1 \end{bmatrix}$$

$$K = 8EI \begin{bmatrix} 24 & 0 & -12 & 3 \\ 0 & 2 & -3 & \frac{1}{2} \\ -12 & -3 & 12 & -3 \\ 3 & \frac{1}{2} & -3 & 1 \end{bmatrix}$$

Following the procedures of section 4.2

$$\omega_1 = 3.5177\sqrt{\frac{EI}{\rho A}}, \quad \omega_2 = 22.2215\sqrt{\frac{EI}{\rho A}}$$

$$\omega_3 = 75.1571\sqrt{\frac{EI}{\rho A}}, \quad \omega_4 = 218.138\sqrt{\frac{EI}{\rho A}}$$

From continuous theory, the natural frequencies of a cantilevered beam are

$$\omega_i = \beta_i\sqrt{\frac{EI}{\rho A}} \text{ where } \beta_1 = 3.51601, \ \beta_2 = 22.0345, \ \beta_3 = 61.6972, \ \beta_4 = 120.9019.$$

The predictions of the first two natural frequencies are quite accurate while the predictions of the third and fourth natural frequencies are terrible.

8.25 Calculate the finite element model of a cantilevered beam one meter in length using three elements. Calculate the natural frequencies and compare them to those obtained in Problem 8.23 and with the exact values listed in Table 6.4.

Solution: Define u_i using the following figure;
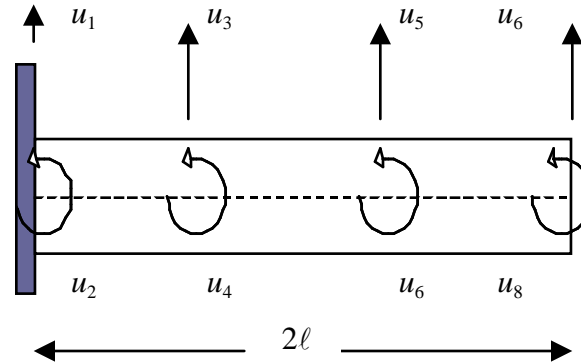
The equation for element one is

$$\frac{\rho A l}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \mathbf{0}$$

The equation for element two is

$$\frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \mathbf{0}$$

The equation for element 3 is the same as for element 2 but with the vector

$[u_3\ u_4\ u_5\ u_6]^{\mathrm{T}}$ replaced with $[u_5\ u_6\ u_7\ u_8]^{\mathrm{T}}$.

Combining the elemental equation using the superposition of the like coordinates yields

$$\frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13l & 0 & 0 \\ 0 & 8l^2 & 13l & -3l^2 & 0 & 0 \\ 54 & 13l & 312 & 0 & 54 & -13l \\ -13l & -3l^2 & 0 & 8l^2 & 13l & -3l^2 \\ 0 & 0 & 54 & 13l & 156 & -22l \\ 0 & 0 & -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \\ \ddot{u}_7 \\ \ddot{u}_8 \end{bmatrix}$$

$$+\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l & 0 & 0 \\ 0 & 8l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \mathbf{0}$$

which can also be written in the form

$$\frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13 & 0 & 0 \\ 0 & 8 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ l\ddot{u}_4 \\ \ddot{u}_5 \\ l\ddot{u}_6 \\ \ddot{u}_7 \\ l\ddot{u}_8 \end{bmatrix}$$

$$+\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 8 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} u_3 \\ lu_4 \\ u_5 \\ lu_6 \\ u_7 \\ lu_8 \end{bmatrix} = \mathbf{0}$$
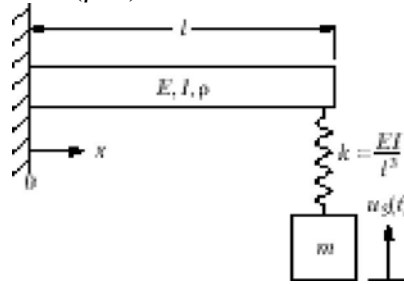
Following the procedure of example 8.3.3

$$\omega_1 = .3907\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\ ,\ \omega_2 = 2.456\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

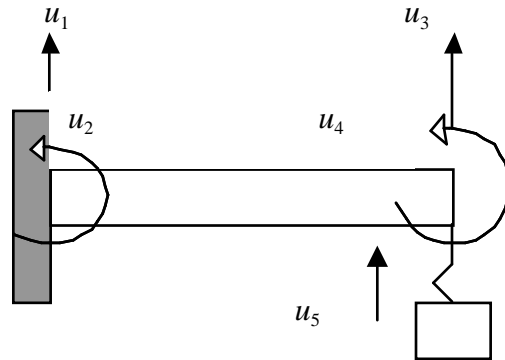$$\omega_3 = 6.941\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\ ,\ \omega_4 = 15.63\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_5 = 29.42\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\ ,\ \omega_6 = 58.64\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.26 Consider the cantilevered beam of Figure P8.26 attached to a lumped spring-mass system. Model this system using a single finite element and calculate the natural frequencies. Assume $m = (\rho A l)/420$.

Solution: Define u_i using the following figure:

The model for the spring mass system is

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_5 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_3 \\ u_5 \end{bmatrix} = \mathbf{0}$$

The single element model for the beam is

$$\frac{\rho A l}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \mathbf{0}$$

Superimposing like coordinates yields

$$\frac{\rho A l}{420}\begin{bmatrix} 156 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 13 & -6l & -1 \\ -6l & 4l^2 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix} = \mathbf{0}$$

The equation of motion may also be written

$$\frac{\rho A l^4}{420 EI}\begin{bmatrix} 156 & -22 & 0 \\ -22 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ l\ddot{u}_4 \\ \ddot{u}_5 \end{bmatrix} + \begin{bmatrix} 13 & -6 & -1 \\ -6 & 4 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_3 \\ lu_4 \\ u_5 \end{bmatrix} = \mathbf{0}$$

The eigenvalue/eigenvector problem is then

$(A - \lambda I)\mathbf{v} = 0$

where

$$A = M^{-1}K\frac{\rho A l^4}{420EI}, \quad \lambda = \frac{\rho A l^4}{420EI}\omega^2$$

$$A = \begin{bmatrix} -.5714 & .4571 & -.0286 \\ -4.6429 & 3.5143 & -.1571 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\lambda_1 = .0294, \ \lambda_2 = 1, \ \lambda_3 = 2.9134$$

$$\omega_1 = 3.52\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = 20.49\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_3 = 34.98\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.27 Repeat Problem 8.26 using two finite elements for the beam and compare the frequencies.

Solution:

A two element model of a cantilevered beam has been created in example 8.3.3.

Superimposing like coordinates for this example with the spring mass model yields

$$\frac{\rho A l}{840}\begin{bmatrix} 312 & 0 & 54 & -6.5l & 0 \\ 0 & 2l^2 & 6.5l & -.75l^2 & 0 \\ 54 & 6.5l & 156 & -11l & 0 \\ -6.5l & -.75l^2 & -11l & l^2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \\ \ddot{u}_7 \end{bmatrix}$$

$$+\frac{8EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 3l & 0 \\ 0 & 2l^2 & -3l & \frac{1}{2}l^2 & 0 \\ -12 & -3l & 12+\frac{1}{8} & -3l & -\frac{1}{8} \\ 3l & \frac{1}{2}l^2 & -3l & l^2 & 0 \\ 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}=\mathbf{0}$$

Note that the coordinate vector for the spring mass system has changed from $[u_3\ u_5]^{\mathrm{T}}$ to $[u_5\ u_6]^{\mathrm{T}}$.

As in (8.26), the equations may be written in the form

$$\frac{\rho A l^4}{6720 EI} \begin{bmatrix} 312 & 0 & 54 & -6.5 & 0 \\ 0 & 2 & 6.5 & -.75 & 0 \\ 54 & 6.5 & 156 & -11 & 0 \\ -6.5 & -.75 & -11 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{u}_3 \\ l\ddot{u}_4 \\ \ddot{u}_5 \\ l\ddot{u}_6 \\ \ddot{u}_7 \end{bmatrix}$$

$$+ \frac{8EI}{l^3} \begin{bmatrix} 24 & 0 & -12 & 3 & 0 \\ 0 & 2 & -3 & \dfrac{1}{2} & 0 \\ -12 & -3 & 12+\dfrac{1}{8} & -3 & -\dfrac{1}{8} \\ 3 & \dfrac{1}{2} & -3 & 1 & 0 \\ 0 & 0 & -\dfrac{1}{8} & 0 & \dfrac{1}{8} \end{bmatrix} \begin{bmatrix} u_3 \\ lu_4 \\ u_5 \\ lu_6 \\ u_7 \end{bmatrix} = \mathbf{0}$$

The eigenvalue/eigenvector problem is then

$$(A - \lambda I)\mathbf{v} = 0$$

where

$$A = M^{-1}K \frac{\rho A l^4}{6720 EI}, \quad \lambda = \frac{\rho A l^4}{6720 EI}\omega^2$$

$$A = \begin{bmatrix} .2878 & .0640 & -.2907 & .0868 & -.0004 \\ 3.6700 & 2.1247 & -5.9000 & 1.5919 & -.0062 \\ .9274 & .2094 & -1.0368 & .3516 & -.0041 \\ 17.8241 & 4.8163 & -20.7187 & 6.6253 & -.0519 \\ 0 & 0 & -.0625 & 0 & .0625 \end{bmatrix}$$

$$\lambda_1 = .0427, \ \lambda_2 = .2455, \ \lambda_3 = .2772, \ \lambda_4 = .9173, \ \lambda_5 = 2.6614$$

$$\omega_1 = \frac{3.50}{l^2}\sqrt{\frac{EI}{\rho A}}, \quad \omega_2 = \frac{20.12}{l^2}\sqrt{\frac{EI}{\rho A}}, \quad \omega_3 = \frac{22.73}{l^2}\sqrt{\frac{EI}{\rho A}}$$

The one element (3 DOF) model predicted the first 2 natural frequencies well. The prediction of the third natural frequency was extremely poor using only one element.

8.28 Calculate the natural frequencies of a clamped-clamped beam for the physical parameters $l = 1$m, $E = 2 \times 10^{11}$ N/m^2, $\rho = 7800$ kg/m^3, $I = 10^{-6}$ m^4, and $A = 10^{-2}$ m^2, using the beam theory of Chapter 6 and a four-element finite element model of the beam.

Solution:

Using VTB8_1

$$M = \begin{bmatrix} 14.49 & 0 & 2.5071 & -.151 & 0 & 0 \\ 0 & .0232 & .0151 & -.0087 & 0 & 0 \\ 2.507 & .151 & 14.49 & 0 & 2.507 & -.151 \\ -.151 & -.0087 & 0 & .0232 & .151 & -.0087 \\ 0 & 0 & 2.5071 & .151 & 14.49 & 0 \\ 0 & 0 & -.151 & -.0087 & 0 & .0232 \end{bmatrix}$$

and

$$K = 1 \times 10^5 \begin{bmatrix} 3072 & 0 & -1536 & 192 & 0 & 0 \\ 0 & 64 & -192 & 16 & 0 & 0 \\ -1536 & -192 & 3072 & 0 & -1536 & 192 \\ 192 & 16 & 0 & 64 & -192 & 16 \\ 0 & 0 & -1536 & -192 & 3072 & 0 \\ 0 & 0 & 192 & 16 & 0 & 64 \end{bmatrix}$$

Remember to zero the x translations since we are not interested in the extensional deformations. The natural frequencies are then found to be

$\omega_1 = 1134$ rad/s, $\omega_2 = 3152$ rad/s, $\omega_3 = 6253$ rad/s, $\omega_4 = 11830$ rad/s, $\omega_5 = 19565$ rad/s, $\omega_6 = 31524$ rad/s

From distributed theory

$\omega_1 = 1132.9$ rad/s, $\omega_2 = 3122.9$ rad/s, $\omega_3 = 6122.2$ rad/s, $\omega_4 = 10120$ rad/s, $\omega_5 = 15118$ rad/s, $\omega_6 = 21115$ rad/s

8.29 Repeat Problem 8.28 with two elements and compare the frequencies with the four-element model. Calculate the frequencies of a clamped-clamped beam using one element. Any comment?

Solution:

Since only two of the six degrees of freedom are free, the mass and stiffness matrices are simply

$$M = \frac{2\rho A \frac{l}{2}}{420}\begin{bmatrix} 156 & 0 \\ 0 & 4\left(\frac{l}{2}\right)^2 \end{bmatrix}$$

and

$$K = \frac{2EI}{\left(\frac{l}{2}\right)^3}\begin{bmatrix} 12 & 0 \\ 0 & 4\left(\frac{l}{2}\right)^2 \end{bmatrix}$$

where $l = 1$ m. The natural frequencies are then

$$\omega_1 = \sqrt{\frac{\dfrac{192EI}{l^3}}{\dfrac{156\rho Al}{420}}} = 22.736\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} = 1151 \text{ rad/s}$$

$$\omega_2 = \sqrt{\frac{\dfrac{16EI}{l^3}}{\dfrac{\rho Al}{420}}} = 81.96\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} = 4151 \text{ rad/s}$$

If you are only interested in the first natural frequency, a two degree of freedom model is adequate. However, the six degree of freedom model is much more accurate and can better predict the second mode. (In general, a finite element model must have twice as many degrees of freedom as the number of modes you want to predict).

8.30 Estimate the first natural frequency of a clamped-simply supported beam. Use a single finite element.

Solution: Since we are using only one element, we need only take the finite element matrix for a single element and strike out the rows and columns corresponding to the fixed degrees of freedom to get the global matrices. This yields

$$M = \frac{4l^3 \rho A}{420}, \quad K = \frac{4l^2 EI}{l^3}$$
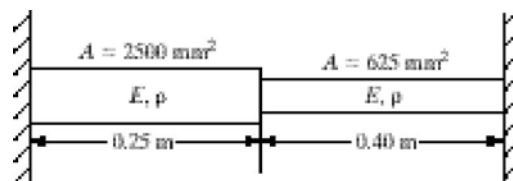
Since there is only a single degree of freedom

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{420} \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}} = 20.49 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}} \quad \text{rad/s}$$

Distributed theory yields

$$\omega_n = 15.42 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}$$

One degree of freedom is not enough to predict the first natural frequency.

8.31 Consider the stepped beam of Figure P8.31 clamped at each end. Both pieces are made of aluminum. Use two elements, one for each step, and calculate the natural frequencies.


Solution: Only a single degree of freedom is free. The mass and stiffness matrices are therefore scalars.

$$K = \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} = 809375000 \text{ N/m}$$

$$M = \frac{1}{3}\left(\frac{\rho_1 A_1}{l_1} + \frac{\rho_2 A_2}{l_2}\right) = 10.41 \text{ kg}$$

$$\omega = \sqrt{\frac{K}{M}} = 8819.2 \text{ rad/s}$$

8.32 Use a two-element model of nonuniform length to estimate the first few natural frequencies of a clamped-clamped beam. Use the spacing indicated in Figure P8.32. Compare the result to the actual frequencies and to those of Problem 8.28 and 8.29.
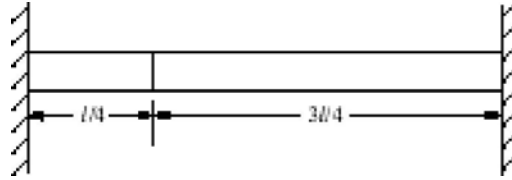
Solution: Since it has been shown in example 8.3.3 that the variable l can be factored outside of the mass and stiffness matrices, we can substitute the percentage of total length of each element into the mass and stiffness matrices and get the correct natural frequencies.

$$M = \frac{\rho A(.25l)}{420}\begin{bmatrix} 156 & -22 \times .25 \\ -22 \times .25 & 4 \times .25^2 \end{bmatrix} + \frac{\rho A(.75l)}{420}\begin{bmatrix} 156 & 22 \times .75 \\ 22 \times .75 & 4 \times .75^2 \end{bmatrix}$$

$$= \frac{\rho A l}{420}\begin{bmatrix} 156 & 11 \\ 11 & 1.75 \end{bmatrix}$$

Similarly,

$$K = \frac{EI}{(.25l)^3}\begin{bmatrix} 12 & -6 \times .25 \\ -6 \times .25 & 4 \times .25^2 \end{bmatrix} + \frac{EI}{(.75l)^3}\begin{bmatrix} 12 & 6 \times .75 \\ 6 \times .75 & 4 \times .75^2 \end{bmatrix}$$

$$= \frac{EI}{l^3}\begin{bmatrix} 796.4 & -85.3 \\ -85.3 & 21.33 \end{bmatrix}$$

$$\omega = \sqrt{eig\left(\tilde{M}^{-1}\tilde{K}\right)}\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

where $\tilde{M}$ and $\tilde{K}$ represent the mass and stiffness matrices with the variables E, I, l, ρ and A factored out.

$$\omega_1 = 25.31\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}, \quad \omega_2 = 132.6\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

This is not nearly as good as the two element model where ω_1 was found to be

$$\omega_1 = 22.74\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

as opposed to the "actual" (from distributed parameter theory) value of

$$\omega_1 = 22.37\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.33 Calculate the first natural frequency of a clamped-pinned beam using first one, then two elements.

Solution:

From problem 8.30, using one element yields

$$\omega_1 = 20.49 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}$$

Using the vibration toolbox and the method described in 8.3.3 (also in the README.8 file) the two element model yields

$$\omega_1 = 15.56 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = 58.41 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}$$

$$\omega_3 = 155.6 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}$$

Problems and Solutions Section 8.4 (8.34 through 8.43)

8.34   Refer to the tapered bar of Figure P8.13. Calculate a lumped-mass matrix for this system and compare it to the solution of Problem 8.13. Since the beam is tapered, be careful how you divide up the mass.

Solution: The lumped mass at node 2 should be the total mass between $x = .25$ and $x = .75$. Therefore

$$M_2 = 2700 \int_{.25}^{.75} \frac{\pi}{4} \left[h_1^2 + \left(\frac{h_2 - h_1}{l} \right)^2 x^2 + 2h_1 \left(\frac{h_2 - h_1}{l} \right) x \right] dx$$

$$= 26.5$$

likewise for node 3

$$M_3 = 2700 \int_{.75}^{1} \frac{\pi}{4} \left[h_1^2 + \left(\frac{h_2 - h_1}{l} \right)^2 x^2 + 2h_1 \left(\frac{h_2 - h_1}{l} \right) x \right] dx$$

$$= 7.289$$

The mass matrix is then

$$M = \begin{bmatrix} 26.5 & 0 \\ 0 & 7.289 \end{bmatrix}$$

and the natural frequencies are

ω_1 = 6670 rad/s and ω_2 = 13106 rad/s.

For the distributed mass system

ω_1 = 7414 rad/s and ω_2 = 20368 rad/s.

The first natural frequency found by the distributed mass model is slightly better than the lumped mass model when compared to the three element distributed mass model derived in problem 13.

8.35   Calculate and compare the natural frequencies obtained for a tapered bar by using first, the consistent-mass matrix (Problem 8.12), and second, the lumped-mass matrix (Problem 8.34).

Solution:

See solution for Problem 8.34.

8.36 Consider again the machine punch of Problem 8.16 and Figure P8.15. Calculate the natural frequencies of this system using a lumped-mass matrix and compare the results to those obtained with the consistent-mass matrix.

Solution:

The lumped mass matrix is

$$M = \begin{bmatrix} \dfrac{\rho_1 A_1 l_1}{2} + \dfrac{\rho_2 A_2 l_2}{2} & 0 \\ 0 & \dfrac{\rho_2 A_2 l_2}{2} \end{bmatrix}$$

$$= rl \begin{bmatrix} A_1 + A_2 & 0 \\ 0 & A_2 \end{bmatrix}$$

$$= \begin{bmatrix} .078 & 0 \\ 0 & .039 \end{bmatrix}$$

The natural frequencies are

$\omega_1 = 38756$ rad/s and $\omega_2 = 93565$ rad/s.

The results for the consistent mass matrix were

$\omega_1 = 40798.6$ rad/s and $\omega_2 = 142525$ rad/s.

The first natural frequency is within 5% for both predictions. For this case, the inconsistent mass matrix is adequate for the 1st mode.

8.37 Consider again the bridge support of Figure P8.17 discussed in connection with Problem 8.17. Develop a four-element finite element model of this structure using a lumped-mass approximation and calculate the natural frequencies. Use constant area elements.

Solution:

We will use elements which each have constant cross section by finding the average area for each element. Elements are numbered from one to four from bottom to top.

$$A_1 = \frac{1}{.25l} \int_0^{.25l} A(x)dx = \frac{A_0}{.25l}\left(-le^{-\frac{x}{l}}\right)\Big|_0^{.25l}$$

$$= -4A_0\left(e^{-.25} - 1\right) = .8848A_0$$

likewise

$$A_2 = .6891A_0, \ A_3 = .5367A_0, \ A_4 = .4179A_0$$

Assembling the stiffness matrix yields

$$K = \frac{EA_0}{.25l}\begin{bmatrix} 1.5739 & -.6891 & 0 & 0 \\ -.6891 & 1.2258 & -.5367 & 0 \\ 0 & -.5367 & .9546 & -.4179 \\ 0 & 0 & -.4179 & .4179 \end{bmatrix}$$

To find the mass matrix, we will assume again that the elements have constant cross section. This yields

$$M = \frac{\rho A_0 l}{8}\begin{bmatrix} 1.5739 & 0 & 0 & 0 \\ 0 & 1.2258 & 0 & 0 \\ 0 & 0 & .9546 & 0 \\ 0 & 0 & 0 & .4179 \end{bmatrix}$$

The natural frequencies are then

$$\omega_1 = 1.86\frac{1}{l}\sqrt{\frac{E}{\rho}}, \ \omega_2 = 4.50\frac{1}{l}\sqrt{\frac{E}{\rho}}, \ \omega_3 = 6.62\frac{1}{l}\sqrt{\frac{E}{\rho}}, \ \omega_4 = 7.78\frac{1}{l}\sqrt{\frac{E}{\rho}},$$

8.38 Consider the torsional vibration problem illustrated in Figure P8.20 and discussed in Problem 8.20. Calculate a lumped-mass matrix for the single element.

Solution:

The total mass moment of inertia would be divided between the two degrees of freedom.

Therefore

$$M = \frac{1}{2}\begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix}$$

8.39 Estimate the first three natural frequencies of a clamped-free bar of length l in torsional vibration by using a lumped-mass model and four elements.

Solution:

The stiffness matrix is

$$K = \frac{4G\gamma}{l}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The mass matrix is

$$M = \frac{\rho J l}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The natural frequencies are then

$$\omega_1 = 1.56\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}}, \; \omega_2 = 4.445\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}}, \; \omega_3 = 6.65\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}}, \; \omega_4 = 7.8463\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}}$$

From table 6.3, it can be seen that the first two natural frequencies predicted by the finite element model are good approximations.

8.40 Calculate the natural frequencies of a pinned-pinned beam of length l using one element and the consistent-mass matrix of equation (8.73).

Solution:

The mass matrix is

$$M = \frac{\rho A l^3}{48} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the stiffness matrix is

$$K = \frac{EI}{l} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

Finding the natural frequencies gives

$$\omega_1 = 9.798 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}, \quad \omega_2 = 16.971 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}$$

The first natural frequency from distributed theory is

$$\omega_n = 9.869 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}$$

8.41 Calculate the natural frequencies of a pinned-pinned beam of length l using one element and the lumped-mass matrix of equation (8.73). Compare your results to those obtained with at consistent-mass matrix of Problem 8.40.

Solution:

The consistent mass matrix is

$$M = \frac{\rho A l^3}{420} \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$$

which gives

$$\omega_1 = 10.96 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}, \quad \omega_2 = 50.20 \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}$$

which is worse than the inconsistent mass matrix results. (See solution 8.40)

8.42 Calculate a three-element finite element model of a cantilevered beam (see Problem 8.25) using a lumped mass that includes rotational inertia. Also calculate the system's natural frequencies and compare them with those obtained with a consistent-mass matrix of Problem 8.25 and with the values obtained by the methods of Chapter 6.

Solution:

The mass matrix is $M = \rho Al \, \text{diag}\left(1, \dfrac{1}{24}, 1, \dfrac{1}{24}, \dfrac{1}{2}, \dfrac{1}{48} \right)$ using the $[u_1 \; lu_2]$ convention for the displacement vector.

The natural frequencies are then

$$\omega_i = a_i \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}}$$

$a_i = .368, \; 2.00, \; 4.98, \; 10.7, \; 14.5, \; 17.1$

This is not as good as the consistent mass matrix results. From distributed parameter theory $a_1 = .3911$.

8.43 Repeat Problem 8.42 using a lumped-mass matrix that neglects the rotational degree of freedom. Discuss any problems you encounter when trying to solve the related eigenvalue problem.

Solution:

$$M = \rho Al \, \text{diag}\left(1, 0, 1, 0, \frac{1}{2}, 0 \right)$$

The singularity of the mass matrix does not allow a solution to be found.

Problems and Solutions Section 8.5 (8.44 through 8.49)

8.44 Derive a consistent-mass matrix for the system of Figure 8.9. Compare the natural frequencies of this system with those calculated with the lumped-mass matrix computed in Section 8.5.

Solution: Using the vibration toolbox

$$M = \rho Al \begin{bmatrix} .6857 & 0 \\ 0 & .7238 \end{bmatrix}$$

The natural frequencies are then

$$\omega_1 = .8311\frac{1}{l}\sqrt{\frac{E}{\rho}} \text{ and } \omega_1 = 1.479\frac{1}{l}\sqrt{\frac{E}{\rho}}$$

These are higher than those predicted with the inconsistent mass matrix

8.45 Consider the two beam system of Figure P8.45. Use VTB8_1 to create a two-element, rod/beam element model and compute the first three natural frequencies. Use $A = 0.0004$ m^2, $I = 1.33 \times 10^{-8}$ m^4, and the properties of aluminum. Assume that nodes 1 and 3 are clamped.
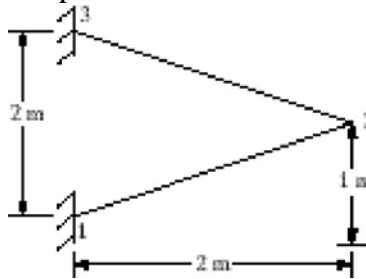
Solution:
```
%scipt file for problem 8.45
node=[0 0;1 .5;2 1;1  1.5;0 2];
ncon=[1 2 69e10 .004 1.33e-8 0 2700;
      2 3 69e10 .004 1.33e-8 0 2700;
      3 4 69e10 .004 1.33e-8 0 2700;
      4 5 69e10 .004 1.33e-8 0 2700];
zero=[1 1;
      1 2;
      1 3;
      5 1;
      5 2;
      5 3];
conm=[];
force=[];
save VTB8_45.con
```

Running this yields that the first three natural frequencies are given as 377.5, 8763.7 and 10951.2 rad/s.

8.46 Follow the procedure of Problem 8.45 using two elements for each beam. Compare the natural frequencies and mode shapes of the four element model produced here to those of the two-element model of Problem 8.45. State which model is better and why.

Solution: Use the script file from 8.45 ending in VTB8_46.con
The first five natural frequencies are 286.8, 419.1, 1074.5, 1510.8, and 2838.9 rad/s. The result from the four element model is probably better because the additional elements allow the first few modes to be found in more detail. Notice the difference in the result for the first mode. The first mode is primarily a rotation of the joint between the two beams. The two element model shows this to be the only significant motion (load the .out data file to observe the mode shape vector). The four element model shows that the middle of each beam displaces and rotates as well.

The eight element model predicts the first five natural frequencies to be 284.3, 413.0 ,925.6, 1147.3, and 1959.7 rad/s, the first four of which agree well with the four element model results.

8.47 Determine a finite element model of the three-bar truss of Figure P8.47 using a lumped-mass matrix.
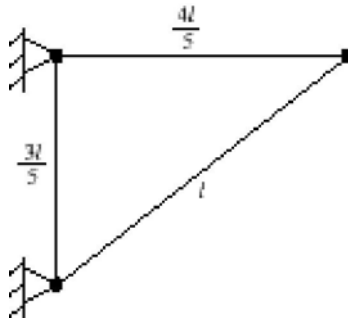
Solution:

Using VTB8_1

$$K = \frac{EA}{l}\begin{bmatrix} 1.89 & .48 \\ .48 & .36 \end{bmatrix}$$

The inconsistent mass matrix is

$$M = \rho Al\begin{bmatrix} .9 & 0 \\ 0 & .9 \end{bmatrix}$$

8.48   Determine a finite element model for the three-bar truss of Figure P8.47 using a consistent-mass matrix.

Solution:

Using VTB8_1 the consistent mass matrix is

$$M = \rho Al \begin{bmatrix} .6137 & -.0183 \\ -.0183 & .6549 \end{bmatrix}$$

However, this mass matrix is created using beam/rod elements. Using simple rod elements gives a consistent mass matrix

$$M = \rho Al \begin{bmatrix} .48 & .16 \\ .16 & .12 \end{bmatrix}$$

8.49   Compare the frequencies obtained for the system of Problem 8.48 with those of Figure P8.47.

Solution:

The natural frequencies using the consistent mass matrix are

$\omega_1 = 1.7321 \quad \omega_2 = 2.1651$

The natural frequencies using the inconsistent mass matrix are

$\omega_1 = .4966 \quad \omega_2 = 1.5012$

These results are terribly inconclusive, but since we have seen in previous examples that the consistent mass matrix generally yields the better results, one would expect the same to be true in this case.

Problems and Solutions Section 8.6 (8.50 through 8.54)

8.50 Consider the machine punch of Figure P8.15. Recalculate the fundamental natural frequency by reducing the model obtained in Problem 8.16 to a single degree of freedom using Guyan reduction.

Solution:

From the results of 8.16

$$K = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \times 10^8, \quad M = \begin{bmatrix} .052 & .013 \\ .013 & .026 \end{bmatrix}$$

From (8.104)

$$Q^T M Q = .052 + .013 + .013 + .026 = .104$$

From (8.105)

$$Q^T K Q = (4 - 2) \times 10^8 = 2 \times 10^8$$

$$\omega = \sqrt{\frac{2 \times 10^8}{.104}} = 43852.9 \text{ rad/s}$$

which is a poor prediction of the first natural frequency. If we reorder K and M (reducing to coordinate 2) we get

$$Q^T M Q = .026 + .013 + .013 = .052$$

$$Q^T K Q = (2 - 1) \times 10^8 = 1 \times 10^8$$

$$\omega = 43852.9 \text{ rad/s}$$

which is the same result as reducing to coordinate 1.

8.51 Compute a reduced-order model of the three-element model of a cantilevered bar given in Example 8.3.2 by eliminating u_2 and u_3 using Guyan reduction. Compare the frequencies of each model to those of the distributed model given in Window 8.1.

Solution:

$$M = \frac{\rho Al}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$K = \frac{3EI}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Let $\tilde{M}$ and $\tilde{K}$ be the matrices with the coefficients factored out.

$$\tilde{M}_{11} = 4, \ \tilde{M}_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \tilde{M}_{12}^T, \ M_{22} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\tilde{K}_{11} = 2, \ \tilde{K}_{21} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \tilde{K}_{11}^T, \ K_{22} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Using equations (8.104) and (8.105)

$$\tilde{M}_r = Q^T MQ = 14$$
$$\tilde{K}_r = Q^T KQ = 1$$
and

$$\omega_n = \sqrt{\dfrac{\dfrac{3EA}{l}}{\dfrac{14\rho Al}{18}}} = 1.964\frac{1}{l}\sqrt{\frac{E}{\rho}}$$

as compared to the distributed model value of

$$\omega_1 = 1.57\frac{1}{l}\sqrt{\frac{E}{\rho}}$$

8.52 Consider the system defined by the matrices

$$M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K = \begin{bmatrix} 20 & -1 & 0 & 0 \\ -1 & 20 & -3 & 0 \\ 0 & -3 & 20 & -17 \\ 0 & 0 & 17 & 17 \end{bmatrix}$$

Use mass condensation to reduce this to a two-degree-of-freedom system with a nonsingular mass matrix.

Solution:

Following the same procedure as example 8.6.1

$$M_r = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } K_r = \begin{bmatrix} 19.95 & -.15 \\ -.15 & 36.55 \end{bmatrix}$$

8.53 Recall the punch press problem modeled in Figure 4.28 and treated in Example 4.8.3. The mass and stiffness matrices are given by

$$M = \begin{bmatrix} 0.4 \times 10^3 & 0 & 0 \\ 0 & 2.0 \times 10^3 & 0 \\ 0 & 0 & 8.0 \times 10^3 \end{bmatrix} \quad K = \begin{bmatrix} 30 \times 10^4 & 30 \times 10^4 & 0 \\ 30 \times 10^4 & 38 \times 10^4 & 8 \times 10^4 \\ 0 & 8 \times 10^4 & 88 \times 10^4 \end{bmatrix}$$

Recalling that the only external force acting on the machine is at the $x_1(t)$ coordinate, reduce this to a single-degree-of-freedom system using Guyan reduction to remove x_2 and x_3. Compare this single frequency with those of Example 4.8.3.

Solution:

Following the same procedure as example 8.6.1

$M_r = 1.7385 \times 10^3, K_r = 5.8537 \times 10^4$ and the natural frequency is

$$\omega_n = \sqrt{\frac{K_r}{M_r}} = 5.803 \,\text{rad/s}$$

Example 4.8.3 gave the first natural frequency as $\omega_1 = 5.387$ rad/s which is within 10% of the Guyan reduced prediction.

8.54. Consider the beam example given in Example 7.6.2. Using the values given there (An aluminum beam: 0.5128 m x 25.5 mm x 3.2 mm, $E = 6.9 \times 10^{10}$ N/m^2, $\rho = 2715$ kg/m^3, $A = 8.16$ m^2 and $I = 6.96 \times 10^{-11}$ m^4), compute the first 4 natural frequencies as accurately as possible and compare them to both the analytical values and the measured values.